Fundamental Graph Algorithms

Part One

Announcements

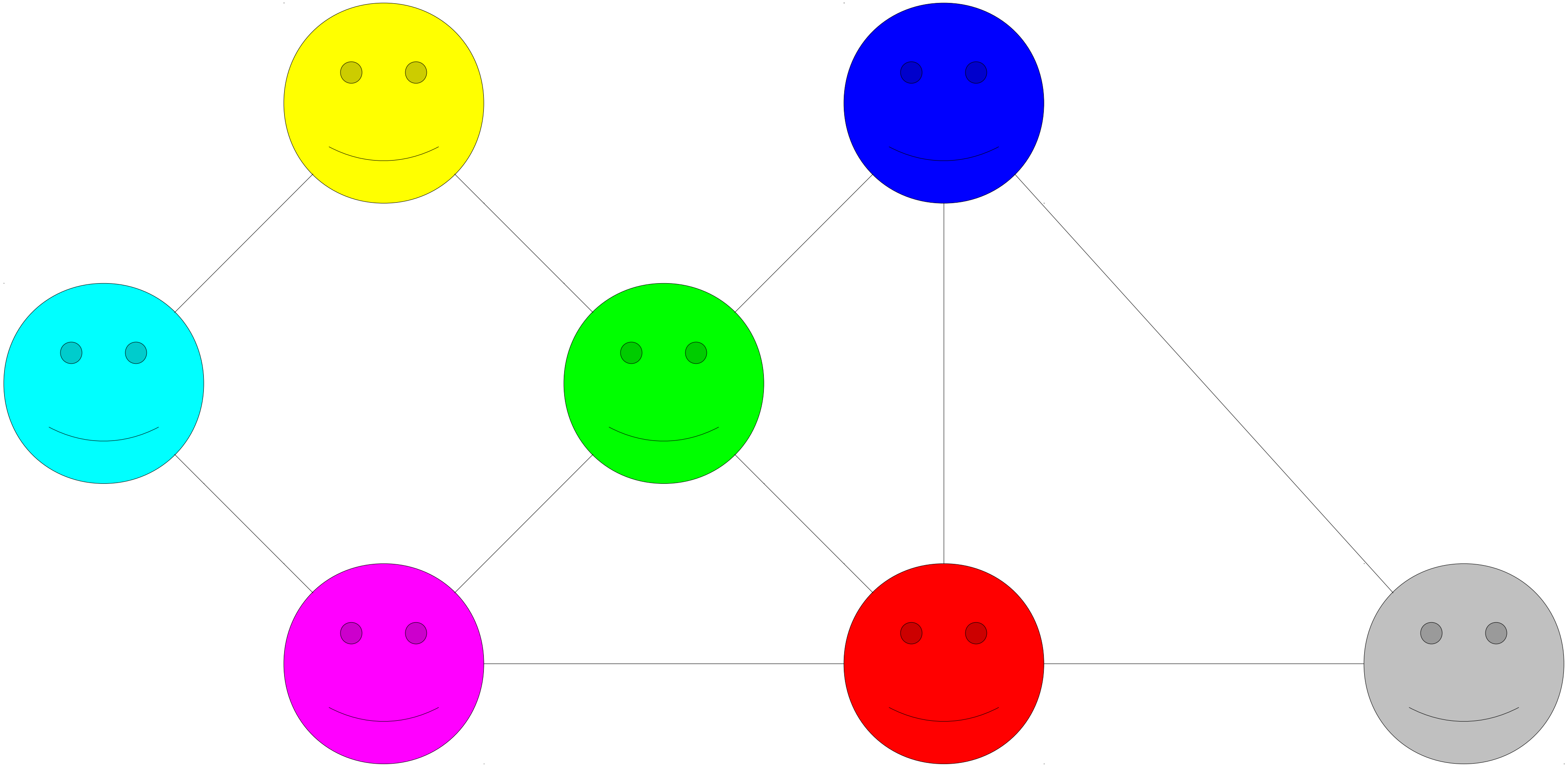
- Problem Set One out, due **Wednesday, July 3**.
  - Play around with O, Ω, and Θ notations!
  - Get your feet wet designing and analyzing algorithms.
  - Explore today's material on graphs.
- Can be completed using just material from the first two lectures.
- We suggest reading through the handout on how to approach the problem sets. There's a lot of useful information there!
- Office hours schedule will be announced tomorrow.

