Lecture 7 - SISO Loop Analysis

SISO = Single Input Single Output

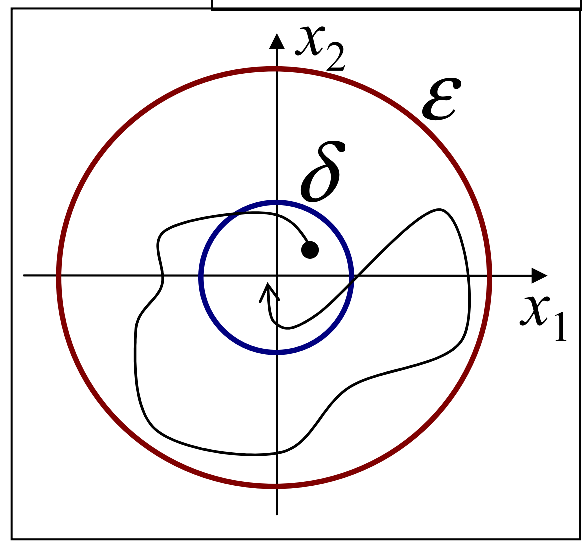
Analysis:

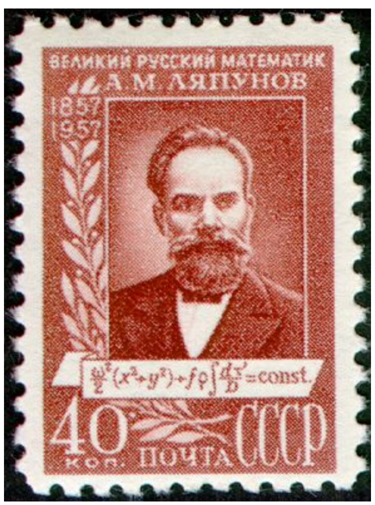
- Stability
- Performance
- Robustness

ODE stability

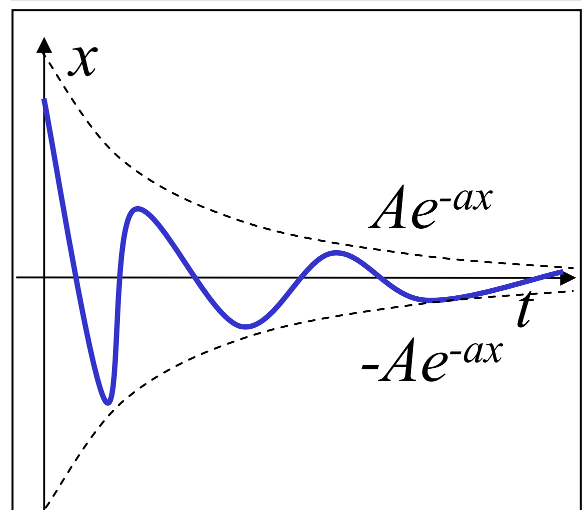
$$\dot{x} = f(x,t)$$

- Lyapunov's mathematical stability theory - nonlinear systems
  - stability definition
  - first (direct) method
    - exponential convergence
  - second method: Lyapunov function
    - generalization of energy dissipation





- Lyapunov's exponent
  - dominant exponent of the convergence
  - for a nonlinear system
  - for a linear system defined by the poles

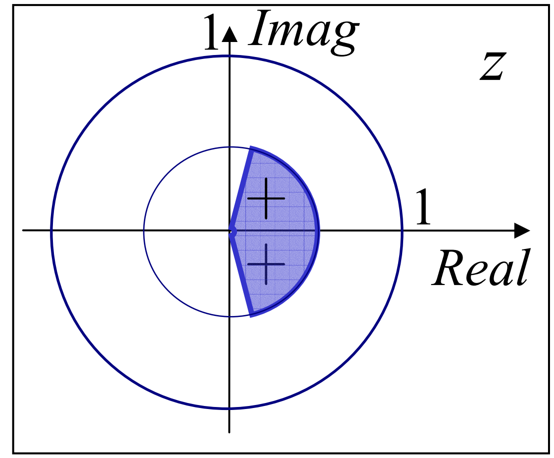


Stability: poles

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

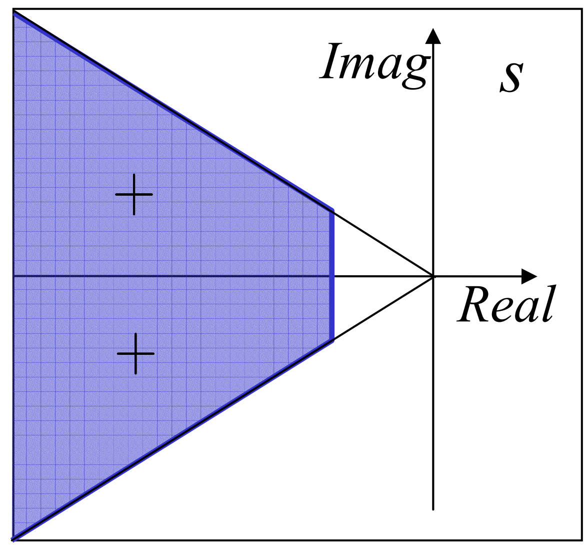
$$y = H(s)\cdot u$$

$$H(s) = C(Is - A)^{-1}B + D$$

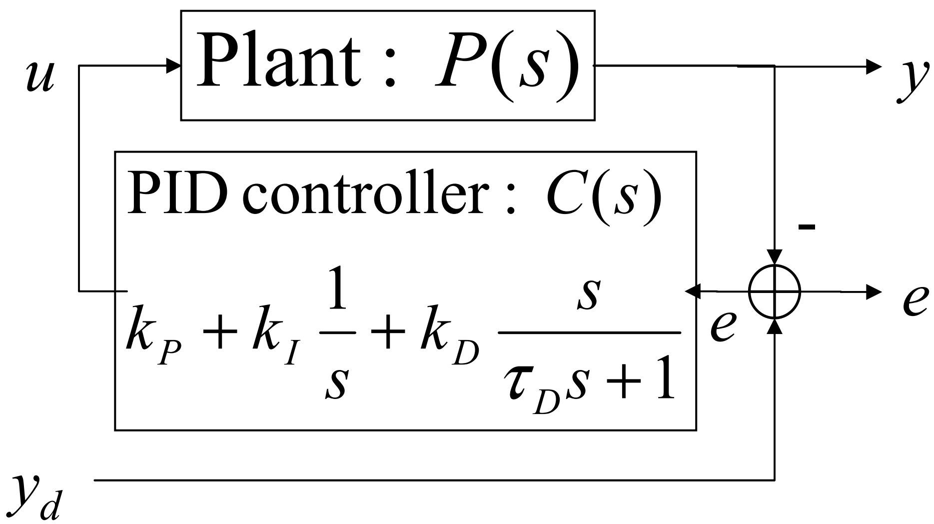


- Characteristic values = transfer function poles
  - l.h.p. for continuous time
  - unit circle for sampled time
- I/O model vs. internal dynamics

$$H(s) = \frac{N(s)}{D(s)} = \frac{g_1}{s - p_1} + \ldots + \frac{g_n}{s - p_n} + g_0$$



Stability: closed loop



$$y = P(s)u$$

$$u = -C(s)e; \qquad e = y - y_d$$

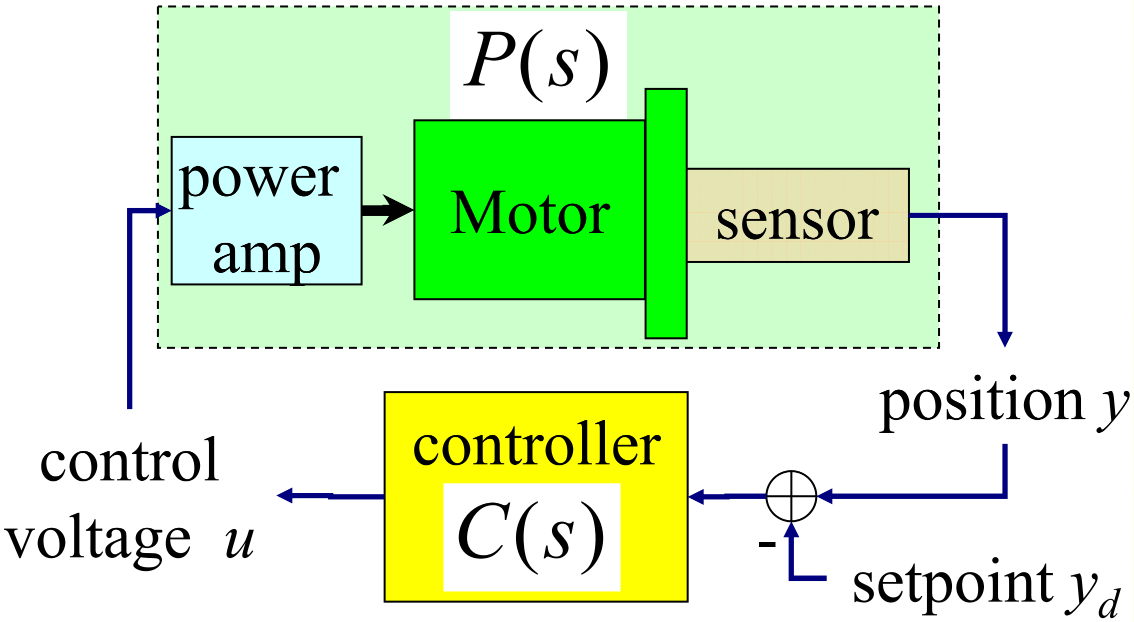
$$e = \underbrace{\left[1 + P(s)C(s)\right]^{-1}}_{S(s)} y_d$$

