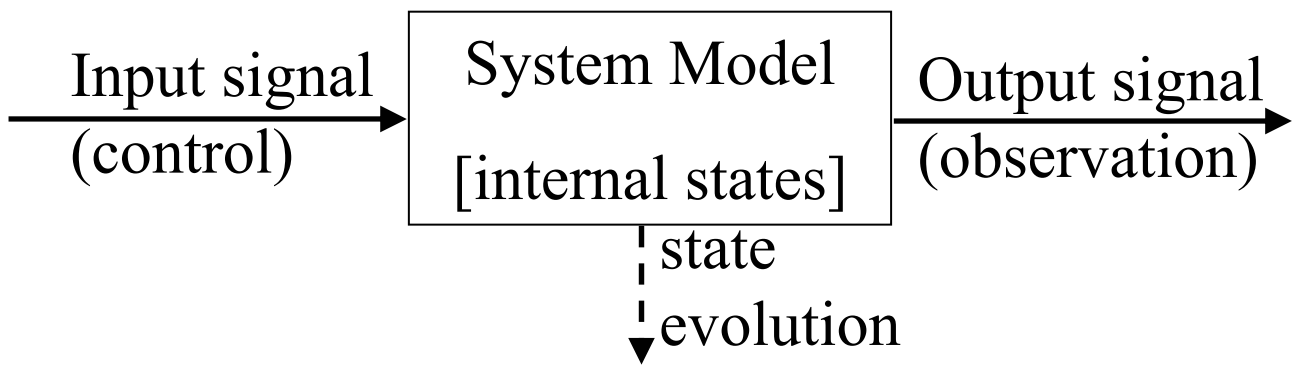
Lecture 2 – Linear Systems

This lecture: EE263 material recap + some controls motivation

- Continuous time (physics)
- Linear state space model
- Transfer functions
- Black-box models; frequency domain analysis
- Linearization

Modeling and Analysis



This lecture considers

- Linear models. More detail on modeling in Lecture 7
- Simulation: computing state evolution and output signal
- Stability: does the solution diverge after some time?
- Approximate linear models

Linear Models

- Model is a mathematical representations of a system
  - Models allow simulating the system
  - Models can be used for conceptual analysis
  - Models are never exact

- Linear models
  - Have simple structure
  - Can be analyzed using powerful mathematical tools
  - Can be matched against real data using known procedures
  - Many simple physics models are linear
  - They are just models, not the real systems

State space model

- Generic state space model
  - is described by ODEs
  - e.g., physics-based system model
  - state vector x
  - observation vector y
  - control vector u

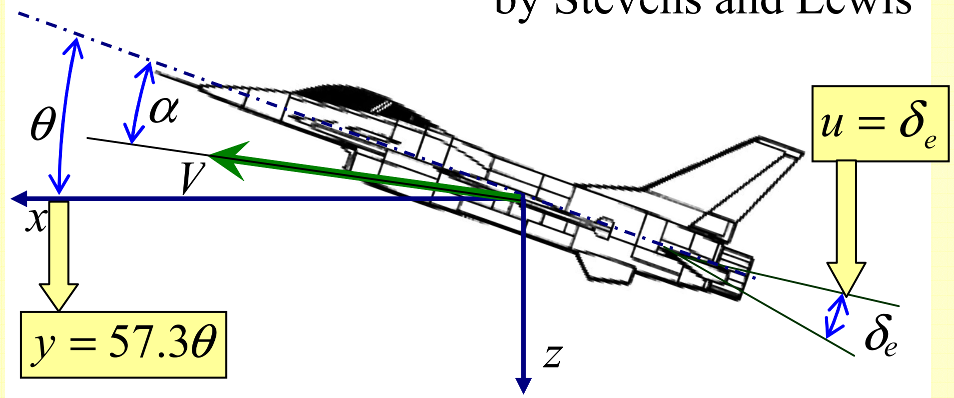
$$\frac{dx}{dt} = f(x,u,t)$$ → state evolution

$$y = g(x,t)$$ → observation

x_1 - velocity V [ft/sec]
x_2 - angle of attack α [rad]
x_3 - pitch angle θ [rad]
x_4 - pitch rate q [rad/sec]
δ_e - elevator deflection [deg]

Example: F16 Longitudinal Model

from *Aircraft Control and Simulation*
by Stevens and Lewis



$$\dot{V} = -1.93\cdot 10^{-2} V + 8.82\alpha - 32.2\theta - 0.58q + 0.17\delta_e$$

$$\dot{\alpha} = -2.54\cdot 10^{-4} V - 1.02\alpha + 0.91q - 2.15\cdot 10^{-3}\delta_e$$

$$\dot{\theta} = q$$

$$\dot{q} = 2.95\cdot 10^{-12} V + 0.82\alpha - 1.08q - 0.18\delta_e$$

Linear state space model

- Linear Time Invariant (LTI) state space model:

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx \end{aligned}$$

→ state evolution

→ observations

- Can be integrated analytically or numerically (simulation)
- Can be well analyzed: stability, response

Example: F16 Longitudinal Model

$$\frac{dx}{dt} = \begin{bmatrix} -1.93\cdot 10^{-2} & 8.82 & -32.2 & -0.48 \\ -2.54\cdot 10^{-4} & -1.02 & 0 & 0.91 \\ 0 & 0 & 0 & 1 \\ 2.95\cdot 10^{-12} & 0.82 & 0 & -1.08 \end{bmatrix} \cdot x + \begin{bmatrix} 0.17 \\ -2.15\cdot 10^{-3} \\ 0 \\ -0.18 \end{bmatrix} \cdot u$$

$$y = \begin{bmatrix} 0 & 0 & 57.3 & 0 \end{bmatrix} \cdot x$$

Integrating linear autonomous system

$$\frac{dx}{dt} = Ax$$

- Matrix exponential
  - Can be computed in Matlab as `expm(A)`
  - The definition corresponds to integrating the ODE by Euler method

Example:

```
>>A = [-1.93e-2   8.82  -32.2  -0.58;
       -2.54e-4  -1.02    0     0.91;
          0         0     0      1;
        2.95e-12  0.82    0    -1.08];

>> expm(A)
ans =
    0.9802    3.1208  -31.8830  -10.2849
   -0.0002    0.5004    0.0031    0.3604
   -0.0000    0.2216    1.0002    0.6709
   -0.0001    0.3239    0.0007    0.4773
```

$$x(T + \Delta t) \approx x(T) + \Delta t A x(t) = \left(I + \Delta t A\right) x(T)$$

$$x(T + n\Delta t) = \left(I + \Delta t A\right)^n x(T) \rightarrow \exp(An\Delta t) x(T)$$

$$\exp(At) = \lim_{\Delta t \rightarrow 0} \left(I + \Delta t A\right)^{t/\Delta t}$$