Announcements

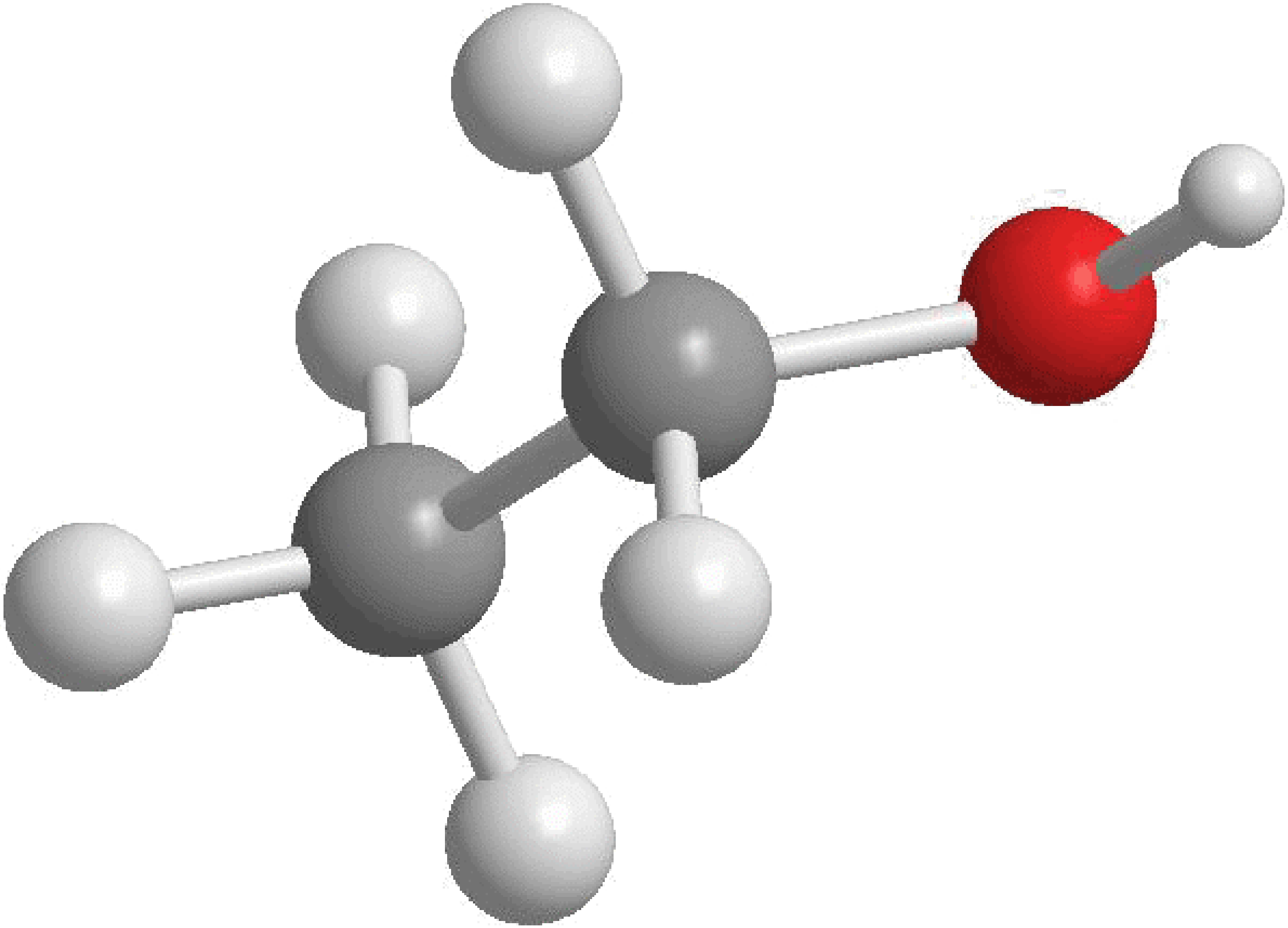
- We will not be writing any code in CS161; we'll focus more on the design and analysis techniques.
- Each week, we will have an optional programming section where you can practice coding up these algorithms.
- Run by TA Andy Nguyen, who coaches Stanford's ACM programming team.
- Meets **Thursdays, 4:15PM – 5:05PM** in **Gates B08**.