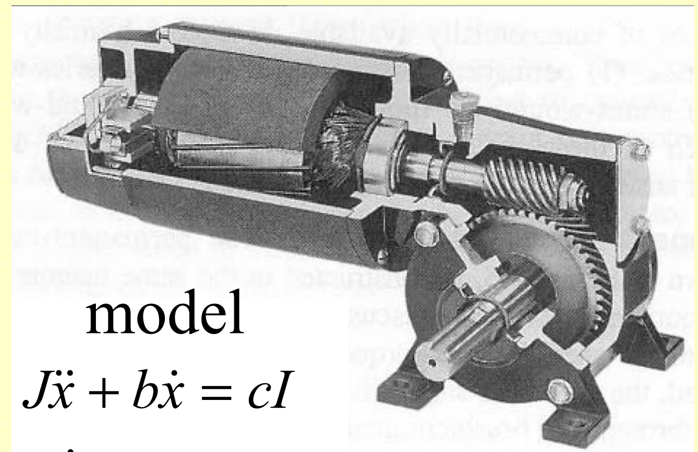
- The transfer function poles are the zeros of

$$1 + P(s)C(s)$$

- Watch for pole-zero cancellations!
- Poles define the closed-loop dynamics (including stability)
- Algebraic problem, easier than state space sim

Servomotor Example





model

$$J\ddot{x} + b\dot{x} = cI$$

$$L\dot{I} + RI = du$$

$$y = \mathrm{LPF}(x)$$

Transfer function

$$y = \frac{G}{(1+T_F s)(1+T_I s)(s+T_J s^2)} u$$

$T_J = 0.1\,\mathrm{sec}$, $T_I = 0.02\,\mathrm{sec}$

$T_F = 0.001\,\mathrm{sec}$ $G = 1$

- The control goal is to track the position setpoint y_d
- Use PID control $\tau_D = 0.01$

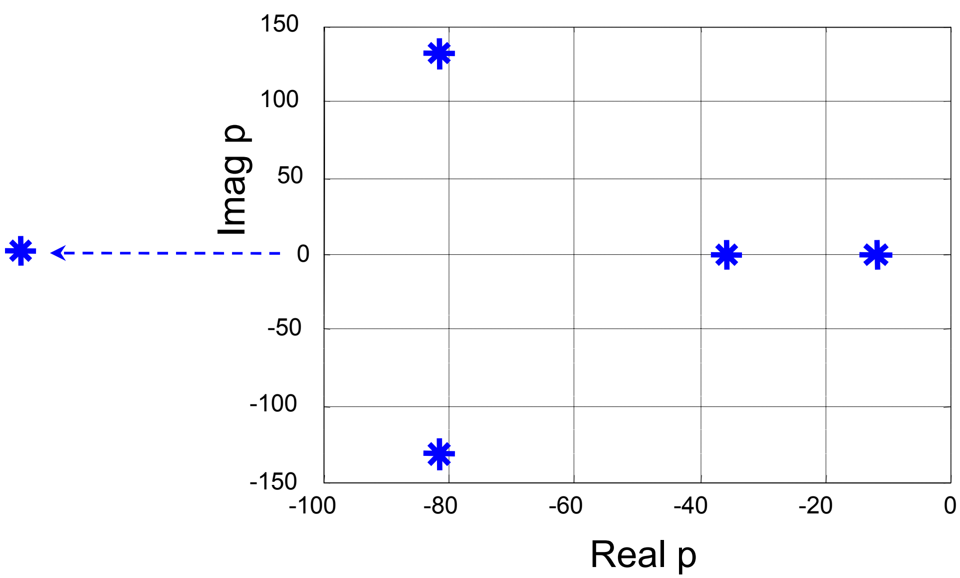
$$C(s) = k_P + k_I \frac{1}{s} + k_D \frac{s}{\tau_D s + 1}$$

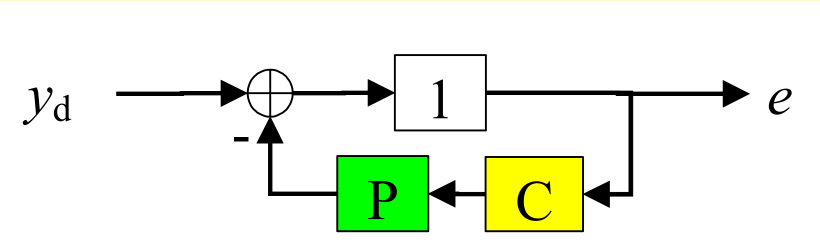
Servomotor Example

- PID controller: $k_P = 10; k_I = 100; k_D = 0.1; \tau_D = 0.01$

$$S(s) = [1 + P(s)C(s)]^{-1}$$

- Stability



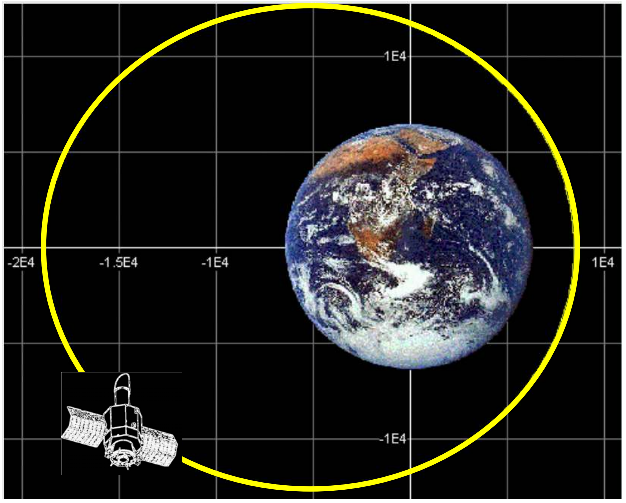



```
>> S = feedback(1,C*P);
>>
>>
>> [z,p,k] = zpkdata(S);
>> plot(p)
```

Stability

For linear system poles describe stability

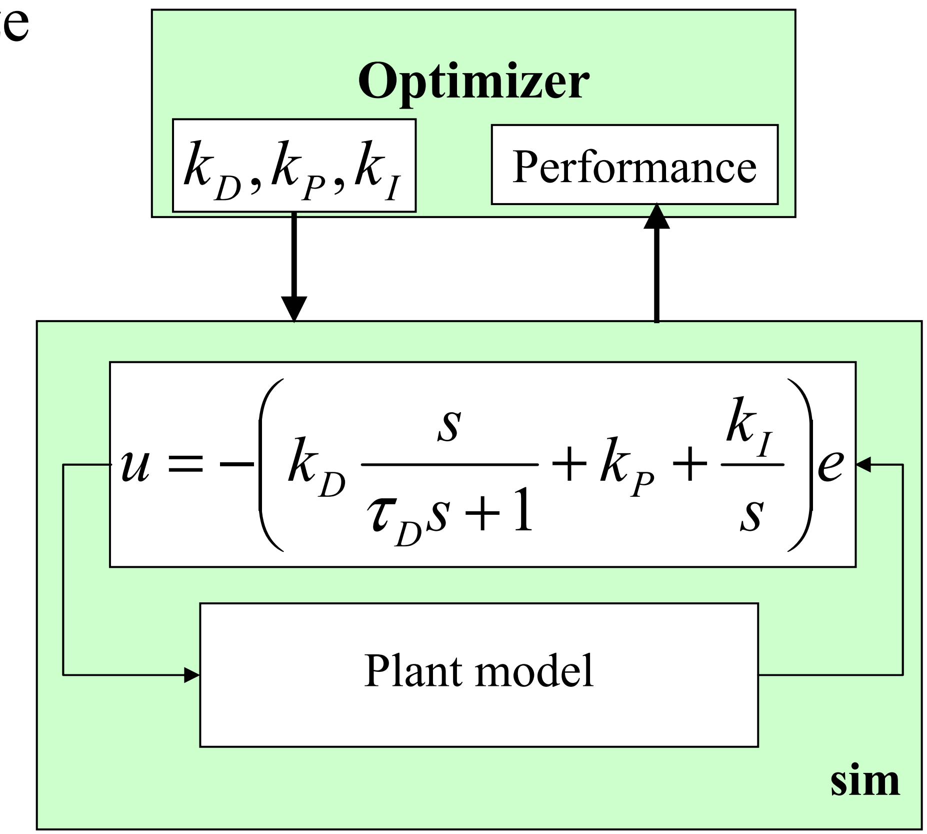
- … almost, except the critical stability
- For nonlinear systems
  - linearize around the equilibrium
  - might have to look at the stability theory - Lyapunov
- Orbital stability:
  - trajectory converges to the desired
  - the state does not - the timing is off
    - spacecraft
    - FMS, 3-D trajectories without aircraft arrival time