Integrating linear autonomous system

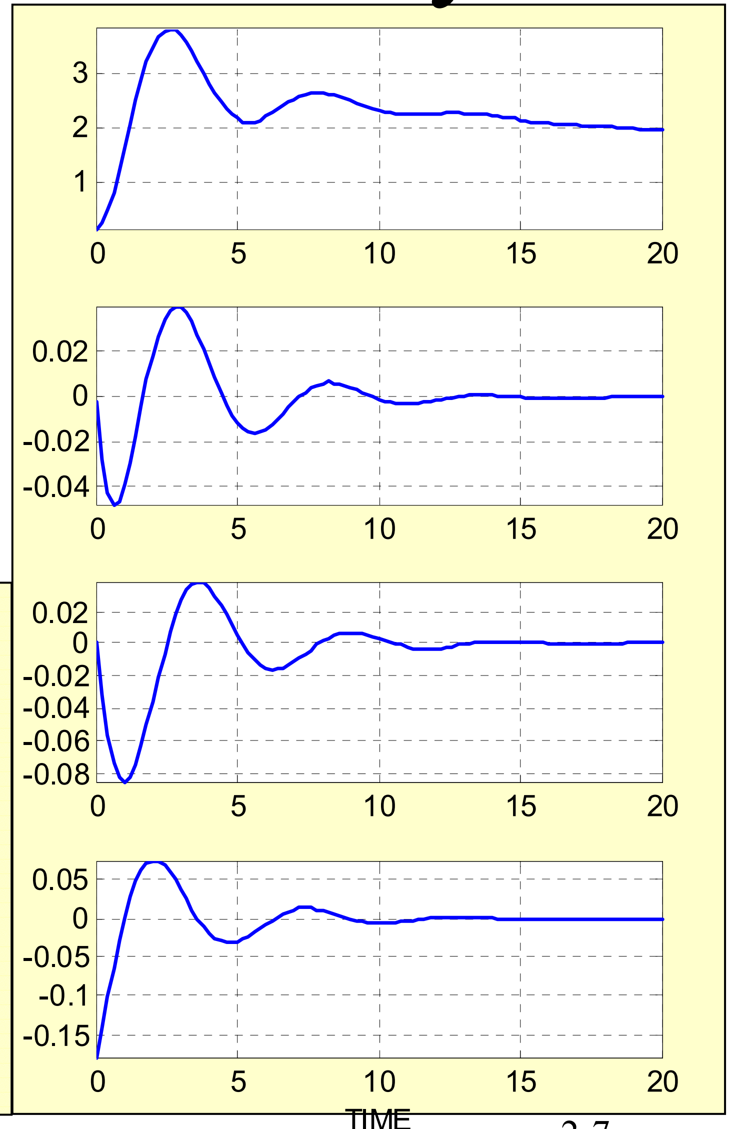
$$\frac{dx}{dt} = Ax, \qquad x(0) = x_0$$

- Initial condition response

$$x(t) = \exp(At)x_0$$

Example:

```
%Take A, B, C from the F16 example
>> k = 0.02
>> G = A + k*B*C;
>> x0 = [0.1700;
        -0.0022;
         0;
        -0.1800];
>>
for j = 1:length(t);
    x(:,j)=expm(G*t(j))*x0; end;
```



Eigenvalues and Stability

Example:

```
% take A from
% the F16 example
>> eig(A)
ans =
  -1.9125
  -0.1519 + 0.1143i
  -0.1519 - 0.1143i
   0.0970
```

- Consider eigenvalues of the matrix A

$$\{\lambda_j\} = \mathrm{eig}(A); \qquad \det(I\lambda - A) = 0$$

- Suppose A has all different and nonzero eigenvalues, then

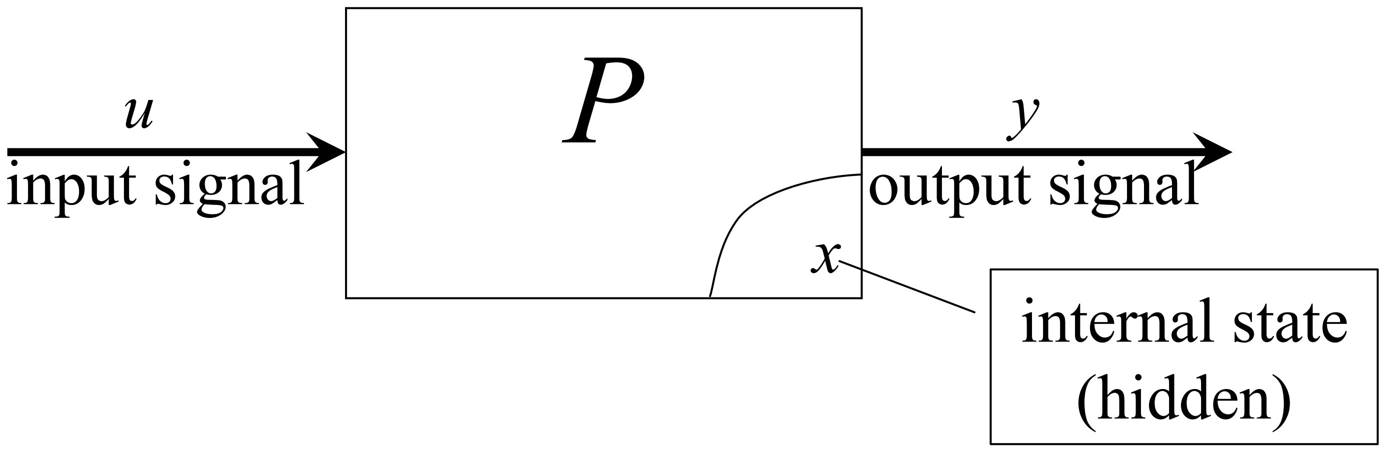
$$A = V \cdot \mathrm{diag}\{\lambda_j\} \cdot V^{-1}$$

$$\exp(At) = V \cdot \mathrm{diag}\{e^{\lambda_j t}\} \cdot V^{-1}$$

- The system solution is exponentially stable if $\mathrm{Re}\,\lambda_j < 0$
- If A has eigenvalues with multiplicity more than 1, things are a bit more complicated: Jordan blocks, polynomials in t
- Still the condition of exponentially stability is $\mathrm{Re}\,\lambda_j < 0$

Input-output models

- Black-box models – describe system *P* as an operator



- Historically (50 years ago)
  – Black-box models → EE
  – State-space models → ME, AA

Linear System (input-output)

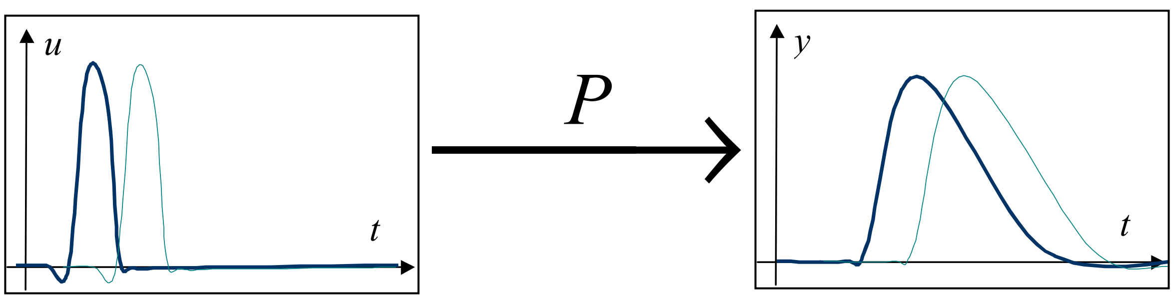
- Linearity

$$u_1(\cdot) \xrightarrow{P} y_1(\cdot) \qquad u_2(\cdot) \xrightarrow{P} y_2(\cdot)$$

$$au_1(\cdot) + bu_2(\cdot) \xrightarrow{P} ay_1(\cdot) + by_2(\cdot)$$

- Linear Time-Invariant systems - LTI

$$u(\cdot - T) \xrightarrow{P} y(\cdot - T)$$



Convolution representation

- Convolution integral

$$y(t) = \int_{-\infty}^{t} h(t-\tau)u(\tau)d\tau$$

signal processing notation $y = h * u$

- Impulse response

$$u(t) = \delta(t) \quad \Rightarrow \quad y(t) = h(t)$$

- Step response: $u = 1$ for $t > 0$

$$g(t) = \int_{0}^{t} h(t-\tau)d\tau = \int_{0}^{t} h(\tau)d\tau \qquad h(t) = \frac{d}{dt}g(t)$$

Impulse Response for State Space Model

$$\frac{dx}{dt} = Ax + Bu \quad \rightarrow \text{state evolution}$$
$$y = Cx \quad \rightarrow \text{observation}$$