Graphs

A Social Network



Chemical Bonds



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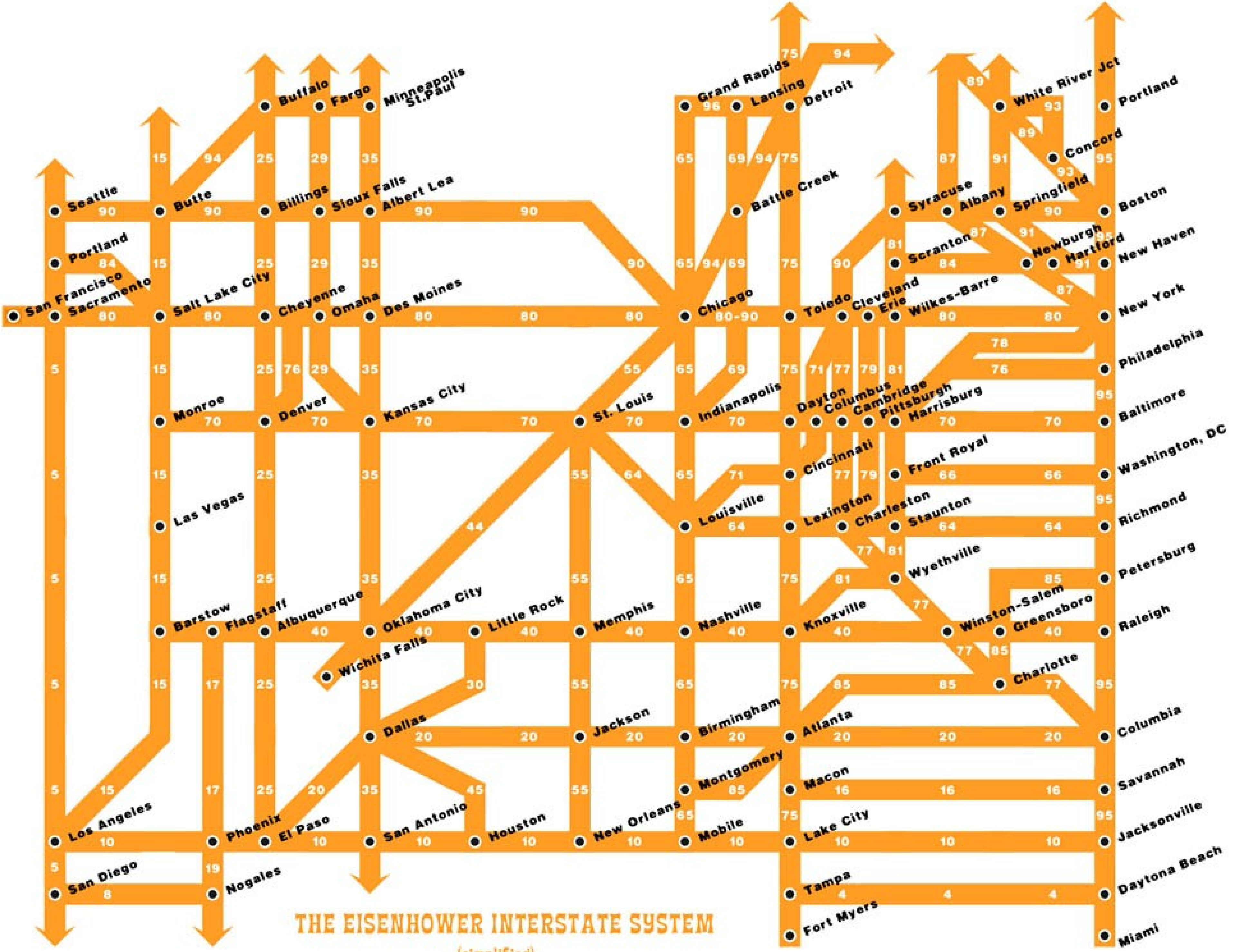
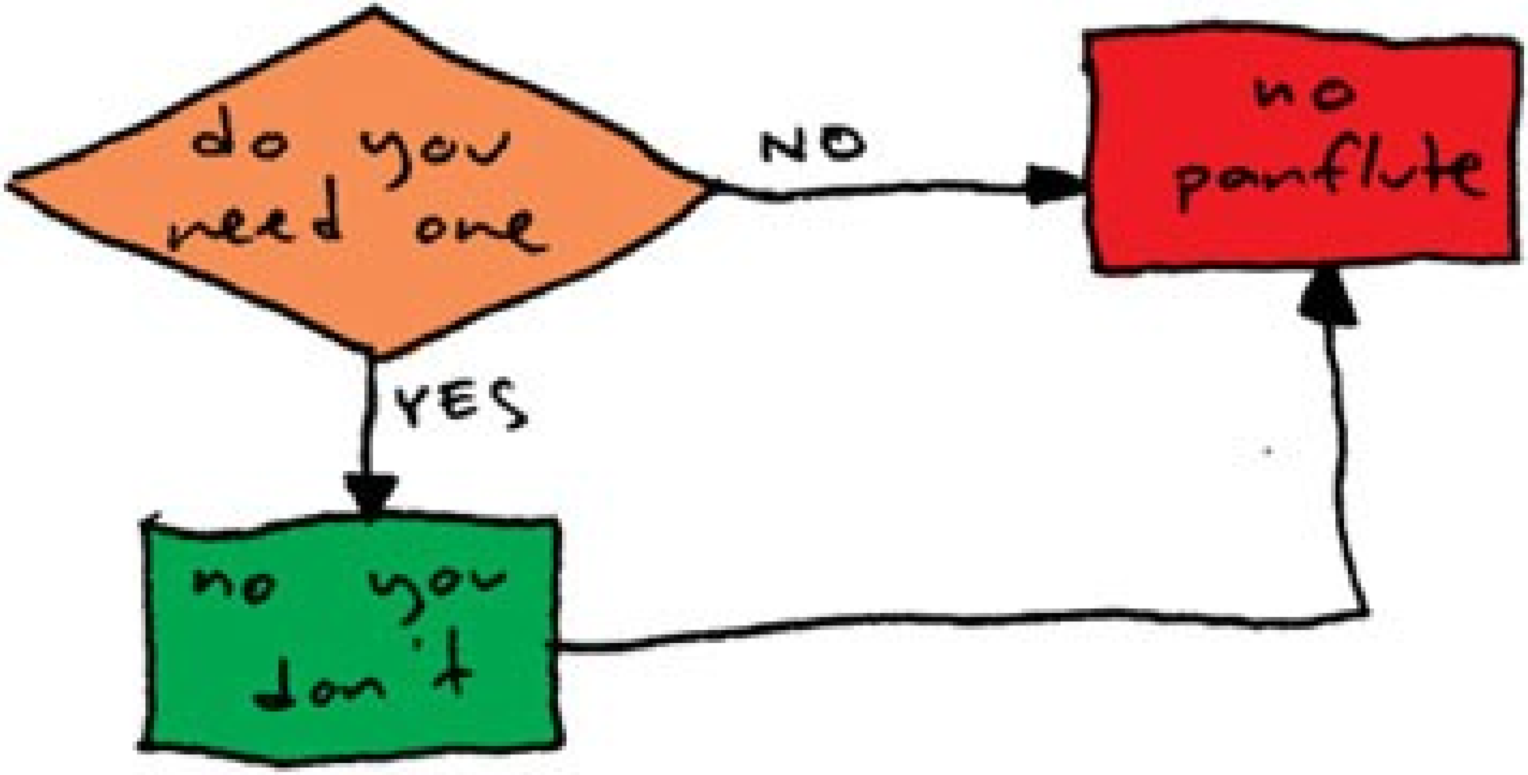