Performance

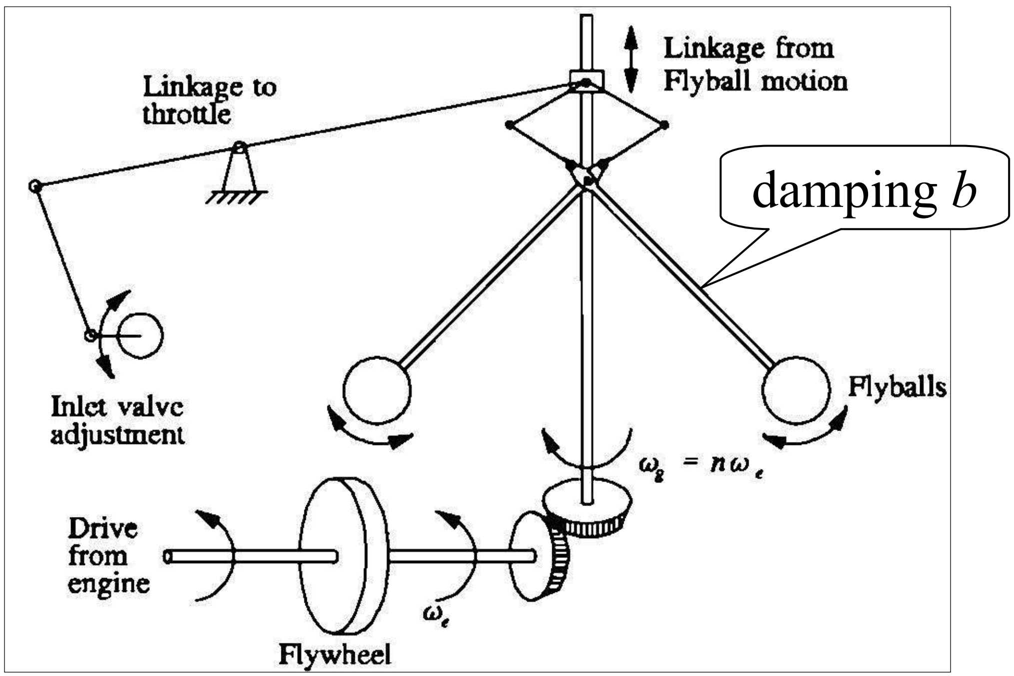
- Need to describe and analyze performance so that we can design systems and tune controllers
- What is the performance index?

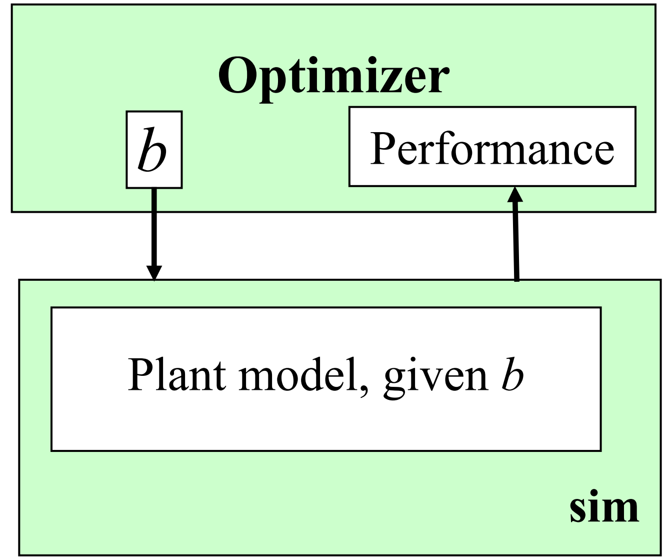
- There are usually many conflicting requirements
- Engineers look for a reasonable trade-off

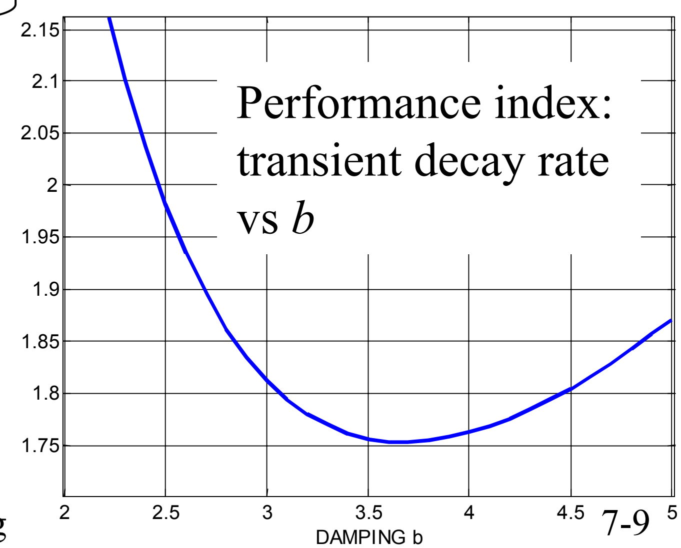


Performance: Example

- Selecting optimal *b* in the Watt's governor - HW Assignment 1



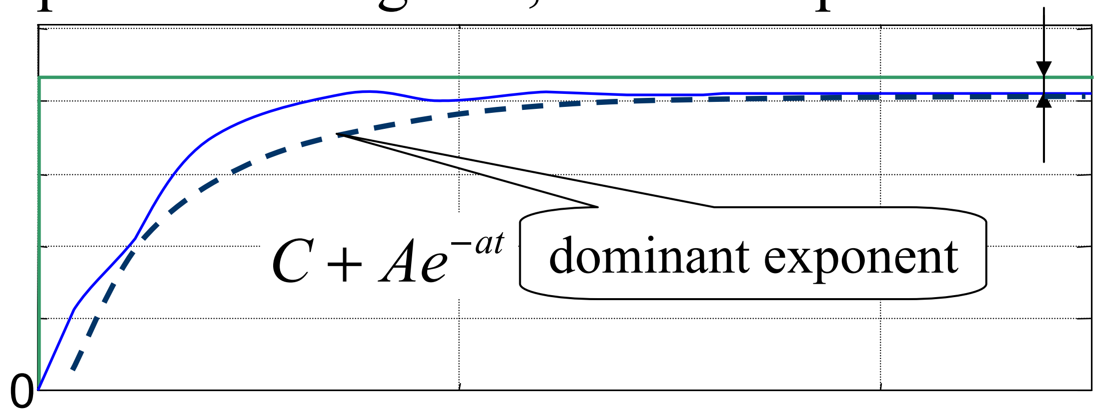




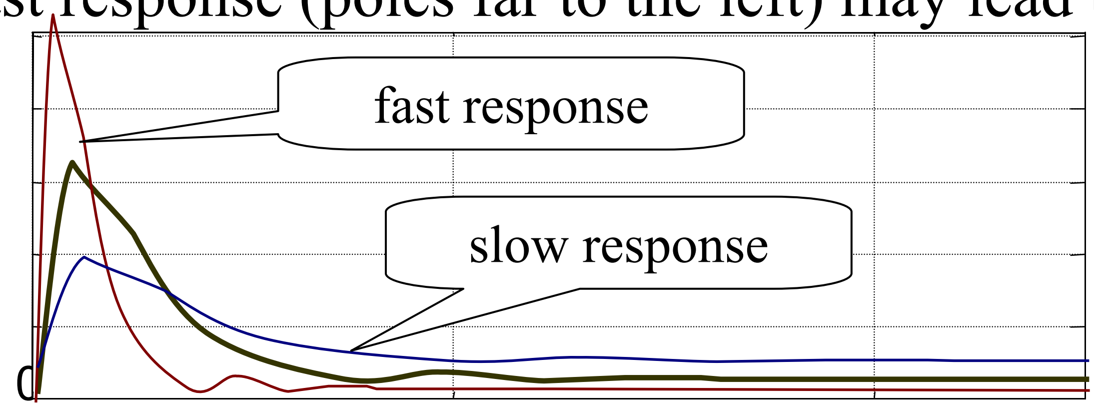
Performance - poles

- Steady state error: study transfer functions at s=0.
- Step/pulse response convergence, dominant pole