- Impulse response for the state x

$$x(\Delta t) \approx \Delta t A x(0) + B \underbrace{\Delta t u}_{1}$$

$$x(\Delta t) \approx B \Delta t u = B$$

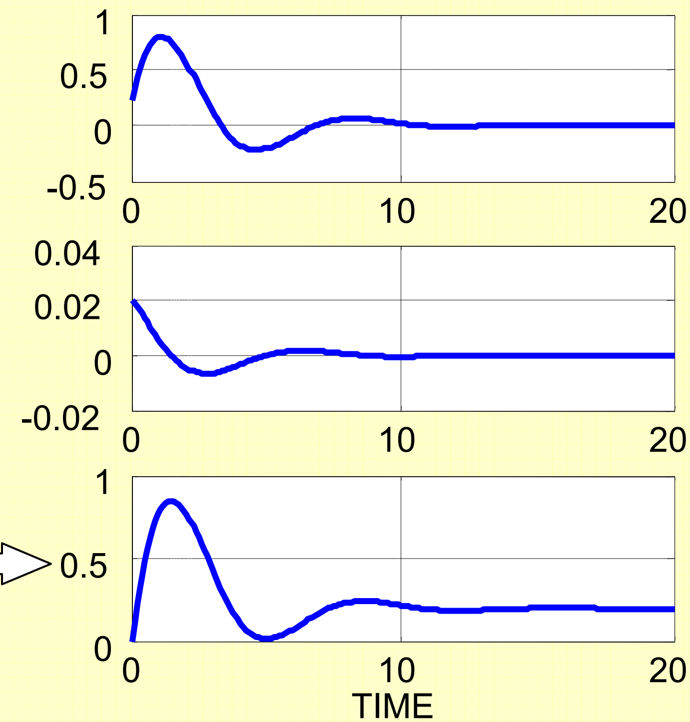
$$x(t) = \exp(At)B$$

- System impulse response

$$h(t) = C \exp(At) B$$

Example:

```
>> A =  [-0.3130   56.7000  0;
         -0.0139   -0.4260  0;
            0      56.7000  0];
>> B =  [0.232;  0.0203;  0];
>> C =  [0,   0,    1];
```



Formal transfer function

- Rational transfer function
  = IIR (Infinite Impulse Response) model
  - Broad class of input-output linear models

$$a_1 \frac{d^m y}{dt^m} + a_2 \frac{d^{m-1} y}{dt^{m-1}} + \ldots + a_{m+1} y = b_1 \frac{d^n u}{dt^n} + b_2 \frac{d^{n-1} u}{dt^{n-1}} + \ldots + b_{n+1} u$$

- Differentiation operator $\frac{d}{dt} \to s$

- Formal transfer function – rational function of s

$$y = H(s) \cdot u = \frac{N(s)}{D(s)} \cdot u \qquad \begin{aligned} N(s) &= a_1 s^m + \ldots + a_m s + a_{m+1} \\ D(s) &= b_1 s^n + \ldots + b_n s + b_{n+1} \end{aligned}$$

- For a causal system $m \le n$

Poles, Impulse Response

$$y = \frac{N(s)}{D(s)} \cdot u \qquad\qquad \begin{array}{ll} N(s) = 0 & \text{- zeros} \\ D(s) = 0 & \text{- poles} \end{array}$$

- Expand $D(s) = (s - p_1)^{M_1} \cdot \ldots \cdot (s - p_K)^{M_K}$

Then $$y = \frac{N_1(s)}{(s - p_1)^{M_1}} \cdot u + \ldots + \frac{N_K(s)}{(s - p_K)^{M_K}} \cdot u$$

- Quasi-polynomial impulse response – see a textbook

$$h(t) = \left(c_{1,1} t^{M_1 - 1} + \ldots + c_{1,M_1+1}\right) e^{p_1 t} + \ldots + \left(c_{K,1} t^{M_K - 1} + \ldots + c_{K,M_K+1}\right) e^{p_K t}$$

- Example: $\dfrac{d^2 y}{dt^2} = u$ Transfer function: $y = \dfrac{1}{s^2} \cdot u$

Impulse response: $h(t) = v_0 t + x_0$

Transfer Function for State Space

- Formal transfer function for a state space model

$$\frac{d}{dt} \to s \qquad \boxed{\begin{array}{ccc} sx & = & Ax + Bu \\ y & = & Cx \end{array}} \qquad \begin{array}{l} y = (sI - A)^{-1} B \cdot u \\ H(s) = C(sI - A)^{-1} B \end{array}$$

- Characteristic polynomial

$$N(s) = \det(sI - A) = 0$$

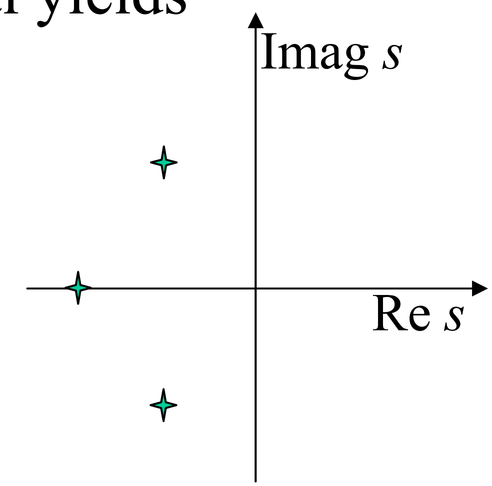
$$\text{Poles} \;\Leftrightarrow\; \det(sI - A) = 0 \;\Leftrightarrow\; \text{eigenvalues}$$

- Poles are the same as eigenvalues of the state-space matrix A
- For stability we need $\operatorname{Re} p_k = \operatorname{Re} \lambda_k < 0$

Laplace transform

- Laplace integral transform: $x(t) \rightarrow \hat{x}(s) = \int_0^\infty x(t)e^{st}dt$
- Laplace transform of the convolution integral yields

$$\hat{y}(s) = H(s)\hat{u}(s) \qquad H(s) = \int_0^\infty h(t)e^{st}dt$$

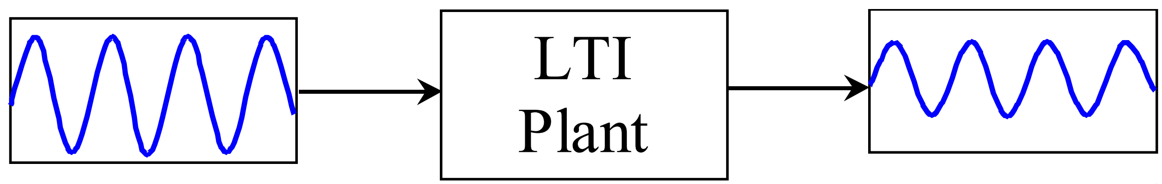


- Transfer function: $H(s)$
  - function of complex variable s
  - analytical in a right half-plane $\mathrm{Re}\, s \geq a$
  - for a stable system $a \leq 0$
  - for an IIR model $H(s) = \dfrac{N(s)}{D(s)}$

$$\frac{dx}{dt} \rightarrow s\hat{x}(s)$$

Frequency decomposition

- Sinusoids are eigenfunctions of an LTI system: $y = H(s)u$



$e^{i\omega t}$ $\qquad$ $H(i\omega)e^{i\omega t}$

$$s\left(e^{i\omega t}\right) \rightarrow \frac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t}$$