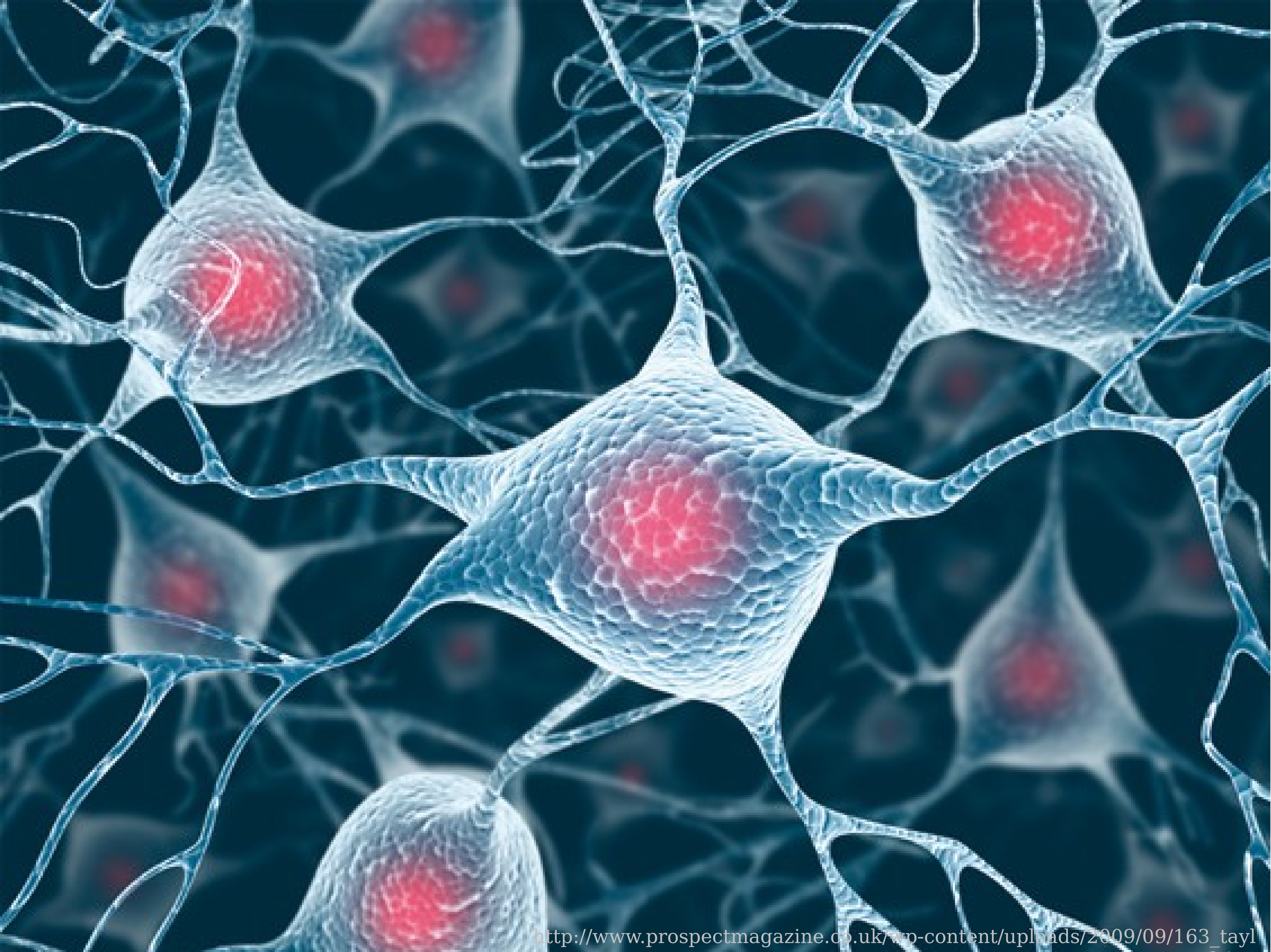
THE EISENHOWER INTERSTATE SYSTEM

(simplified)