$$a = \min\{\operatorname{Re} p_j\}_{j=1}^{n}$$

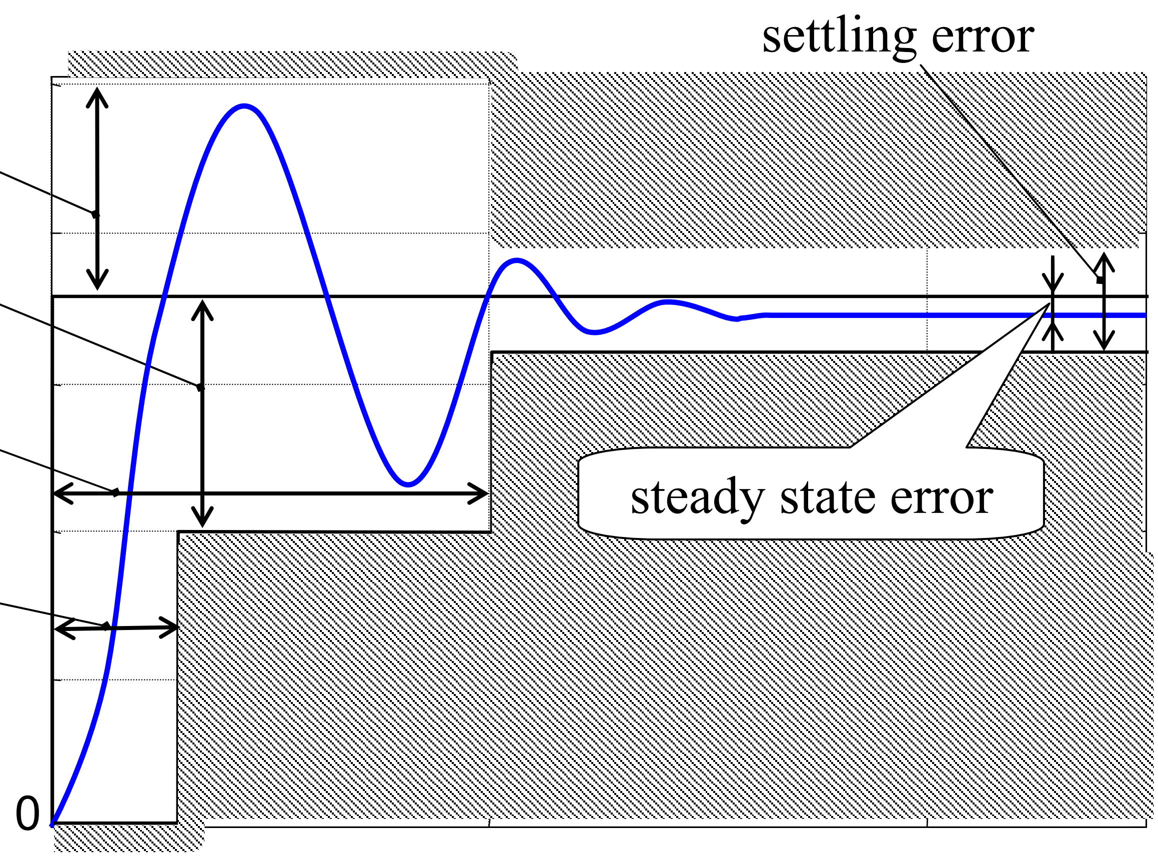


- Caution! Fast response (poles far to the left) may lead to peaking



Performance - step response

- Step response shape characterization:
  - overshoot
  - undershoot
  - settling time
  - rise time

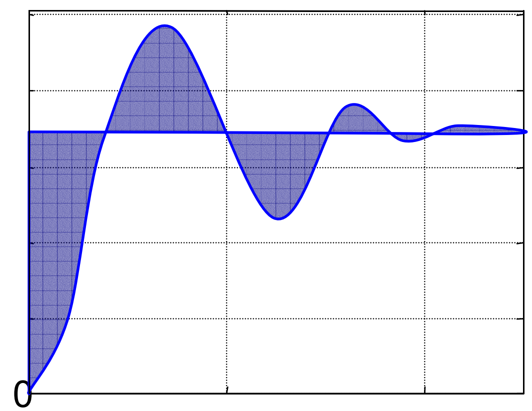


Performance - quadratic index

- Quadratic performance
  - response, in frequency domain

$$J = \int_{t=0}^{\infty} \left| y(t) - y_d(t) \right|^2 dt = \frac{1}{2\pi} \int_{\infty}^{\infty} \left| \tilde{e}(i\omega) \right|^2 d\omega =$$

$$\frac{1}{2\pi} \int \left| S(i\omega) \tilde{y}_d(i\omega) \right|^2 d\omega = \frac{1}{2\pi} \int \left| S(i\omega) \right|^2 \underbrace{\frac{1}{\omega^2}}_{\text{STEP}} d\omega$$



$$S(s) = \left[1 + P(s)C(s) \right]^{-1}$$

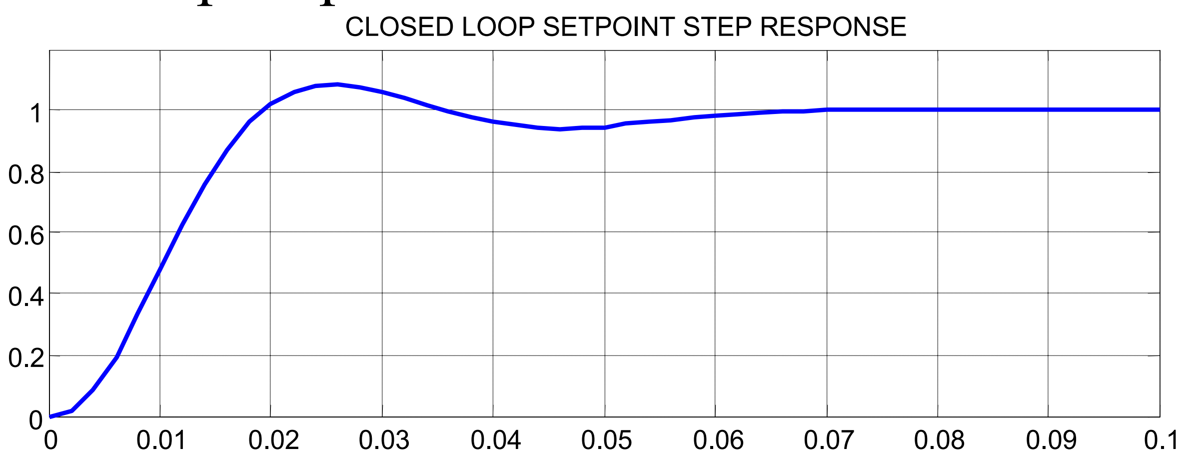
- For $y_d(t)$ a zero mean random process with spectral power $Q(i\omega)$

$$J = E\left(\int_{t=0}^{\infty} \left| y(t) - y_d(t) \right|^2 dt \right) = \frac{1}{2\pi} \int \left| S(i\omega) \right|^2 Q(i\omega) d\omega$$

