Randomized Algorithms

Part One

Announcements

- Problem Set 2 due right now if you're using a late period.
  - Solutions released right after lecture.
- Julie's Tuesday office hours this week will be remote office hours. Details emailed out tomorrow.

Outline for Today

- **Randomized Algorithms**
  - How can randomness help solve problems?
- **Quickselect**
  - Can we do away with median-of-medians?
- **Techniques in Randomization**
  - Linearity of expectation, the union bound, and other tricks.

Randomized Algorithms

Deterministic Algorithms

- The algorithms we've seen so far have been deterministic.
- We want to aim for properties like
  - Good worst-case behavior.
  - Getting exact solutions.
- Much of our complexity arises from the fact that there is little flexibility here.
- Often find complex algorithms with nuanced correctness proofs.

Randomized Algorithms

- A **randomized algorithm** is an algorithm that incorporates randomness as part of its operation.
- Often aim for properties like
  - Good *average-case* behavior.
  - Getting exact answers *with high probability*.
  - Getting answers that are *close* to the right answer.
- Often find very simple algorithms with dense but clean analyses.

Where We're Going

- Motivating examples:
  - Quickselect and quicksort are **Las Vegas algorithms**: they always find the right answer, but might take a while to do so.
  - Karger's algorithm is a **Monte Carlo algorithm**: it might not always find the right answer, but has dependable performance.
  - Hash tables with universal hash functions are **randomized data structures** that have high performance due to randomness.

Our First Randomized Algorithm: **Quickselect**

The Selection Problem

- Recall from last time: the selection problem is to find the *k*th largest element in an unsorted array.
- Can solve in O(*n* log *n*) time by sorting and taking the *k*th largest element.
- Can solve in O(*n*) time (with a large constant factor) using the "median-of-medians" algorithm.

Comparison of Selection Algorithms

Array Size	Sorting	Median of Medians
10000000	0.92	0.37
20000000	1.9	0.74
30000000	2.9	1.05
40000000	3.94	1.43
50000000	5.01	1.83
60000000	6.06	2.12
70000000	7.16	2.54
80000000	8.26	2.89
90000000	9.3	3.2

Partition-Based Selection

- Recall: The median-of-medians algorithm belongs to a family of algorithms based on the partition algorithm:
  - Choose a pivot.
  - Use partition to place it correctly.
  - Stop if the pivot is in the right place.
  - Recurse on one piece of the array otherwise.
- With no constraints on how the pivot is chosen, runtime is $\Omega(n)$ and $O(n^2)$.

Randomized Selection

- Silly question: What happens if you pick pivots completely at random?
- Intuitively, gives reasonably good probability of picking a good pivot.
- This algorithm is called **quickselect**.

Analyzing Quickselect

- When analyzing a randomized algorithm, we typically are interested in learning the following:
  - What is the average-case runtime of the function?
  - How likely are we to achieve that average-case runtime?
- We'll answer these questions in a few minutes.
- For now, let's start off with a simpler question...

The Worst Case

- In the worst-case, a partition-based selection algorithm can take O(n^2) time.
- Recall: What triggers the worst-case behavior of the selection algorithm?
- **Answer:** Continuously pick the largest or smallest element on each iteration.
- Since quickselect picks pivots randomly, what is the probability that this happens in quickselect?

Triggering the Worst Case

- Let $\mathcal{E}_k$ be the event that we pick the largest or smallest element of the array when there are k elements left.
- Let event $\mathcal{E}$ correspond to the worst-case runtime of quickselect occurring.
- We can then define $\mathcal{E}$ as the event

$$\mathcal{E} = \bigcap_{i=1}^{n} \mathcal{E}_i$$

- Question: What is $P(\mathcal{E})$?

Triggering the Worst Case

- We have

$$P(\mathcal{E}) = P\left(\bigcap_{i=1}^{n} \mathcal{E}_i\right)$$

- Since all $\mathcal{E}_i$'s are independent (we make independent random choices at each level), this simplifies to

$$P(\mathcal{E}) = P\left(\bigcap_{i=1}^{n} \mathcal{E}_i\right) = \prod_{i=1}^{n} P(\mathcal{E}_i)$$

- If $i > 1$, then $P(\mathcal{E}_i) = 2 / i$. $P(\mathcal{E}_1) = 1$. Thus

$$P(\mathcal{E}) = \prod_{i=1}^{n} P(\mathcal{E}_i) = \prod_{i=2}^{n} \frac{2}{i} = \frac{2^{n-1}}{n!}$$

Eensy Weensy Numbers

- The probability of triggering the worst-case behavior of quickselect is

$$P(\mathcal{E}) = \frac{2^{n-1}}{n!}$$

- To put that in perspective: if $n = 31$, then $2^{n-1} \approx 10^9$ and $n! \approx 8 \times 10^{33}$.
- This is *extremely* unlikely!

