Dynamic Programming

Part Three

Announcements

- Problem Set Five due right now, or due Wednesday with a late period.
- Problem Set Six out, due next Monday.
  - Explore dynamic programming across different application domains!
  - Get a feel for how to structure DP solutions!
  - You may use a late day on Problem Set Six, but be aware this will overlap with the final project.
- Handout: “Guide to Dynamic Programming” also available.

Final Project Logistics

- Final project will go out next Monday and be due on **Saturday, August 17** at **12:15PM** (note the different time).
- Format: Three algorithms questions, each of which combine two or more different techniques from the quarter.
  - No collaboration permitted with other students.
  - No outside sources may be consulted.
  - Course staff will only answer clarifying questions about the problems.

Please evaluate this course on Axess.

Your feedback really makes a difference.

Outline for Today

- **Shortest Paths Revisited**
  - What if the edge weights are negative?
- **The Bellman-Ford Algorithm**
  - A simple and elegant algorithm for finding shortest paths.
- **The Floyd-Warshall Algorithm**
  - Finding shortest paths between all pairs of points.

Negative Edge Weights

The Recurrence

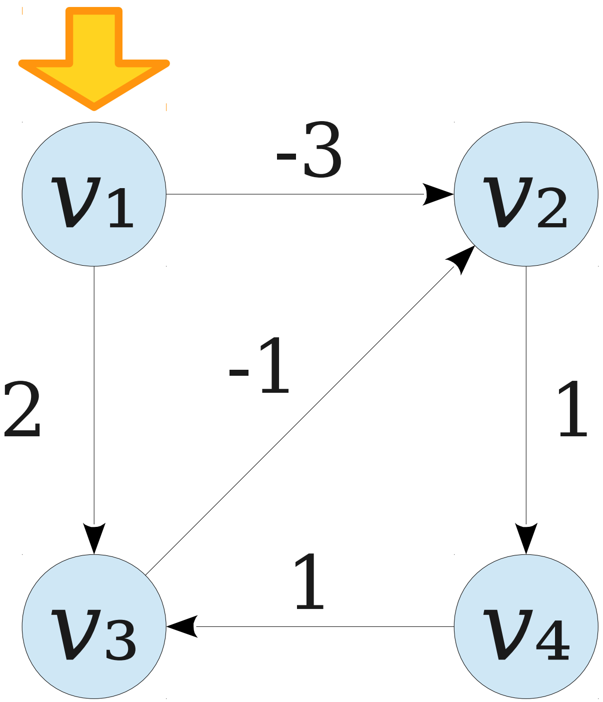
- **Idea:** Find paths of lengths at most 0, 1, 2, …, n.
- Let $w(u, v)$ denote the weight of edge (u, v).
- Let s be our start node. Let OPT(v, i) be the length of the shortest s – v path whose length is at most i, or ∞ if no path exists.
- **Claim:** OPT(v, i) satisfies the following recurrence:

$$\mathrm{OPT}(v,i)=\begin{cases} 0 & \text{if } i=0 \text{ and } v=s \\ \infty & \text{if } i=0 \text{ and } v\neq s \\ min\left\{\begin{array}{l} \mathrm{OPT}(v,i-1), \\ \min\limits_{(u,v)\in E}\{\mathrm{OPT}(u,i-1)+w(u,v)\} \end{array}\right\} & \text{otherwise} \end{cases}$$

The Bellman-Ford Algorithm

- The **Bellman-Ford algorithm** evaluates this recurrence bottom-up:
  - Create a table DP of size $n \times n$.
  - Set DP[v][0] = ∞ for all $v \neq s$.
  - Set DP[s][0] = 0
  - For $i = 1$ to $n - 1$, for all $v \in V$:
    - Set DP[v][i] =
      min {
        DP[v][i – 1],
        min { DP[u][i – 1] + $w(u, v)$ } (where $(u, v) \in E$)
      }
  - Return row n of DP.

	v_1	v_2	v_3	v_4
3	**0**	**-3**	**-1**	**-2**
2	**0**	**-3**	**2**	**-2**
1	**0**	**-3**	**2**	**∞**
0	**0**	**∞**	**∞**	**∞**



$$\mathrm{OPT}(v,i)=\begin{cases} 0 & \text{if } i=0 \text{ and } v=s \\ \infty & \text{if } i=0 \text{ and } v\neq s \\ min\left\{ \begin{array}{l} \mathrm{OPT}(v,i-1), \\ \underset{(u,v)\in E}{min}\{\mathrm{OPT}(u,i-1)+w(u,v)\} \end{array} \right\} & \text{otherwise} \end{cases}$$

Analyzing Time Complexity

- What is the time complexity of this algorithm?
  - Create a table DP of size $n \times n$.
  - Set DP[v][0] = ∞ for all $v \neq s$.
  - Set DP[s][0] = 0
  - For i = 1 to n – 1, for all $v \in V$:
    - Set DP[v][i] =
min {
DP[v][i – 1],
min { DP[u][i – 1] + $w(u, v)$ } *(where $(u, v) \in E$)*
}
  - Return row n of DP.
- Answer: **O(*mn*)**, i you reverse G prior to running the algorithm.