- For $Q(i\omega) = 1$, this is just Parceval's theorem

Servomotor example

- Step response




```
>> T = feedback(C*P,1);
>> step(T)
>>
```

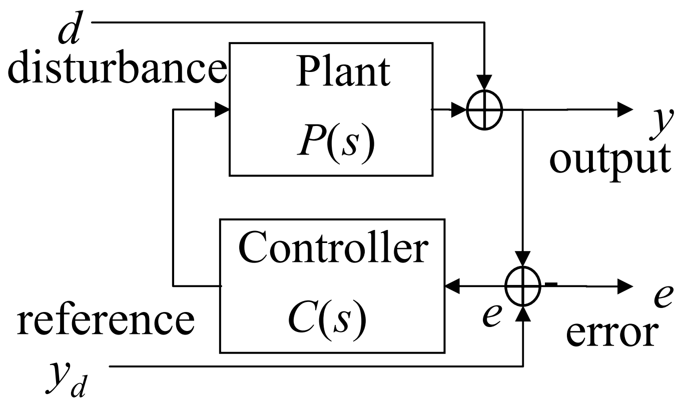
- Quadratic index

$$J = \int_{t=0}^{\infty} \left| y(t) - y_d(t) \right|^2 dt$$

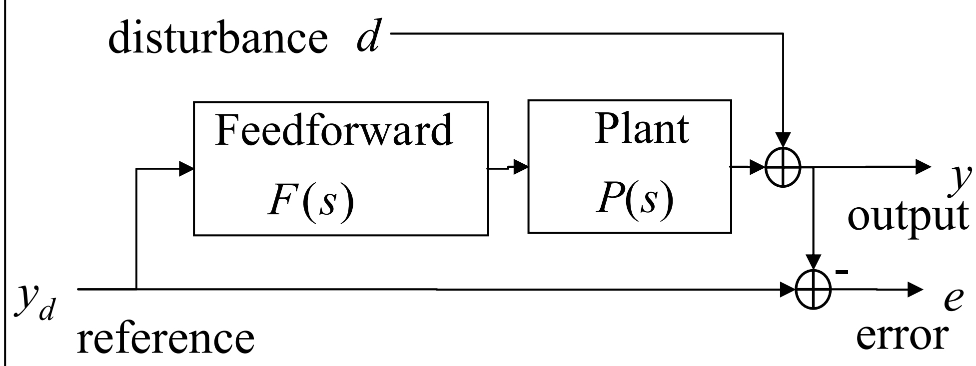
```
>> T  = feedback(C*P,1);
>> dt = 0.01;
>> y  = step(T,0:dt:0.1);
>> J  = sum((y-1).^2*dt)

J =
    0.0081
```

Sensitivities



$$y = S(s)d + T(s)y_d$$



$$y = d + F(s)P(s)y_d$$

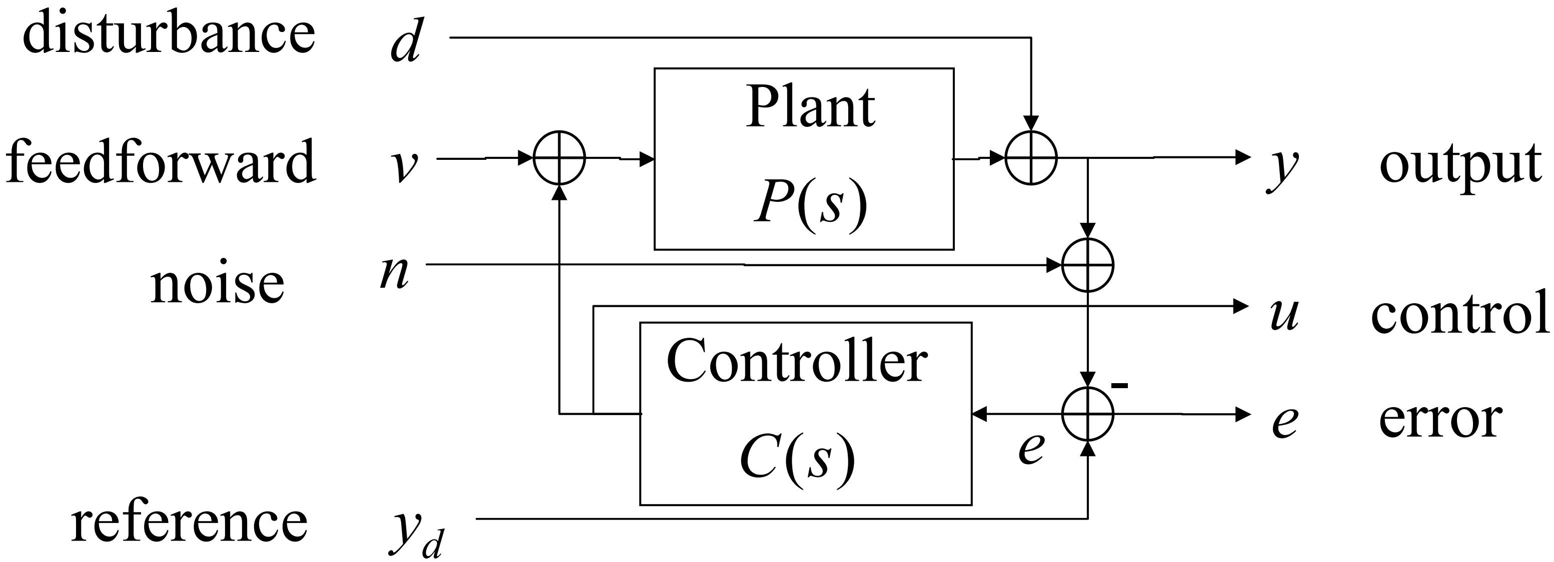
$$S(i\omega) = \frac{1}{1 + L(i\omega)}$$

$$S(s) = [1 + P(s)C(s)]^{-1}$$
$$L(s) = P(s)C(s)$$

$$S_{FF}(i\omega) = 1$$

- Feedback sensitivity
  - $|S(i\omega)| \ll 1$ for $|L(i\omega)| \gg 1$
  - $|S(i\omega)| \approx 1$ for $|L(i\omega)| \ll 1$
  - can be bad for $|L(i\omega)| \approx 1$ - ringing, instability
- Feedforward sensitivity
  - good for any frequency
  - never unstable

Transfer functions in control loop



$$e = S(s)d - S(s)y_d + T(s)n + S_y(s)v$$

$$y = S(s)d + T(s)y_d + T(s)n + S_y(s)v$$

$$u = -S_u(s)d + S_u(s)y_d + S_u(s)n + T(s)v$$

Transfer functions in control loop

$$
\begin{aligned}
e &= y - y_d + n \\
y &= P(s)(u + v) + d \\
u &= -C(s)e
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
e &= S(s)d - S(s)y_d + T(s)n + S_y(s)v \\
y &= S(s)d + T(s)y_d + T(s)n + S_y(s)v \\
u &= -S_u(s)d + S_u(s)y_d + S_u(s)n + T(s)v
\end{aligned}
$$

Sensitivity $S(s) = [1 + P(s)C(s)]^{-1}$

Complementary sensitivity $T(s) = [1 + P(s)C(s)]^{-1} P(s)C(s)$

Noise sensitivity $S_u(s) = [1 + P(s)C(s)]^{-1} C(s)$

Load sensitivity $S_y(s) = [1 + P(s)C(s)]^{-1} P(s)$

$$\boxed{S(s) + T(s) = 1}$$

Sensitivity requirements

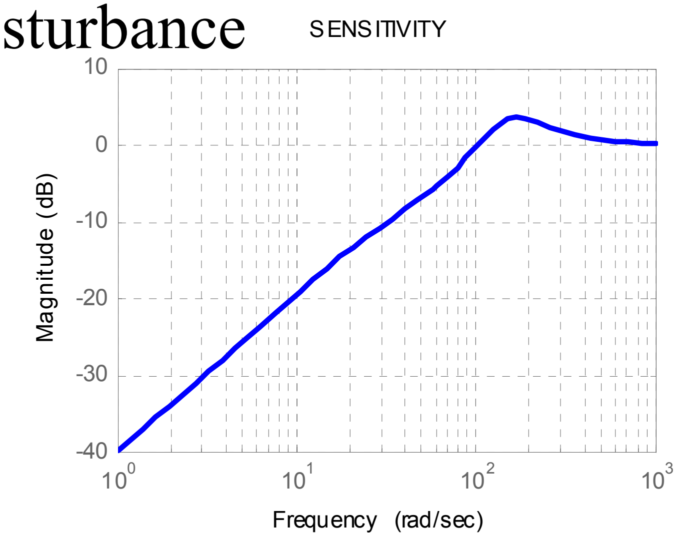
$$e = S(s)d - S(s)y_d + T(s)n + S_y(s)v$$
$$y = S(s)d + T(s)y_d + T(s)n + S_y(s)v$$
$$u = -S_u(s)d + S_u(s)y_d + S_u(s)n + T(s)v$$

$$S(i\omega) = \frac{1}{1 + P(i\omega)C(i\omega)}$$
$$S_y(i\omega) = \frac{P(i\omega)}{1 + P(i\omega)C(i\omega)}$$
$$S_u(i\omega) = \frac{C(i\omega)}{1 + P(i\omega)C(i\omega)}$$

