Assignment 1: Assigned Thu 01/11. Due Thu 01/18

1. **Section 1.1** 10, 20, 36
2. **Section 1.2** 22, 24, 32
3. If you have a system of m linear homogeneous equations in n variables, then justify the following:
   (a) If $m < n$, then the set of solutions is *at least* a $n - m$ dimensional hyperplane. (More precisely, at least $n - m$ variables can be chosen arbitrarily).
   (b) If $m < n$, the system has infinitely many solutions. [This means that if you have *fewer equations than variables*, then the *homogeneous* system always has infinitely many solutions.]
   (c) The system always has at least one solution, no matter what m and n are.
4. Let M be a $m \times n$ matrix, and consider the linear homogeneous system of equations $M\vec{x} = \vec{0}$. (Here the notation means that $\vec{x}$ is an n dimensional vector).
   (a) If $\vec{u} \in \mathbb{R}^n$ is a solution of $M\vec{x} = \vec{0}$, and $\alpha \in \mathbb{R}$ is any number, then show that $\alpha\vec{u}$ is also a solution.
   (b) If $\vec{u}, \vec{v} \in \mathbb{R}^n$ are two solutions to the system, then show that $\vec{u} + \vec{v}$ is also a solution.
   (c) What is wrong with the following logic:

   > By qutesion 3c, we know the system $M\vec{x} = \vec{0}$ has at least one solution, call it $\vec{v}$. Now take any real number α. By the previous subpart, $\alpha\vec{x}$ is also a solution. Thus the system has infinitely many solutions.

   (d) Give an example of a linear homogeneous system of equations with exactly one solution. [If you were not convinced that the logic in the previous was incorrect, then this should convince you.]
5. Let M be an $m \times n$ matrix and $\vec{c} \in \mathbb{R}^n$, and consider the linear system of equations $M\vec{x} = \vec{c}$. Suppose the property in questions 4a and 4b hold: Namely if $\vec{u}, \vec{v} \in \mathbb{R}^n$ are two solutions to $M\vec{x} = \vec{c}$ then so is $\alpha\vec{u}$ and $\vec{u} + \vec{v}$. The show that the system must be homogeneous (i.e. $\vec{c} = \vec{0}$). [You can assume that the system $M\vec{x} = \vec{c}$ has at least one solution]
6. Show that if you perform *column* operations on a matrix M, then you will not change the row rank. [Hint: First convince yourself that this is true for matrices that are already in row echeleon form]

Assignment 2: Assigned Thu 01/18. Due Thu 01/25

1. **Section 2.1** 6, 8, 44, 48. **Section 2.3** 2, 8.
2. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(\vec{x}) = \binom{2x_1+x_3}{3x_2-x_1}$.
   (a) Express $T(\vec{x})$ as multiplication by some matrix A.
   (b) Verify that the columns of A are $T(\vec{e}_1)$, $T(\vec{e}_2)$ and $T(\vec{e}_3)$ respectively.
   (c) Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation obtained by rotating counter clockwise by $30°$. Compute the matrix of R, and $R \circ T$, and verify that the matrix of $R \circ T$ is the product of the matrices of R and T respectively.
3. We know that rotating the plane by an angle of α_1 and then by α_2 should result in a total rotation of $\alpha_1 + \alpha_2$. Verify this by multiplying the appropriate rotation matrices.
4. To make the notation from class a little more conventional, we re-index as follows: Say $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation. Suppose $T(\vec{e}_i) = \sum_j a_{ji}\vec{e}_j$ (i.e. $T(\vec{e}_i)$ is the vector with coordinates $a_{1i}, a_{2i}, \ldots, a_{ni}$). Note how this is different from class. Let A be the $n \times m$ matrix whose entry in the i^{th} row and j^{th} column is a_{ij}. Show that $T(\vec{x}) = A\vec{x}$. [This is pretty much the same as in class, except now the matrix A is written in the more conventional form.]
5. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation.
   (a) If $m < n$ show that T is not injective. Thus if $m < n$, T can not be invertible. [Hint: Remember T can be written as multiplication by some matrix, A. Then try and show there exist infinitely many $\vec{x}$ such that $A\vec{x} = 0$. Previous homework problems will help ...]
   (b) If $m > n$, show that T can not be invertible. [Hint: Suppose T is invertible. Try and apply the previous part to T^{-1}.]
   (c) Conclude that if T is invertible, then $m = n$. The converse however is false. Give an example of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^m$ which is not invertible.