CHRIS YATES 2007

PANFLUTE FLOWCHART





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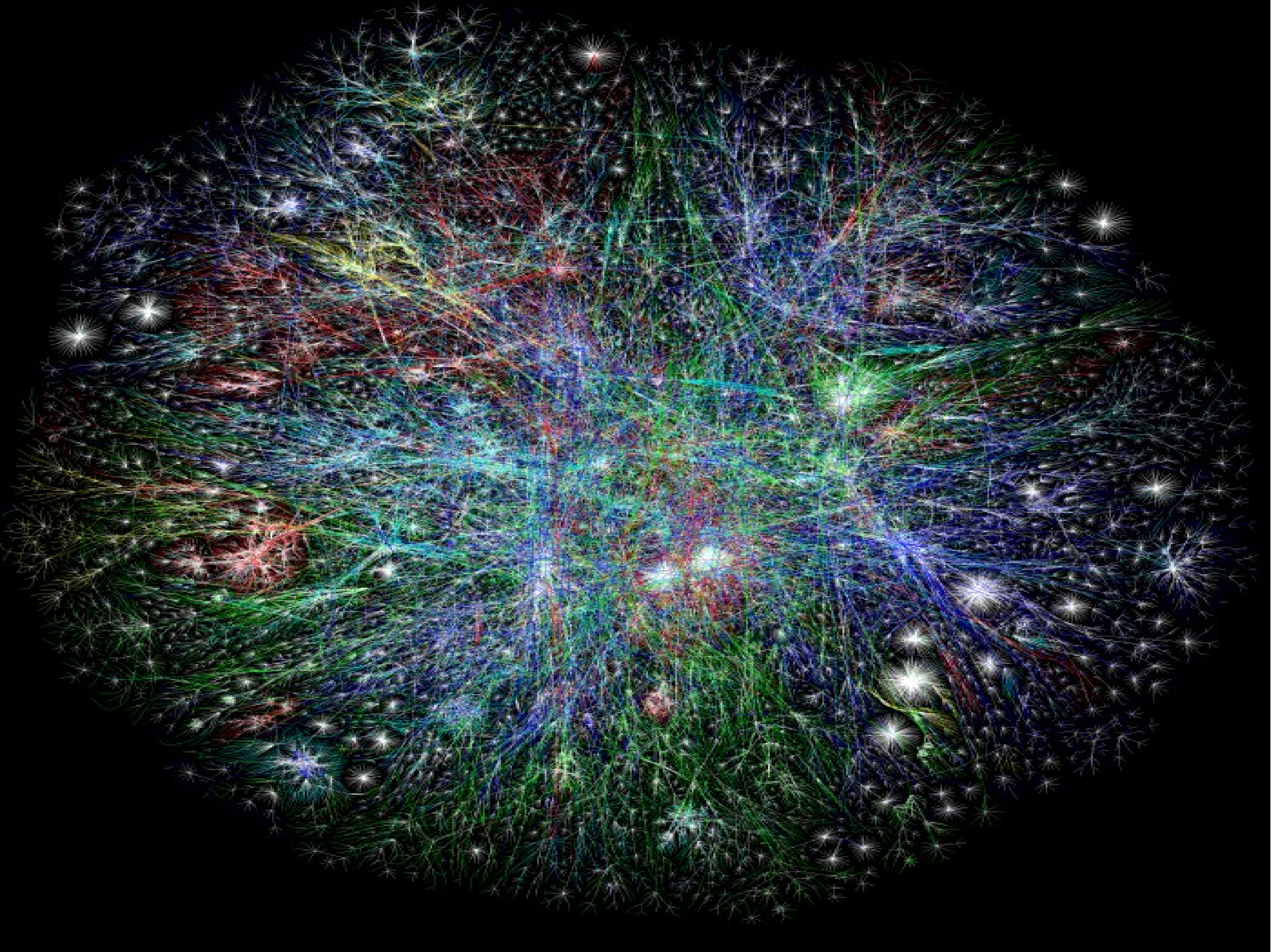

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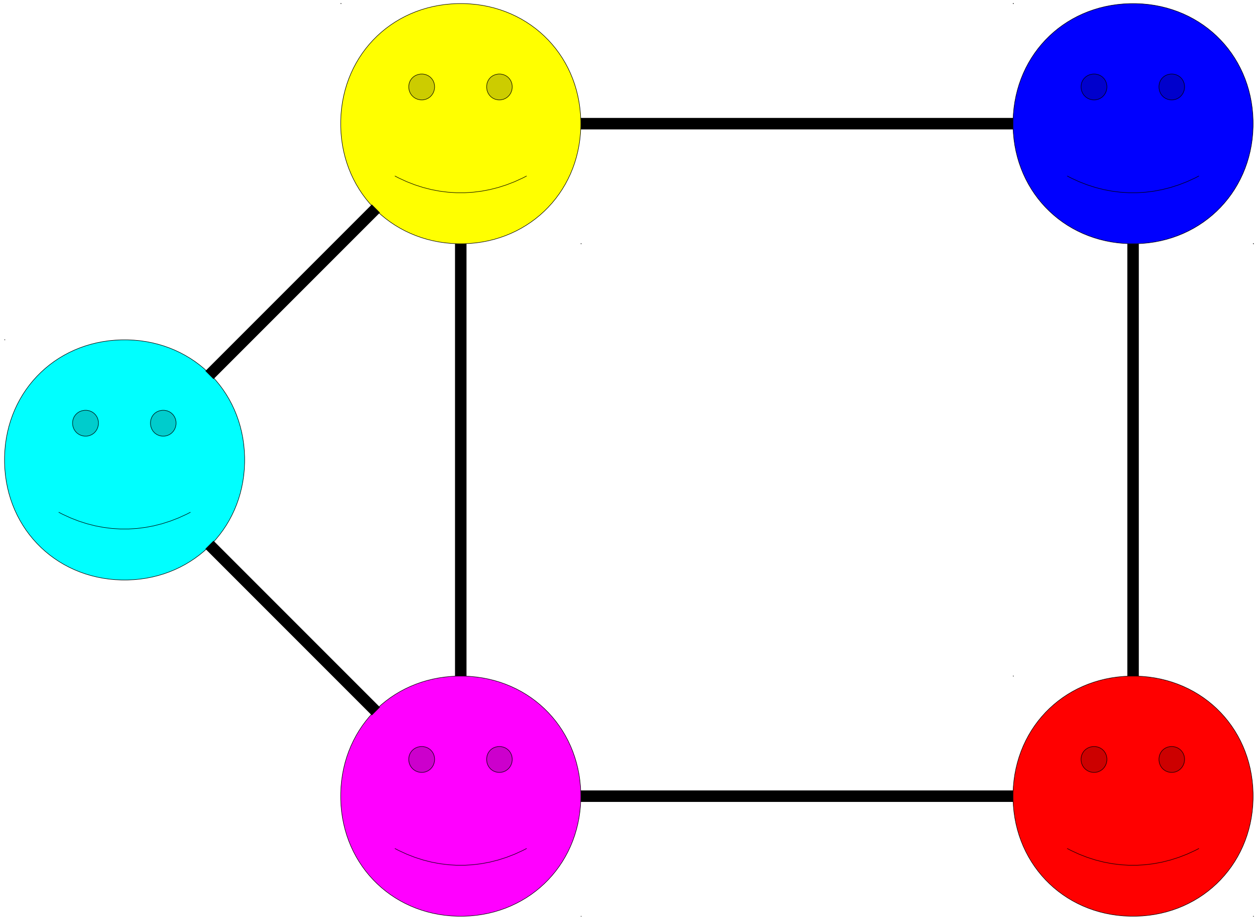