On Average

- We know that the probability of getting a worst-case runtime is vanishingly small.
- But how does the algorithm do *on average*? Is it $\Theta(n)$? $\Theta(n \log n)$? Something else?
- Totally reasonable thing to do: try running it and see what happens!

Comparison of Selection Algorithms

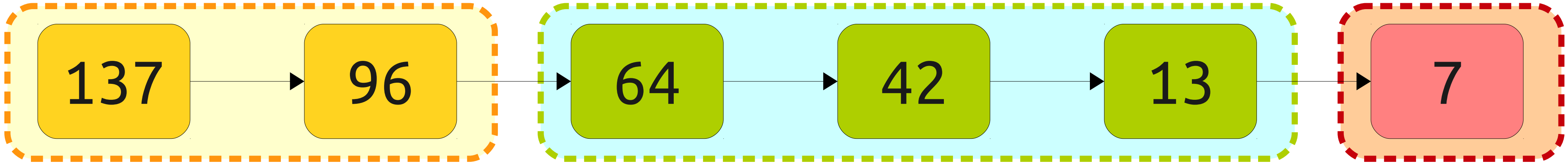
Array Size	Sorting	Median of Medians	Quickselect
10000000	0.92	0.37	0.11
20000000	1.9	0.74	0.14
30000000	2.9	1.05	0.27
40000000	3.94	1.43	0.44
50000000	5.01	1.83	0.53
60000000	6.06	2.12	0.64
70000000	7.16	2.54	0.69
80000000	8.26	2.89	1.01
90000000	9.3	3.2	0.72

An Average-Case Analysis

- Our guess: average runtime is $\Theta(n)$.
- How would we go about proving this?
- Since algorithm is recursive, might want to write a recurrence relation.
- This is challenging: the split size isn't guaranteed, so we have no idea how big our subproblems will be!
- Let's try another approach...

An Accounting Trick

- Because quickselect makes at most one recursive call, we can think of the algorithm as a chain of recursive calls:



- Accounting trick: group multiple calls together into one “phase” of the algorithm.
- The sum of the work done by all calls is equal to the sum of the work done by all phases.
- Goal: Pick phases intelligently to simplify analysis.

Picking Phases

- Let's define one “phase” of the algorithm to be when the algorithm decreases the size of the input array to 75% of the original size or less.
- Why 75%?
    - If array shrinks by any constant factor from phase to phase and only does linear work per phase, total work done is linear.
    - The number 75% has a nice intuition…

Triggering 75% / 25%

- Suppose that we pick a pivot whose value is in the middle 50% of all array values.
- Then 25% of array values are larger and 25% of array values are smaller.
- Guaranteed to get a 75% / 25% split!
- A phase ends as soon as we pick a pivot in the middle 50% of all values.

Analyzing the Runtime

- Number the phases 0, 1, 2, …
- In phase k, the array size is at most $n(3 / 4)^k$.
- Last phase numbered at most $\lceil \log_{4/3} n \rceil$.
- Let X_k be a random variable equal to the number of recursive calls in phase k.
- Work done in phase k is at most

$$X_k \cdot cn\left(\frac{3}{4}\right)^k \qquad \textit{(for some constant c)}$$

- Let W be a random variable denoting the total work done. Then

$$W \leq \sum_{k=0}^{\lceil \log_{4/3} n \rceil} \left(X_k \cdot cn\left(\frac{3}{4}\right)^k \right) = cn \sum_{k=0}^{\lceil \log_{4/3} n \rceil} \left(X_k \left(\frac{3}{4}\right)^k \right)$$

The Average-Case Analysis

- Our goal is to determine the expected runtime for quickselect on an array of size n.
- This is $E[W]$, the expected value of W.
- This is given by

$$E[W] \leq E\left[cn \sum_{k=0}^{\lceil \log_{4/3} n \rceil} \left(X_k \left(\frac{3}{4}\right)^k \right)\right]$$

Properties of Expectation

- The expected value of a constant or non-random variable is just that constant or variable itself:

$$\mathbf{E}[c] = c$$

- Expected value is a linear operator:

$$\mathbf{E}[aX + b] = a\mathbf{E}[X] + b$$

$$\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y]$$

- Note that the second claim holds even if X and Y are dependent variables.