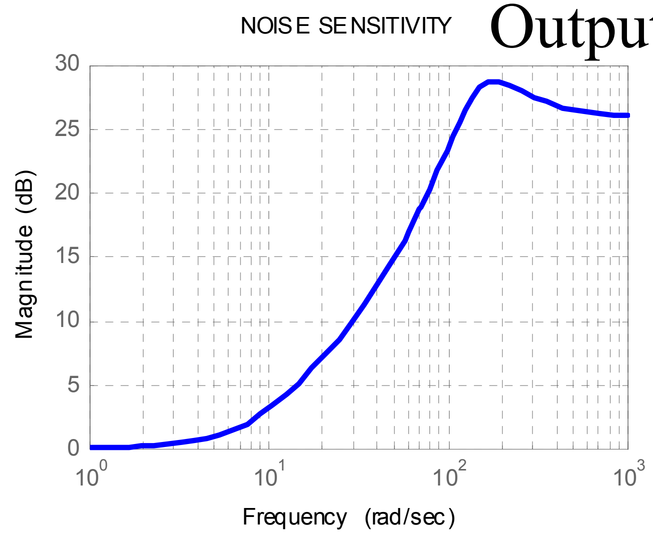
- Disturbance rejection and reference tracking
  - $|S(i\omega)| << 1$ for the disturbance d
  - $|S_y(i\omega)| << 1$ for the input 'noise' v
- Limited control effort
  - $|S_u(i\omega)| << 1$ conflicts with disturbance rejection where $|P(i\omega)| < 1$
- Noise rejection
  - $|T(i\omega)| << 1$ for the noise n, conflicts with disturbance rejection

Servomotor example - sensitivities

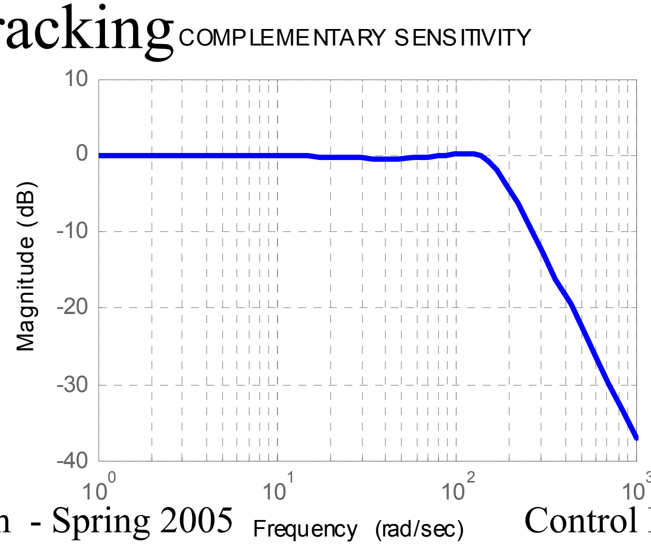
Output disturbance



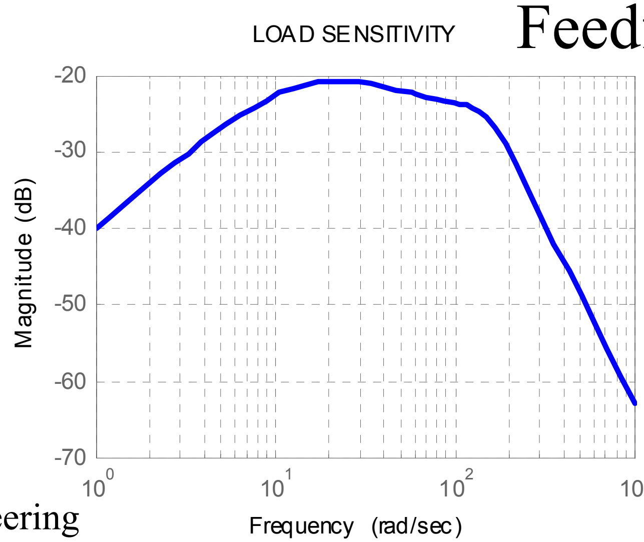
Output noise



Setpoint tracking

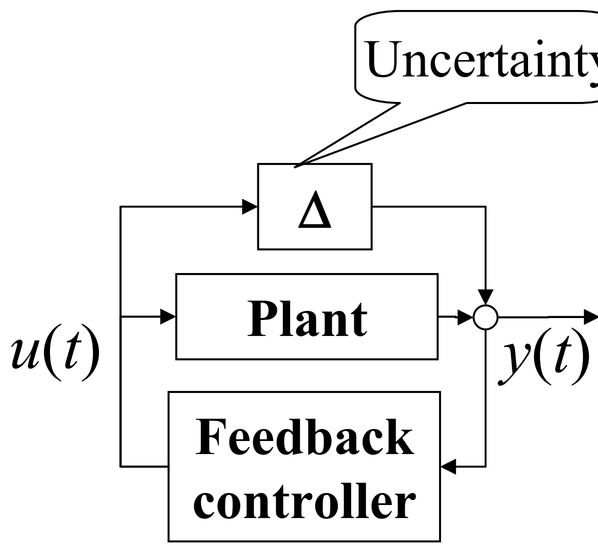


Feedforward



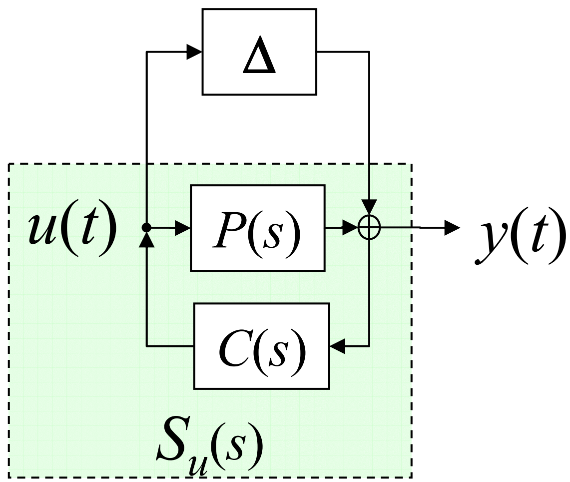
Robustness

- A controller works for a model.
- Will it work for a *real system*?
- Can check that controller works for a *range* of different models and hope that the real system is covered by this range



Robustness

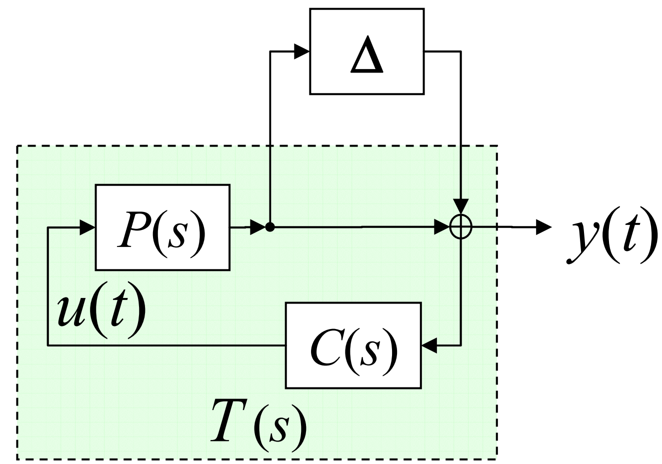
- Additive uncertainty



Condition of robust stability

$$\underbrace{\left|\frac{C(i\omega)}{1+P(i\omega)C(i\omega)}\right|}_{\|S_u\|} \cdot \underbrace{\left|\Delta(i\omega)\right|}_{\|\Delta\|} < 1$$