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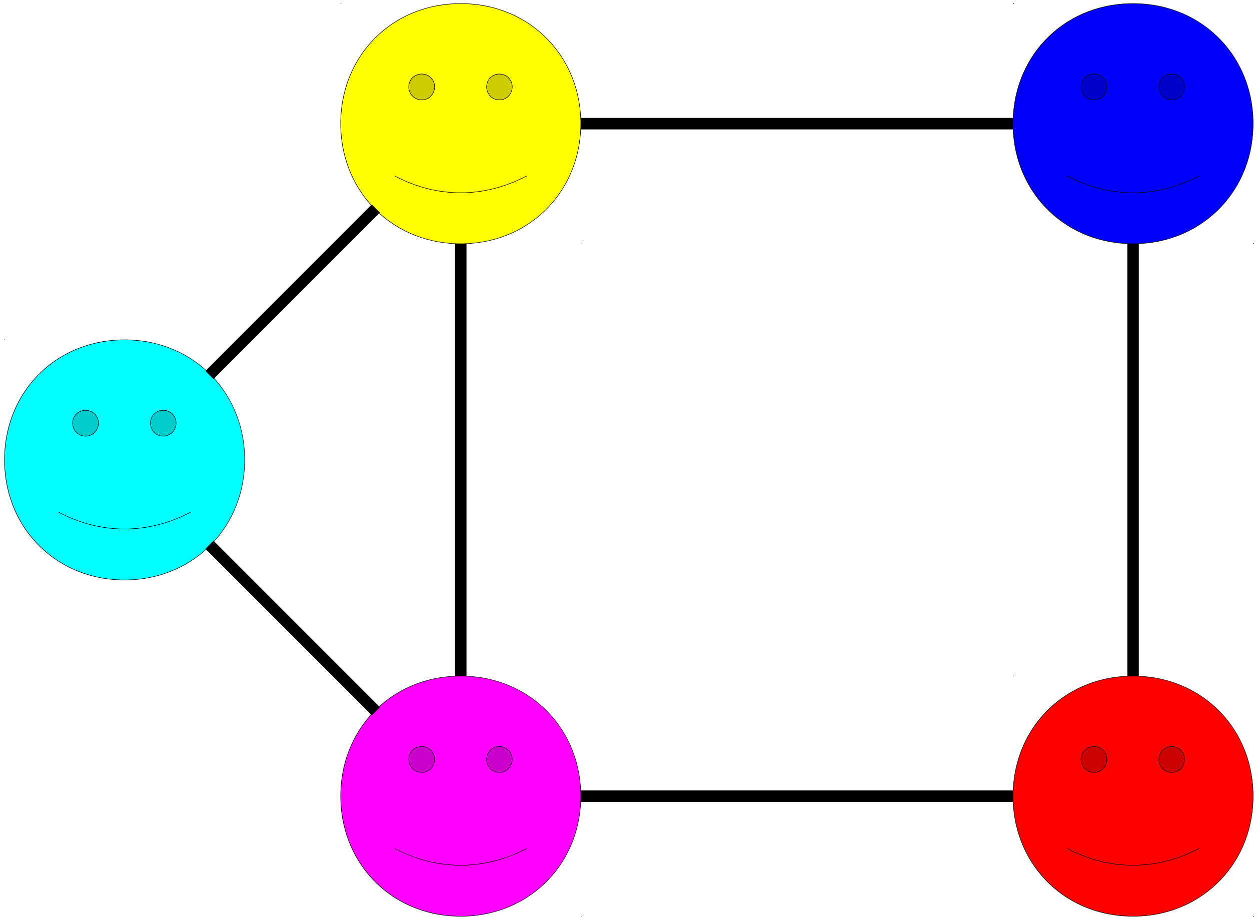