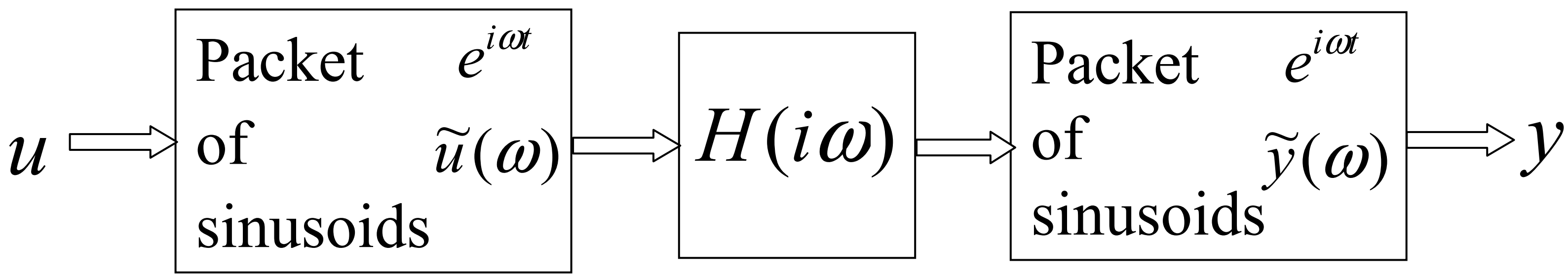
- Frequency domain analysis

$$\sum u_k e^{i\omega_k t} \rightarrow y = \sum u_k H(i\omega_k)e^{i\omega_k t}$$

Frequency domain description

- Frequency domain analysis

$$u = \frac{1}{2\pi}\int \tilde{u}(\omega)e^{-i\omega t}d\omega \Rightarrow y = \frac{1}{2\pi}\int \underbrace{H(i\omega)\tilde{u}(\omega)}_{\tilde{y}(\omega)}e^{-i\omega t}d\omega$$



- Fourier transform – numerical analysis
- Laplace transform – complex analysis

$u(t) = 0$, for $t < 0$

$$\tilde{u}(\omega) = \int_{-\infty}^{\infty} u(t)e^{i\omega t}d\omega = \left(\int_{-\infty}^{\infty} u(t)e^{st}d\omega\right)_{s=i\omega} = \hat{u}(i\omega)$$

Continuous systems in frequency domain

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} x(t)e^{i\omega t}dt$$

- Fourier transform
  $[-\infty, \infty] \to [-\infty, \infty]$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{x}(\omega)e^{-i\omega t}d\omega$$

- Inverse Fourier transform

$$y(t) = \int_{-\infty}^{t} h(t-\tau)u(\tau)dt$$

- I/O impulse response model

$$H(s) = \int_{0}^{\infty} h(t)e^{st}dt$$

- Transfer function

$$\tilde{y}(\omega) = H(i\omega)\tilde{u}(\omega)$$

- System frequency response

Frequency domain description

- Bode plots:

$$M(\omega) = |H(i\omega)|$$

$$\varphi(\omega) = \arg H(i\omega)$$

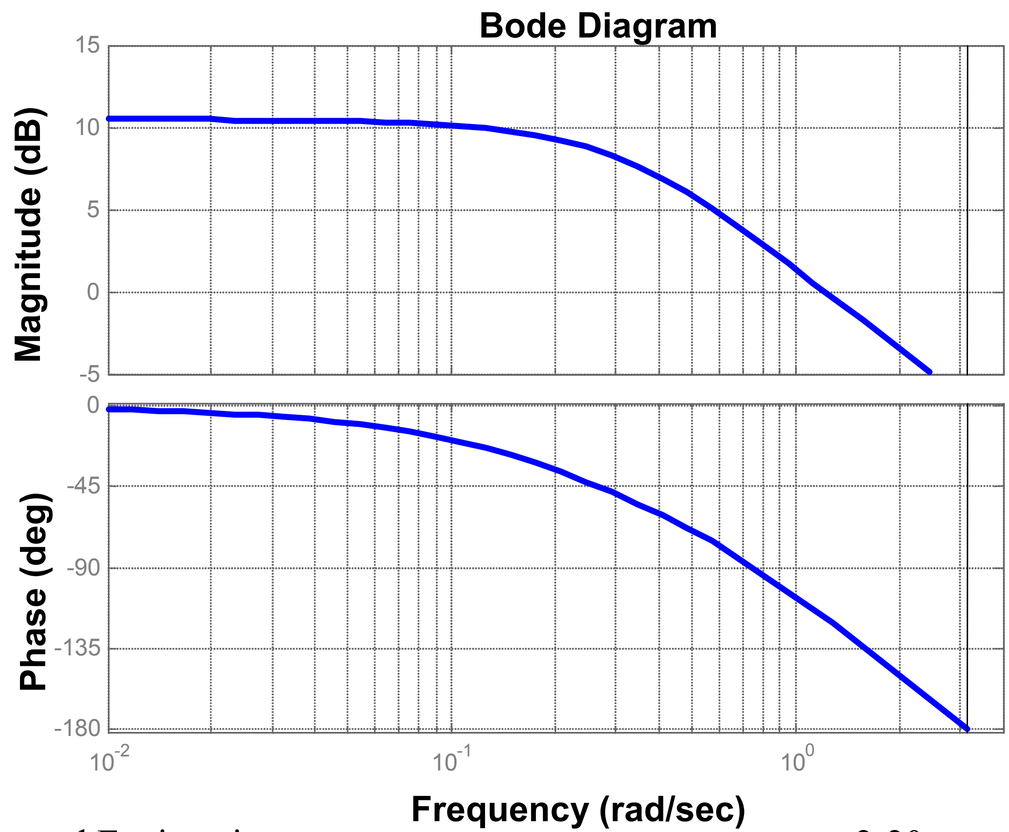
$$u = e^{i\omega t}$$

$$y = H(i\omega) e^{i\omega t}$$

- Example:

$$H(s) = \frac{1}{s - 0.7}$$

- $|H|$ is often measured in dB
  - [dB] = 20 $\log_{10} M$

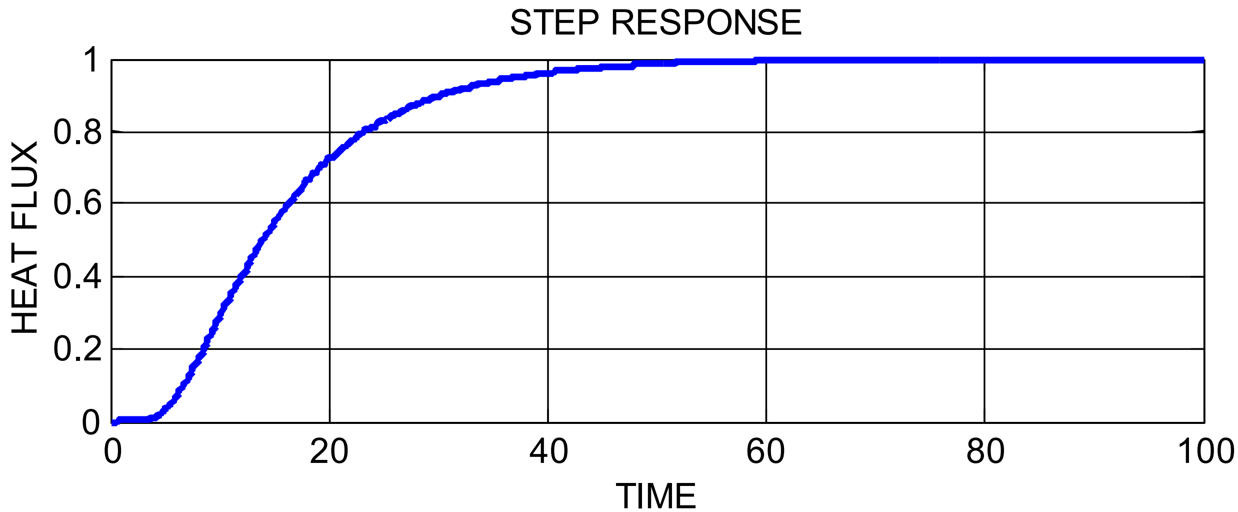


Model Approximation

- Model structure – physics, computational
- Determine parameters from data
- Step/impulse responses are close ←→ the input/output models are close
- Example – fit step response
- Linearization of nonlinear model