- Multiplicative uncertainty

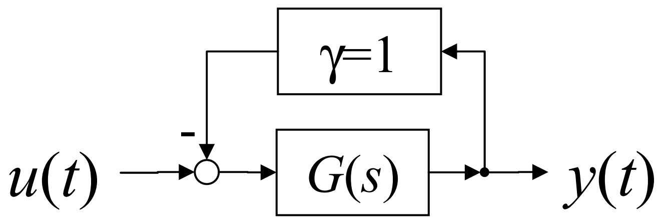


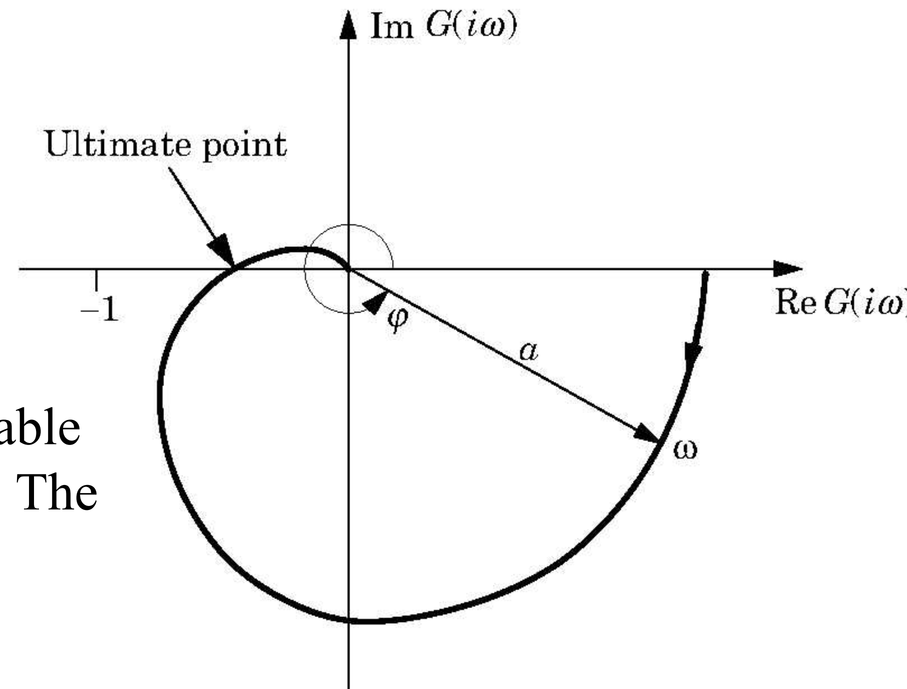
Condition of robust stability

$$\underbrace{\left|\frac{P(i\omega)C(i\omega)}{1+P(i\omega)C(i\omega)}\right|}_{\|T\|} \cdot \underbrace{\left|\Delta(i\omega)\right|}_{\|\Delta\|} < 1$$

Small Gain Theorem: loop gain < 1 → stability

Nyquist stability criterion

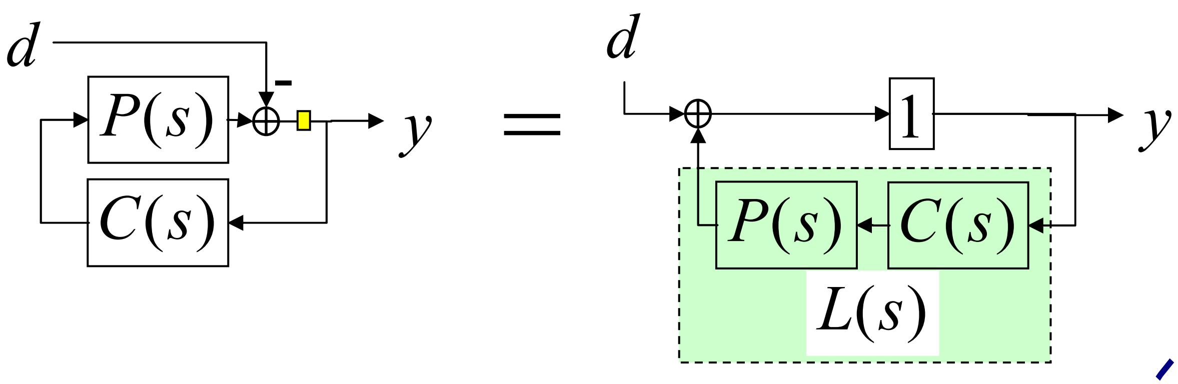




- Homotopy "Proof"
  - $G(s)$ is stable, hence the loop is stable for $\gamma=0$. Gradually increase γ to 1. The instability cannot occur unless $\gamma G(iw)+1=0$ for some $0 \le \gamma \le 1$.
  - $|G(i\omega_{180})|<1$ is a *sufficient* condition
- Subtleties: r.h.p. poles and zeros
  - Formulation and real proof using the agrument principle, encirclements of -1
  - stable → unstable → stable as $0 \to \gamma \to 1$

Compare against Small Gain Theorem:

Gain and phase margins

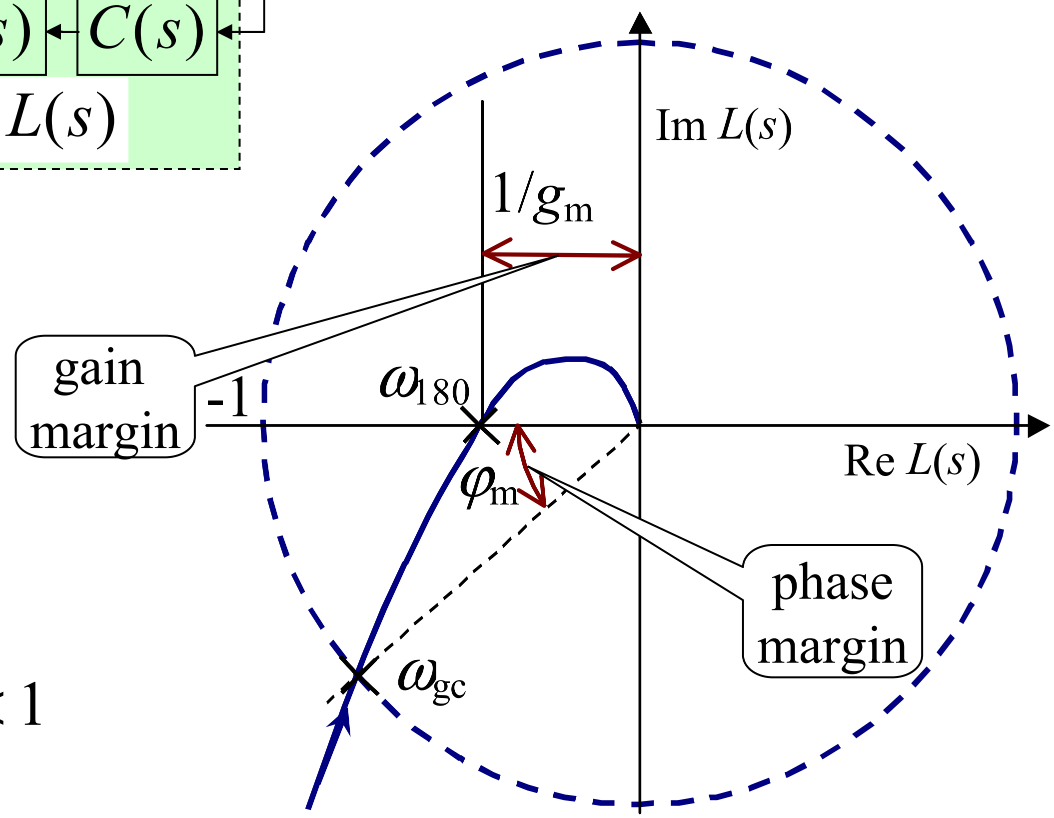


- Loop gain

$$L(s) = P(s)C(s)$$

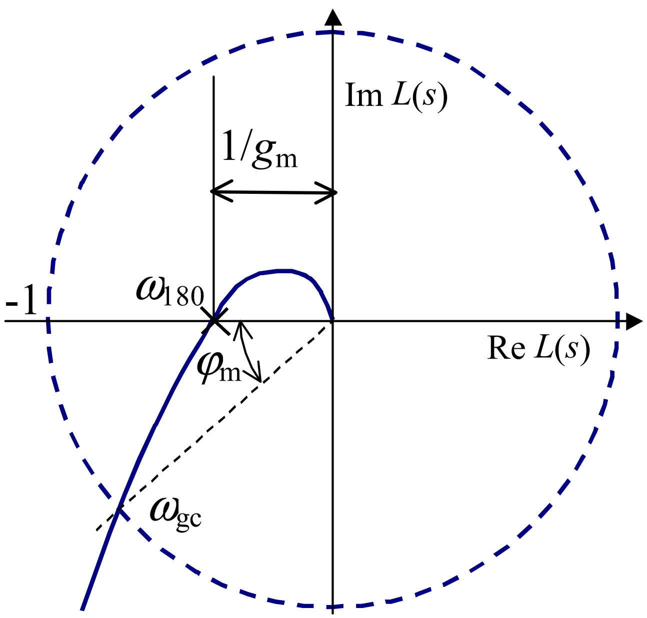
$$S(s) = [1 + L(s)]^{-1}$$

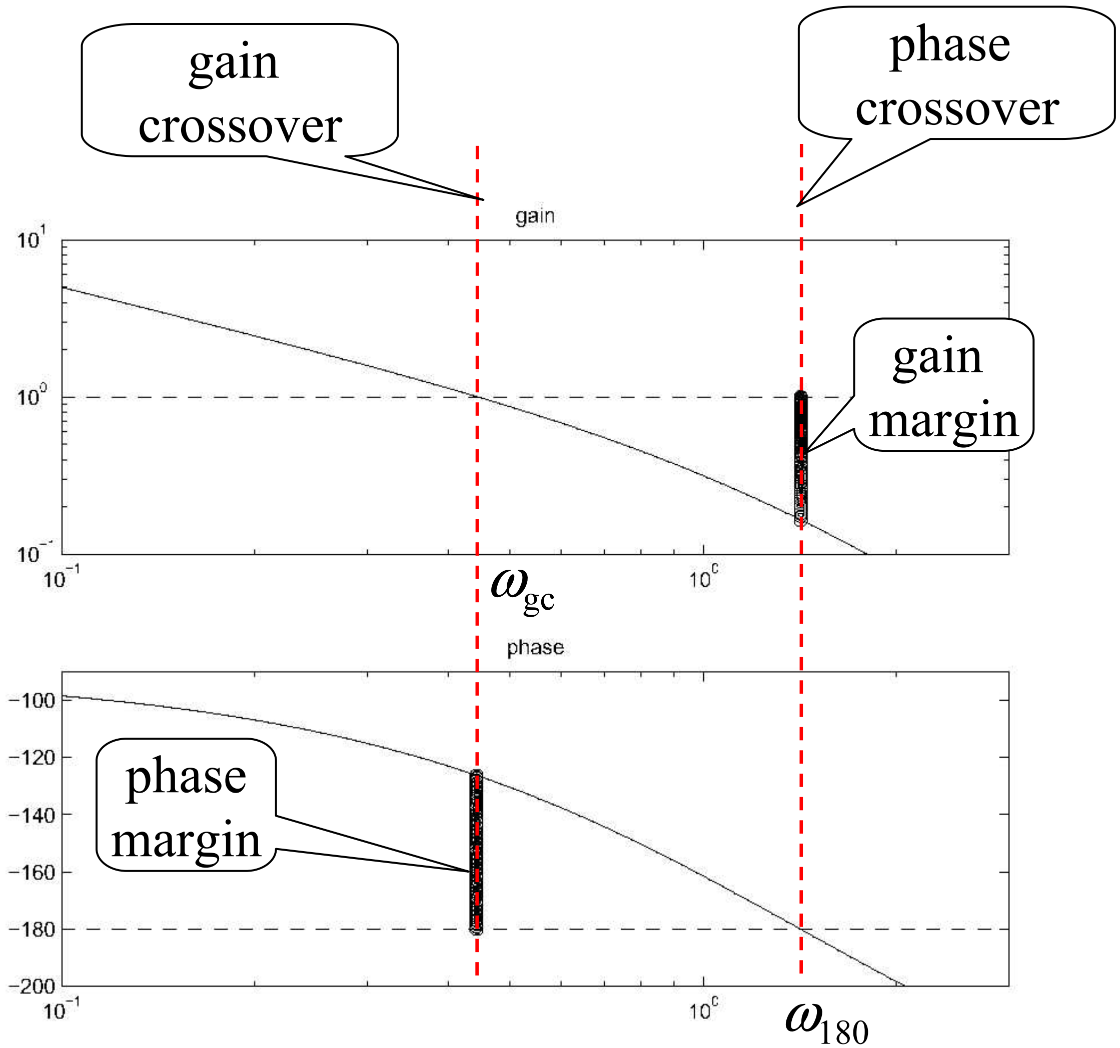
- Nyquist plot for L
  - at high frequency $|L(i\omega)| << 1$



Gain and phase margins

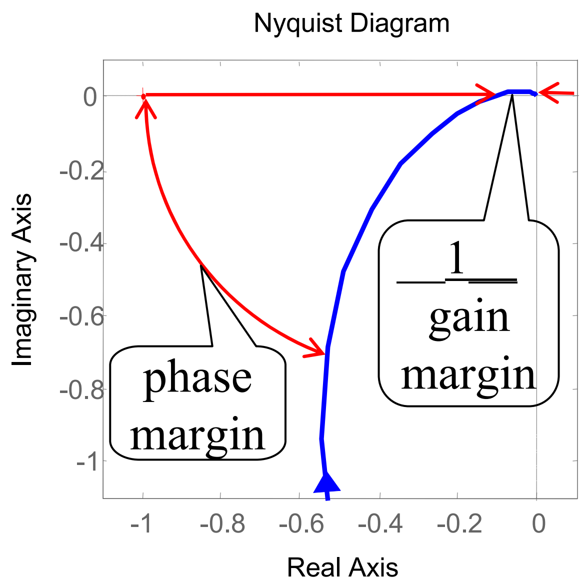
- Bode plots

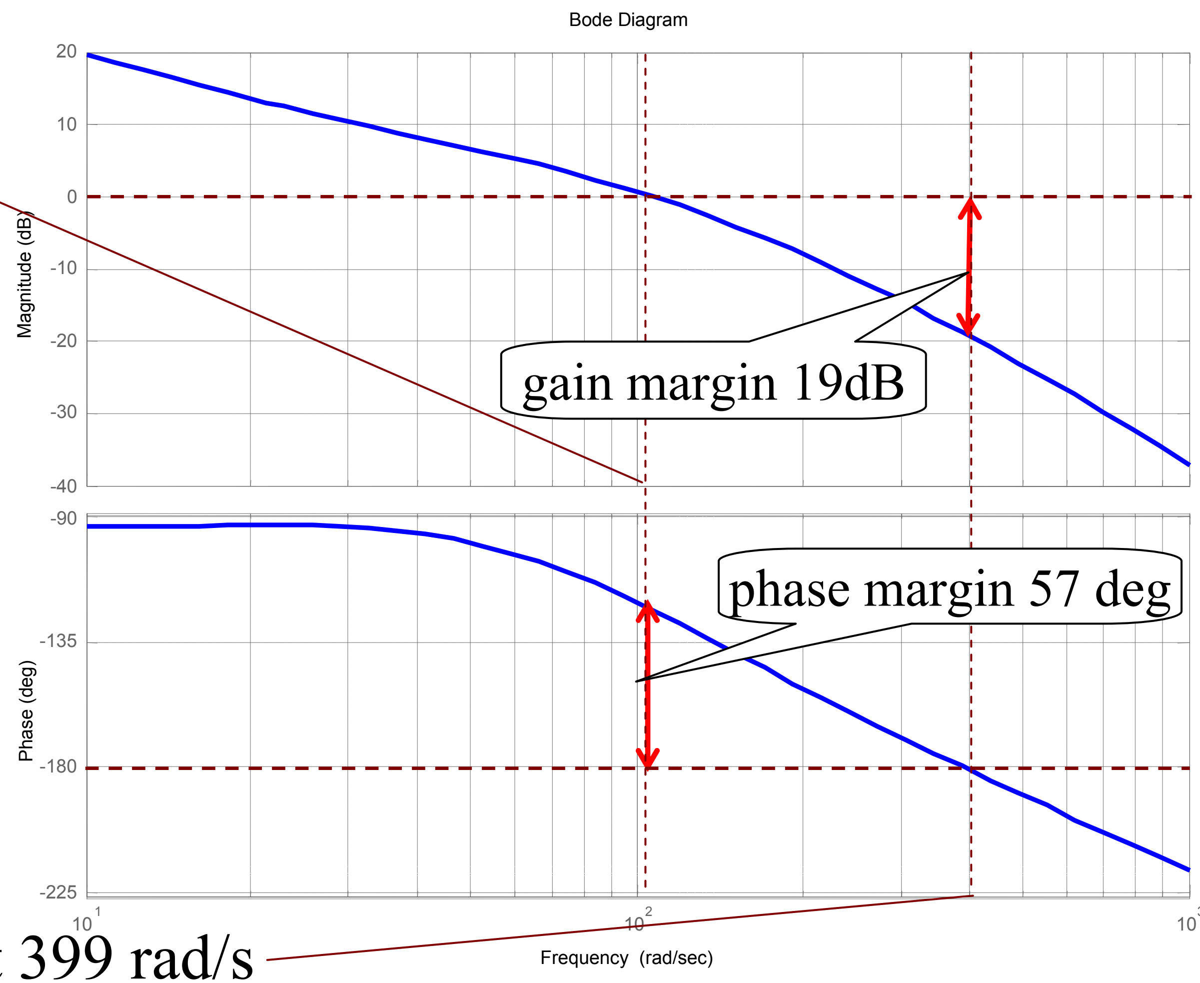




Servomotor example

- Gain crossover at 107 rad/s





- Phase crossover at 399 rad/s