Analyzing Space Complexity

- What is the *space* complexity of this algorithm?
  - Create a table DP of size $n \times n$.
  - Set DP[v][0] = ∞ for all $v \neq s$.
  - Set DP[s][0] = 0
  - For i = 1 to n – 1, for all $v \in V$:
    - Set DP[v][i] =
min {
    DP[v][i – 1],
    min { DP[u][i – 1] + $w(u, v)$ } *(where $(u, v) \in E$)*
}
  - Return row n of DP.
- Answer: **O(n^2)**. *(Can we reduce this?)*

All-Pairs Shortest Paths

Shortest Paths

- Dijkstra's algorithm and the Bellman-Ford algorithm solve the ***single-source shortest paths problem*** in which we want shortest paths starting from a single node.
- The ***all-pairs shortest paths problem*** asks how to find the shortest paths between all possible pairs of nodes.
- Can we already solve this problem?
- How efficient is our solution?

Intermediary Nodes

- A path between *u* and *v* starts at *u*, passes through some set of intermediary nodes, and ends at *v*.
- If there are no negative cycles, there is some shortest path from *u* to *v* where no nodes will be revisited. *(Why?)*

Intermediary Nodes

- Number all nodes v_1, v_2, …, v_n.
- What does a shortest path from u to v look like if no intermediary nodes are allowed?
- What does a shortest path from u to v look like if only node v_1 can be an intermediary node?
- What does a shortest path from u to v look like if only nodes v_1 and v_2 can be intermediary nodes?
- What does a shortest path from u to v look like if only nodes v_1, v_2, and v_3 can be intermediary nodes?

The Recurrence

- Let OPT(i, j, k) be the length of the shortest path from i to j where the only permitted internal nodes are v_1, v_2, …, v_k.
- **Claim:** OPT(i, j, k) satisfies this recurrence:

$$\mathrm{OPT}(i,j,k)=\begin{cases} 0 & \text{if } i=j \text{ and } k=0 \\ w(v_i,v_j) & \text{if } (v_i,v_j)\in E \text{ and } k=0 \\ \infty & \text{otherwise if } k=0 \\ min\left\{\begin{array}{l} \mathrm{OPT}(i,j,k-1), \\ \mathrm{OPT}(i,k,k-1)+ \\ \quad \mathrm{OPT}(k,j,k-1) \end{array}\right\} & \text{if } k\neq 0 \end{cases}$$

The Floyd-Warshall Algorithm

- Let DP be an $n \times n \times (n + 1)$ table.
- For i from 1 to n, j from 1 to n:
  - Set DP[i][j][0] = 0 if $i = j$.
  - Set DP[i][j][0] = $w(v_i, v_j)$ if $i \neq j$ and $(u, v) \in E$.
  - Set DP[i][j][0] = ∞ if $i \neq j$ and $(u, v) \notin E$.
- For k from 1 to n, i from 1 to n, j from 1 to n:
  - Set DP[i][j][k] = min{
    DP[i][j][k – 1],
    DP[i][k][k – 1] + DP[k][j][k – 1]
    }
- Return row k of DP.

Time and Space Complexity

- What is the time complexity of this algorithm?
  - $\mathbf{O(\boldsymbol{n}^3)}$
- What is the space complexity of this algorithm?
  - $\mathbf{O(\boldsymbol{n}^3)}$
- Interestingly, no dependence on the number of edges!

Further Algorithms

- **Johnson's Algorithm** combines Dijkstra's algorithm and Bellman-Ford together to solve the all-pairs shortest paths problem in arbitrary graphs with no negative cycles.
- Runtime is $O(mn + n^2 \log n)$ when implemented with appropriate data structures.
- How does that compare to Floyd-Warshall?
- Come talk to me after lecture for details!

Next Time

- Intractable Problems
- **NP**-Hardness
- Pseudopolynomial Algorithms