Black-box model from data

- Linear black-box model can be determined from the data, e.g., step response data, or frequency response



- Example problem: fit an IIR model of a given order
- This is called model identification
- Considered in more detail in Lecture 8

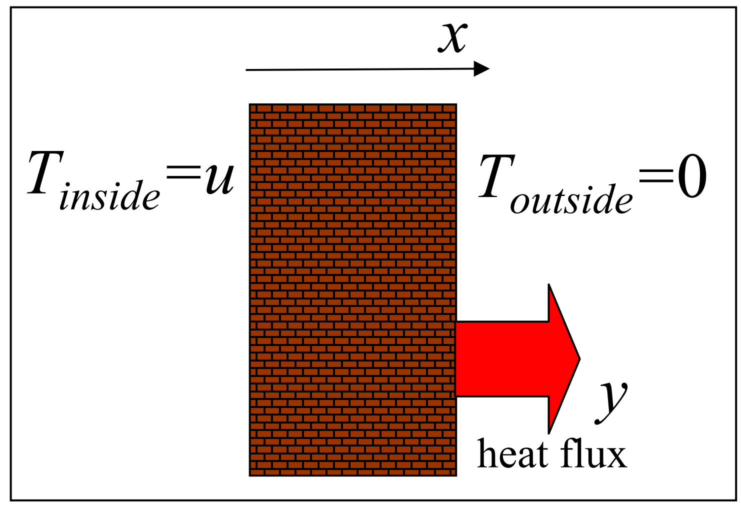
Linear PDE models

- Include functions of spatial variables
  - electromagnetic fields
  - mass and heat transfer
  - fluid dynamics
  - structural deformations
- Example: sideways heat equation

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

$$T(0) = u; \qquad T(1) = 0$$

$$y = \left.\frac{\partial T}{\partial x}\right|_{x=1}$$



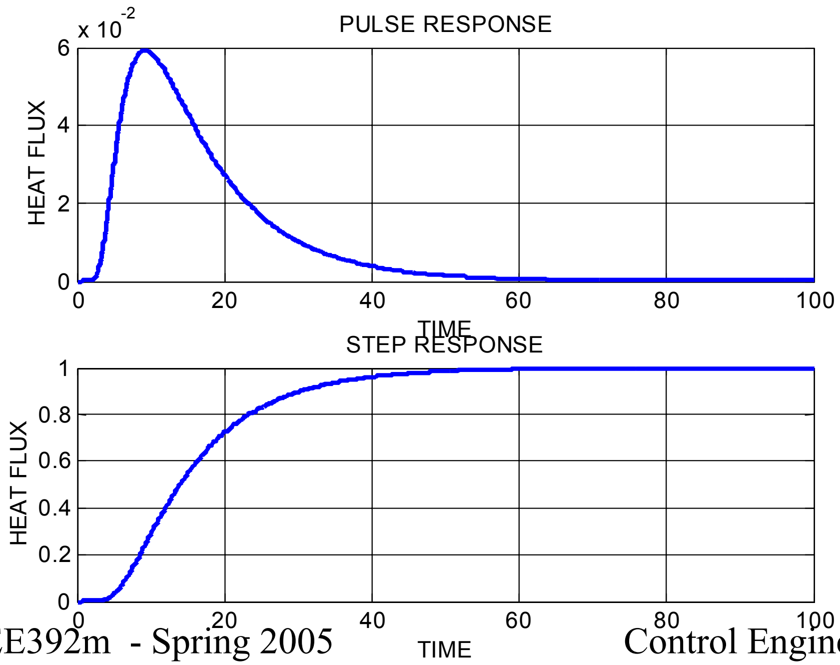
Linear PDE System Example

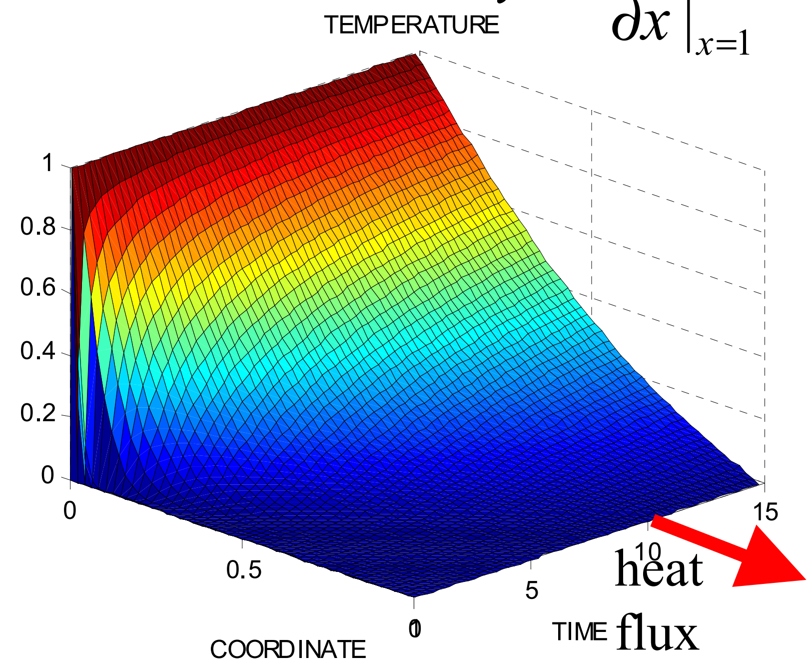
- Heat transfer equation,
  - boundary temperature input u
  - heat flux output y
- Impulse response and step response
- Transfer function is not rational