Simplifying Our Expression

$$\begin{aligned}
\mathrm{E}[W] &\le \mathrm{E}\left[cn \sum_{k=0}^{\lceil \log_{4/3} n \rceil} \left(X_k \left(\frac{3}{4}\right)^k \right) \right] \\
&= cn \cdot \mathrm{E}\left[\sum_{k=0}^{\lceil \log_{4/3} n \rceil} \left(X_k \left(\frac{3}{4}\right)^k \right) \right] \\
&= cn \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} \mathrm{E}\left[X_k \left(\frac{3}{4}\right)^k \right] \\
&= cn \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} \mathrm{E}[X_k] \left(\frac{3}{4}\right)^k
\end{aligned}$$

E[X_k]

- By definition:

$$E[X_k] = \sum_{i=0}^{\infty} i \cdot P(X_k = i)$$

  Recall: X_k is the number of calls within phase k.
- Equivalently: The number of calls before a pivot is chosen in the middle 50% of the elements.
- Can we determine this explicitly?

E[X_k]

- E[X_k] is defined by

$$E[X_k] = \sum_{i=0}^{\infty} i \cdot P(X_k = i)$$

- $P(X_k = i)$ is the probability that the first i – 1 pivots we chose weren't in the middle 50% and that the ith pivot is in the middle 50%.
  - (As an edge case, it's 0 when i = 0.)
- As a simplification: assume that whenever we pick a pivot, we can choose from any of the n elements present at the start of the phase.
- Only makes it harder to end the phase; provides an upper bound on the phase length.

E[X_k]

- Recall: E[X_k] is defined by

$$\mathrm{E}[X_k]=\sum_{i=0}^{\infty} i\cdot P(X_k=i)$$

- Under the assumption that all pivot choices are independent, $P(X_k = i)$ is given by

$$P(X_k = i) = (1 / 2)^i$$

- Probability the first i – 1 pivots are in the outer 50% and the ith pivot was in the inner 50%.
- Therefore

$$\mathrm{E}[X_k] = \sum_{i=0}^{\infty} i\cdot P(X_k=i) \le \sum_{i=1}^{\infty} \frac{i}{2^i} = 2$$

Finalizing the Computation

$$\begin{aligned}
\mathrm{E}[W] &\leq cn \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} \mathrm{E}[X_k] \left(\frac{3}{4}\right)^k \\
&\leq cn \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} 2 \left(\frac{3}{4}\right)^k \\
&= 2cn \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} \left(\frac{3}{4}\right)^k \\
&\leq 2cn \cdot \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k \\
&= 8cn \\
&= \mathrm{O}(n)
\end{aligned}$$

Bounding the Spread

- We now know that quickselect runs in expected O(n) time.
- How *likely* is it that the runtime is O(n)?

Bounding the Spread

- Idea: Devise a formula for the probability that every phase terminates within *r* steps.
- If this happens, quickselect will run in time

$$\begin{aligned} cn \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} X_k \left(\frac{3}{4}\right)^k &\leq cn \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} r \left(\frac{3}{4}\right)^k \\ &= cnr \cdot \sum_{k=0}^{\lceil \log_{4/3} n \rceil} \left(\frac{3}{4}\right)^k \\ &\leq 4cnr \\ &= \mathrm{O}(nr) \end{aligned}$$

- Goal: Find the probability (as a function of *r*) that this occurs.

Bounding the Spread

- We want the probability of the event

All phases terminate within *r* steps.

- Mathematically, it's easier to work with the probability of the *complement* of this event:

At least one phase terminates in at least *r* + 1 steps.

- We can compute the probability of the first event by subtracting the probability of the second event from one.

Long Phase Runtimes

- The probability that phase *k* takes more than *r* steps to finish is given by

$$P(X_k > r)$$

- This is

$$P(X_k > r) = P\left(\bigcup_{i=r+1}^{\infty} X_k = i\right)$$

- Since these events are all mutually exclusive:

$$P(X_k > r) = \sum_{i=r+1}^{\infty} P(X_k = i) \leq \sum_{i=r+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^r}$$

Long Phase Runtimes

- The probability that *any* phase takes more than *r* steps to finish is

$$P\left(\bigcup_{i=0}^{\lceil \log_{4/3} n \rceil} X_i > r\right)$$

- These are not mutually exclusive events – we may have multiple different phases finish in more than *r* steps.
- We can use the **union bound** to get an upper-bound on the true value:

$$P\left(\bigcup_{i=0}^{\infty} \mathcal{E}_i\right) \leq \sum_{i=0}^{\infty} P\left(\mathcal{E}_i\right)$$

Long Phase Runtimes

- The probability that *any* phase takes more than *r* steps to finish is

$$P\left(\bigcup_{i=0}^{\lceil \log_{4/3} n \rceil} X_i > r\right)$$

- Using the union bound:

$$P\left(\bigcup_{i=0}^{\lceil \log_{4/3} n \rceil} X_i > r\right) \leq \sum_{i=0}^{\lceil \log_{4/3} n \rceil} P(X_i > r) \leq \sum_{i=0}^{\lceil \log_{4/3} n \rceil} \frac{1}{2^r} = \frac{\lceil \log_{4/3} n \rceil + 1}{2^r}$$

Long Phase Runtimes

- For any number *r*, the probability that any one phase takes more than *r* steps to finish is

$$\frac{\lceil \log_{4/3} n \rceil + 1}{2^r}$$

- If for any value *s* we pick $r = s + \log_2(\lceil \log_{4/3} n \rceil + 1)$, then the probability that any phase takes more than *r* steps to complete is at most

$$\frac{\lceil \log_{4/3} n \rceil + 1}{2^r} = \frac{\lceil \log_{4/3} n \rceil + 1}{2^{s + \log_2(\lceil \log_{4/3} n \rceil + 1)}} = \frac{\lceil \log_{4/3} n \rceil + 1}{2^s(\lceil \log_{4/3} n \rceil + 1)} = \frac{1}{2^s}$$

Bounding the Runtime

- Recall: If all phases terminate within r steps, the total runtime will be $O(nr)$.
- If we pick $r = s + \log_2(\lceil \log_{4/3} n \rceil + 1)$, then the runtime will be $O(ns + n \log \log n)$ with probability at least $1 - 1/2^s$.
- For any constant k, pick $s = \log_2 n^k = k \log_2 n$. Probability that the runtime is **$O(n \log n)$** is at least **$1 - 1 / n^k$**.
- ***Definition:*** Event $\mathcal{E}$ occurs **with high probability** iff $P(\mathcal{E}) \geq 1 - 1 / n^c$ for some $c \geq 1$.
- Quickselect runs in time at most $O(n \log n)$ with high probability.

Wrap-Up: **Introselect**

Where We Stand

- The median-of-medians algorithm has runtime O(n), but has a large constant factor.
- Quickselect has average-case runtime O(n) with a low constant factor, but isn't guaranteed to run in time O(n).
- Can we get the best of both worlds?

Introspective Selection

- The **introselect** algorithm intelligently combines median-of-medians and quickselect.
- Idea: Run quickselect, but keep track of how many iterations have passed in the current phase.
- If the phase ends before the number of iterations exceeds some constant k, reset the counter and continue.
- Otherwise, run the median-of-medians algorithm to choose a pivot and reset the counter.

Introspective Selection

- Assuming introselect makes good random choices, it is inappreciably slower than normal quickselect.
- If it makes too many bad choices, we do some expensive median-of-medians steps, which is slower but ensures linear time.
- Net result is an algorithm that has worst-case O(n) runtime and on expectation matches quickselect's runtime.

Comparison of Selection Algorithms

Array Size	Sorting	Median of Medians	Quickselect	Introselect
10000000	0.92	0.37	0.11	0.07
20000000	1.9	0.74	0.14	0.17
30000000	2.9	1.05	0.27	0.17
40000000	3.94	1.43	0.44	0.33
50000000	5.01	1.83	0.53	0.42
60000000	6.06	2.12	0.64	0.41
70000000	7.16	2.54	0.69	0.51
80000000	8.26	2.89	1.01	0.56
90000000	9.3	3.2	0.72	0.88

Next Time

- Quicksort
- Indicator Random Variables
- Harmonic Numbers