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

$$u = T(0) \qquad T(1) = 0$$

$$y = \left.\frac{\partial T}{\partial x}\right|_{x=1}$$

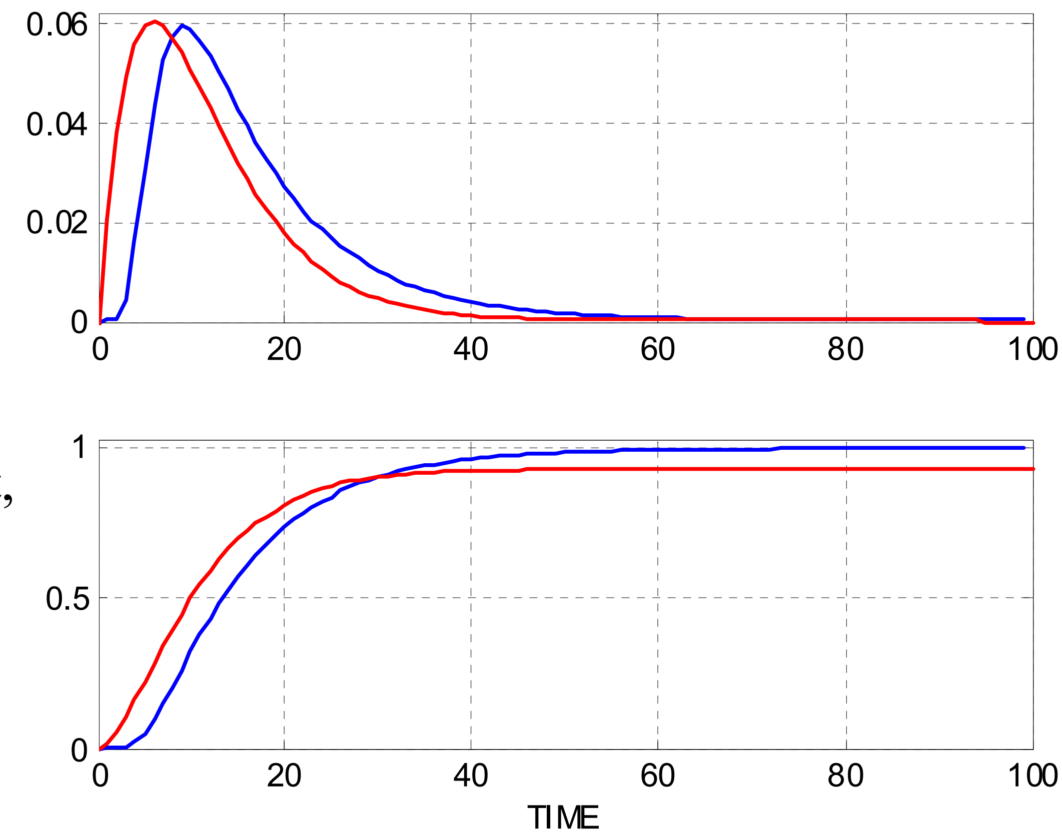




Impulse response approximation

- Approximating impulse and step responses by a low order rational transfer function model
- Higher order model can provide very accurate approximation
- Methods:
  – trial and error
  – sampled time response fit, e.g., Matlab's `prony`
  – identification, Lecture 8
  – formal model reduction approaches - advanced

$$H(s) = 0.01\frac{0.08s^2 - 0.4s + 2.8}{s^2 + 0.34s + 0.03}$$



Validity of Model Approximation

- Why can we use an approximate model instead of the 'real' model?
- Will the analysis hold?
- The input-output maps of two systems are 'close' if the convolution kernels (impulse responses) are 'close'

$$y(t) = \int_{-\infty}^{t} h(t-\tau)u(\tau)\tau$$

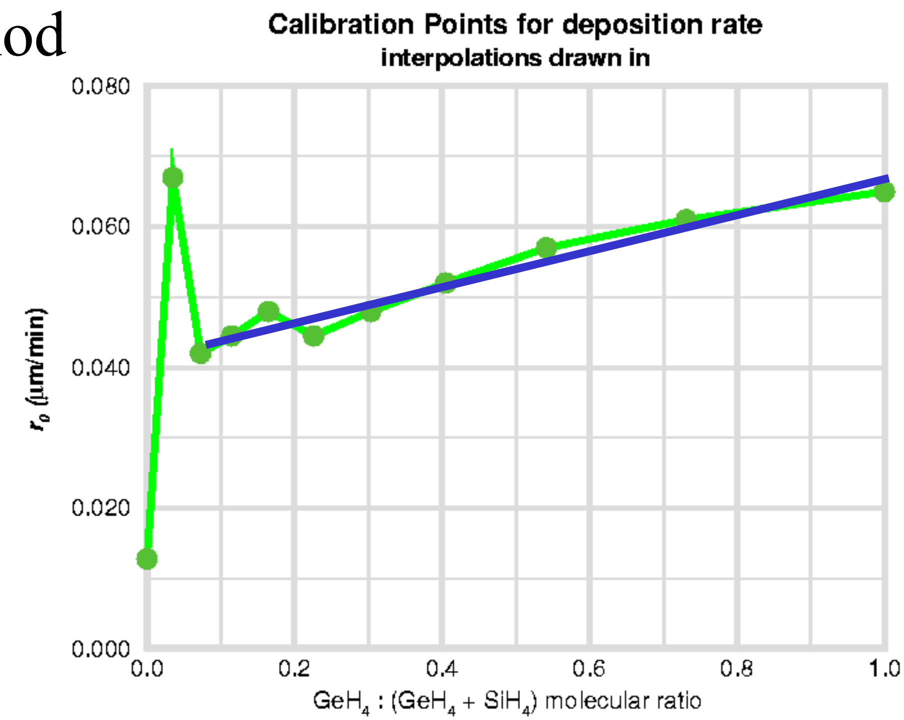
- The closed-loop stability impact of the modeling error
  - Control robustness
  - Will be discussed in Lecture 9

Nonlinear map linearization

- Nonlinear - detailed model
- Linear - conceptual design model
- Differentiation, secant method

- Example:
  static map linearization

$$y = f(u) \approx \frac{\Delta f}{\Delta u}(u - u_0)$$



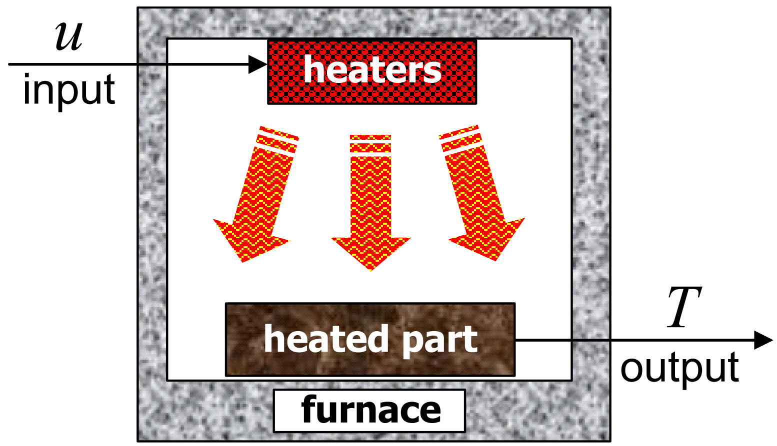
Linearization Example: RTP

- RTP – Rapid Thermal Processing
- Major semiconductor manufacturing process

$$\frac{dT}{dt} = bu - c_1(T^4 - T_F^4) - c_2(T - T_F)$$

T – part temperature
u – IR heater power
T_F – furnace temperature

- Stefan-Boltzmann law nonlinearity
- T_F is assumed to be constant



RTP, cont'd

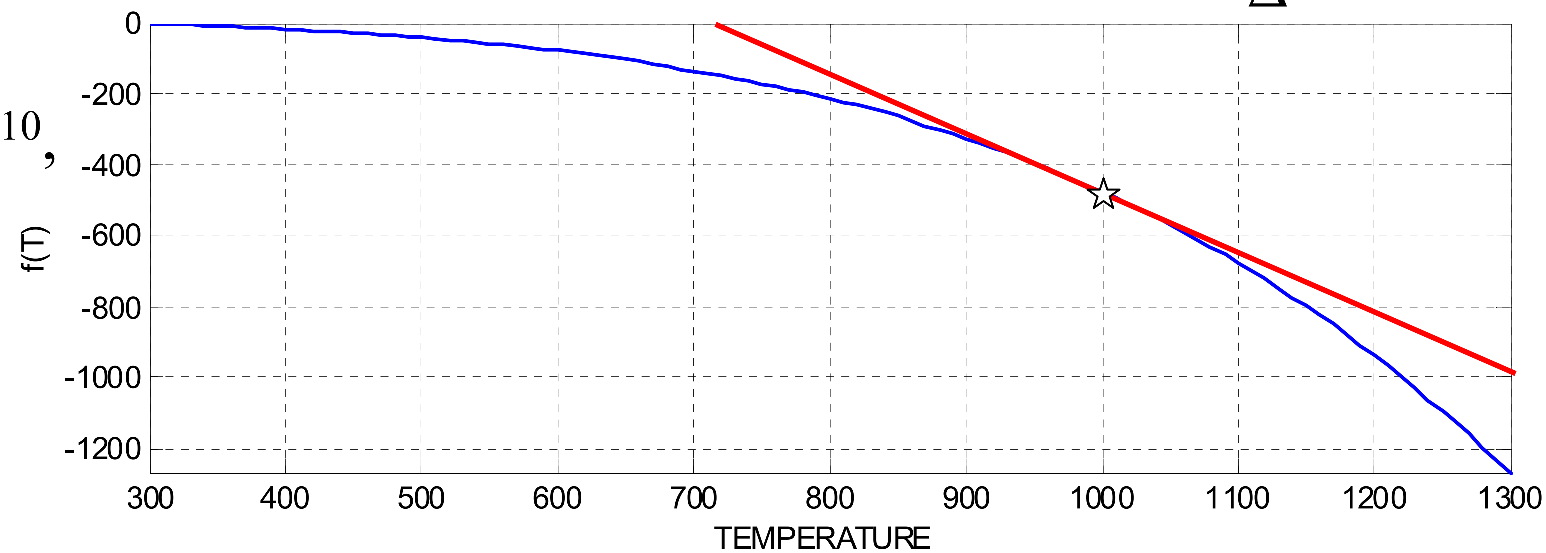
$$\frac{dT}{dt} = f(T) + bu \qquad f(T) = -c_1(T^4 - T_F^4) - c_2(T - T_F)$$

Linearize around a steady state point

$$\frac{dT}{dt} = f_L(T) + bu \qquad f_L(T) = a(T - T_*) + d$$

$$d = f(T_*) \qquad a = f'(T_*) \approx \frac{f(T_* + \Delta) - f(T_*)}{\Delta}$$

$b = 1000,$
$c_1 = 1.1 \cdot 10^{-10},$
$c_2 = 0.8,$
$T_F = 300$



RTP, cont'd

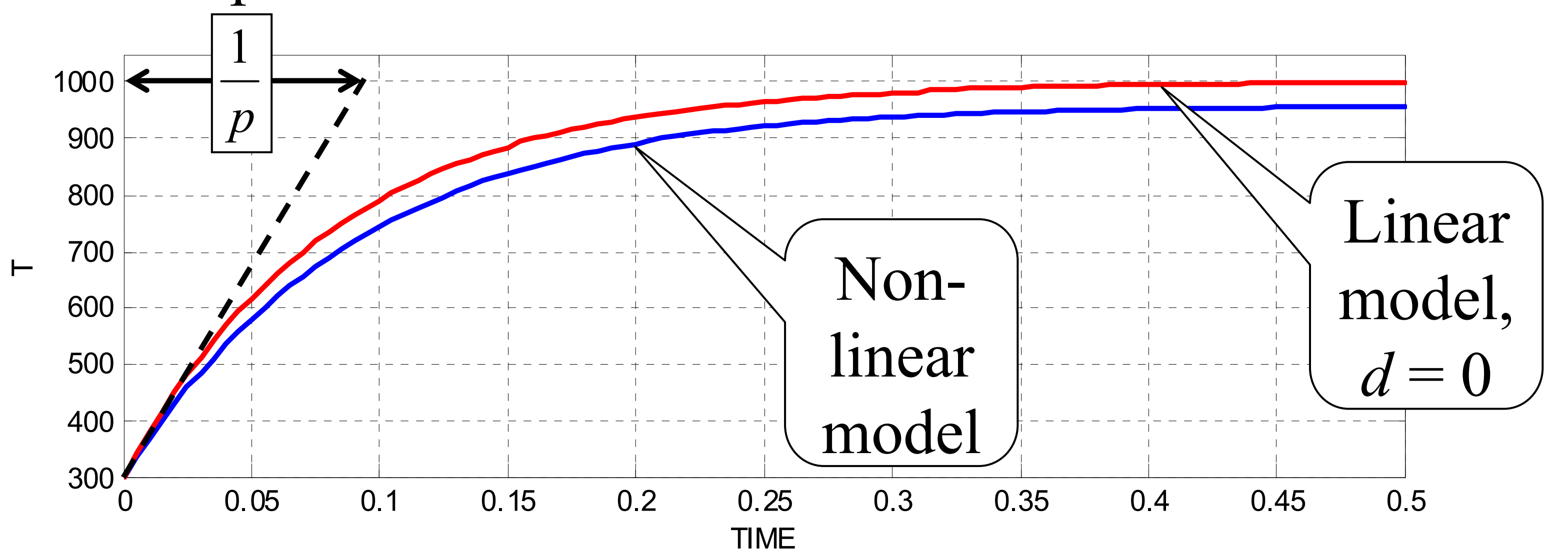
$$\dot{x} = ax + bu + d \qquad x = T - T_*$$

$$u = -kx$$

Linear system with a pole $p = -(a + bk)$

$T_* = 1000$, $a = -1.7425$, $b = 1000$, $k = 0.01$ → $p = -11.7425$

Simulate performance:



Nonlinear state space model linearization

- Linearize the r.h.s. map in a state-space model

$$\dot{x} = f(x,u) \approx \frac{\Delta f}{\Delta x}\underbrace{(x - x_0)}_{q} + \frac{\Delta f}{\Delta u}\underbrace{(u - u_0)}_{v}$$

$$\dot{q} = Aq + Bv$$

- Linearize around an equilibrium $0 = f(x_0, u_0)$
- Secant method

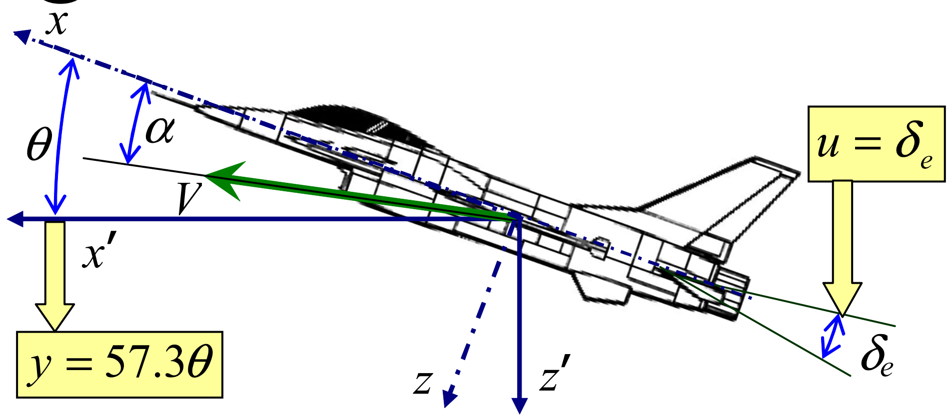
$$\left[\frac{\Delta f}{\Delta x}\right]^j = \frac{f(x_0 + s_j, u_0) - f(x_0, u_0)}{d_j}$$

$$s_j = [0 \quad \ldots \quad \underbrace{d_j}_{\# j} \quad \ldots \quad 0]$$

- This is how Simulink computes linearization

Example: F16 Longitudinal Model

$$\frac{dx}{dt} = f(x,u)$$



- State vector x

x_1 - velocity V [ft/sec]

x_2 - angle of attack α [rad]

x_3 - pitch angle θ [rad]

x_4 - pitch rate q [rad/sec]

- Control input

u - elevator deflection δ_e [deg].

$$x = \begin{bmatrix} V \\ \alpha \\ q \\ \theta \end{bmatrix}$$

$$\dot{V} = \frac{1}{m}(F_x \cos\alpha + F_z \sin\alpha)$$

$$\dot{\alpha} = \frac{1}{mV}(-F_x \sin\alpha + F_z \cos\alpha) + q$$

$$\dot{q} = \frac{M_y}{I_y}$$

$$\dot{\theta} = q$$

$$F_x = rC_{x,t}(\alpha) - mg\sin\theta + T$$

$$F_z = rC_{z,t}(\alpha,\delta_e) + mg\cos\theta$$

$$M_y = RC_{m,t}(\alpha,\delta_e)$$

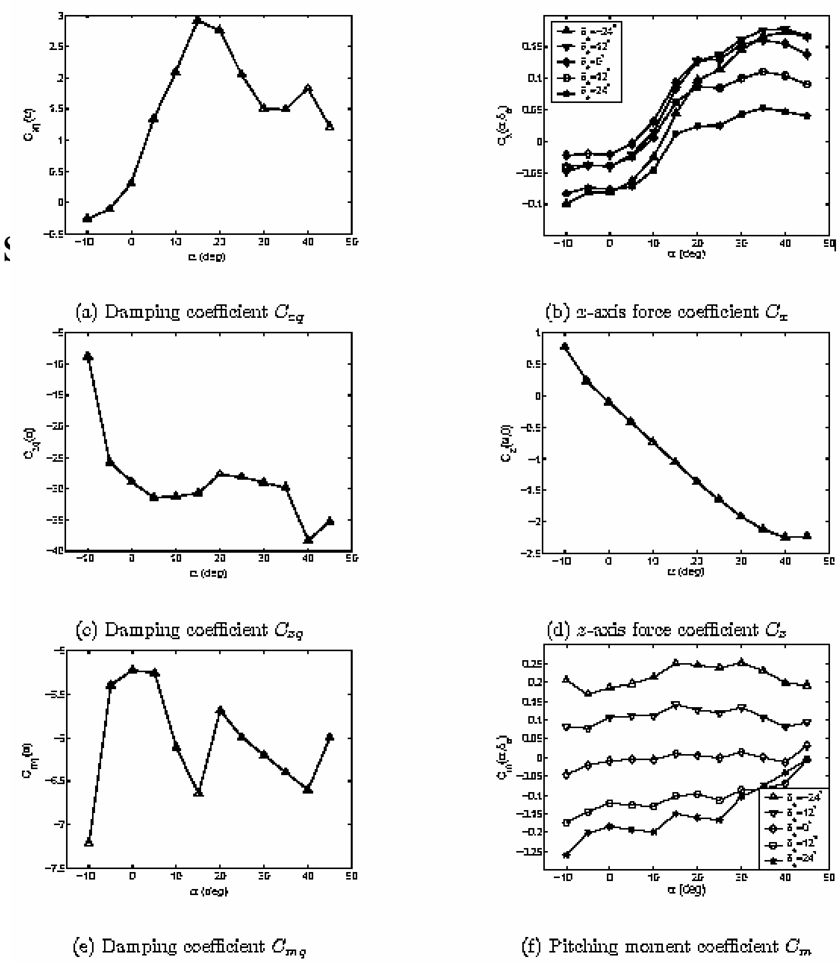
For more detail see: *Aircraft Control and Simulation* by Stevens and Lewis

Nonlinear Model of F16

$$\frac{dx}{dt} = f(x,u)$$ → state evolution

$$y = g(x)$$ → observation

- Aircraft models are understood by groups of people
- Could take many man-years worth of effort
- Aerodynamics model is based on empirical data
- $f(x,u)$ available as a computational function can be used without a deep understanding of the model
- The nonlinear model can be used for simulation, or linearized for analysis



(a) Damping coefficient C_{xq}
(b) x-axis force coefficient C_x
(c) Damping coefficient C_{zq}
(d) z-axis force coefficient C_z
(e) Damping coefficient C_{mq}
(f) Pitching moment coefficient C_m

Linearized Longitudinal Model of F16

- Assume trim condition

$$x_0 = \begin{bmatrix} V_0 \\ \alpha_0 \\ q_0 \\ \theta_0 \end{bmatrix} = \begin{bmatrix} 500 \\ 0.0393 \\ 0 \\ 0.0393 \end{bmatrix} \begin{array}{l} \text{- velocity } V \text{ [ft/sec]} \\ \text{- angle of attack } \alpha \text{ [rad]} \\ \text{- pitch rate } q \text{ [rad/sec]} \\ \text{- pitch angle } \theta \text{ [rad} \end{array}$$

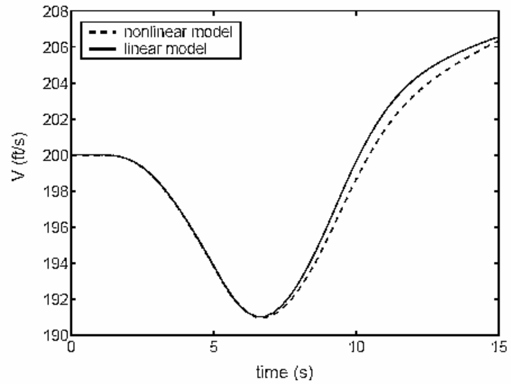
- Linearize the nonlinear function $f(x,u)$ by a finite difference method (secant method). Step = [1 0.001 0.01 0.001]

$$A = \frac{\Delta f}{\Delta x} = \begin{bmatrix} -1.93 \cdot 10^{-2} & 8.82 & -32.2 & -0.48 \\ -2.54 \cdot 10^{-4} & -1.02 & 0 & 0.91 \\ 0 & 0 & 0 & 1 \\ 2.95 \cdot 10^{-12} & 0.82 & 0 & 1.08 \end{bmatrix} \qquad B = \frac{\Delta f}{\Delta u} = \begin{bmatrix} 0.17 \\ -2.15 \cdot 10^{-3} \\ 0 \\ -0.18 \end{bmatrix}$$

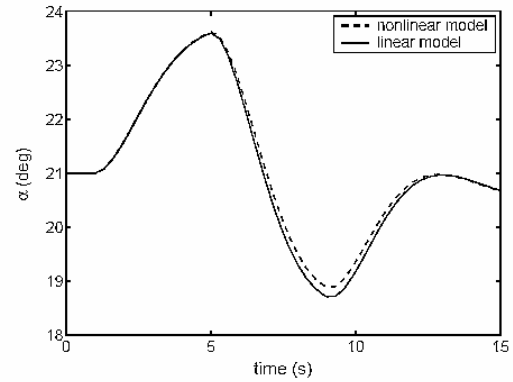
- These are the matrices we considered in the linear F16 model example

Simulation-based validation

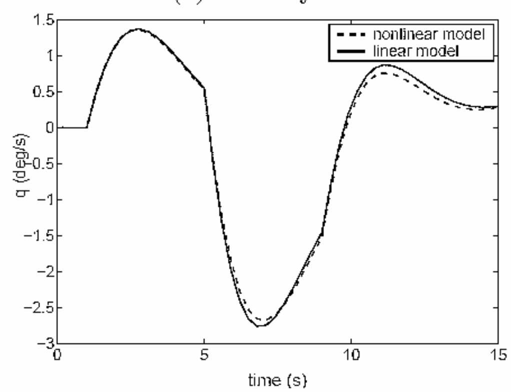
- Simulate with nonlinear model, compare with linear model results
- Doublet response



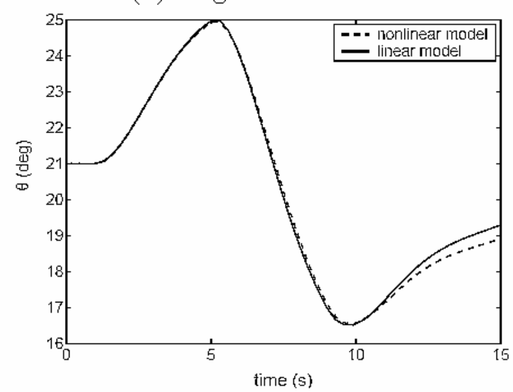
(a) Velocity V



(b) Angle of attack α



(c) Pitch rate q



(d) Pitch angle θ

LTI models - summary

- ODE model
- State space linear model
- Linear system can be described by impulse response or step response
- Linear system can be described by frequency response = Fourier transform of the impulse response
- Linear model approximations can be obtained from more complex models
  - Approximation of a linear model response
  - Linearization of a nonlinear model