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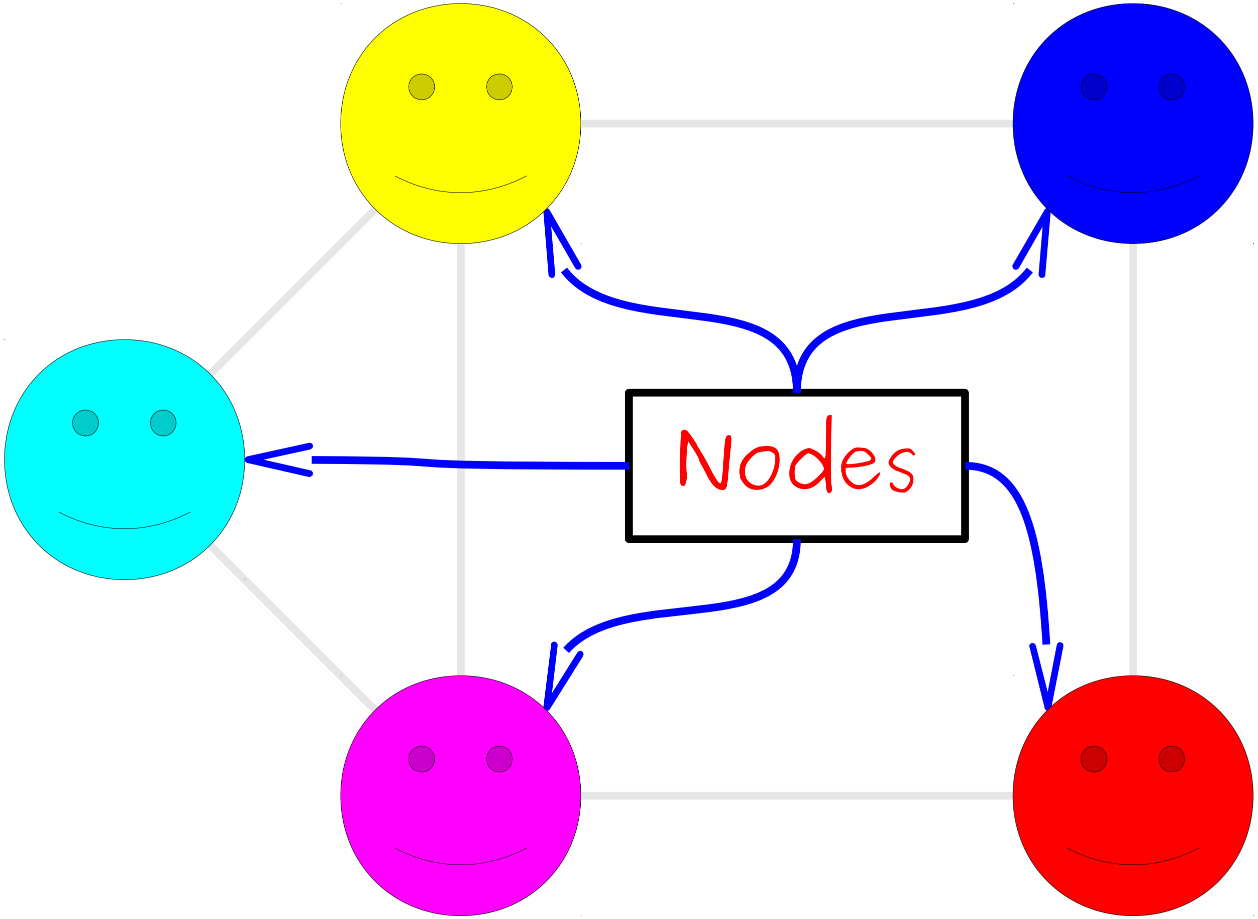

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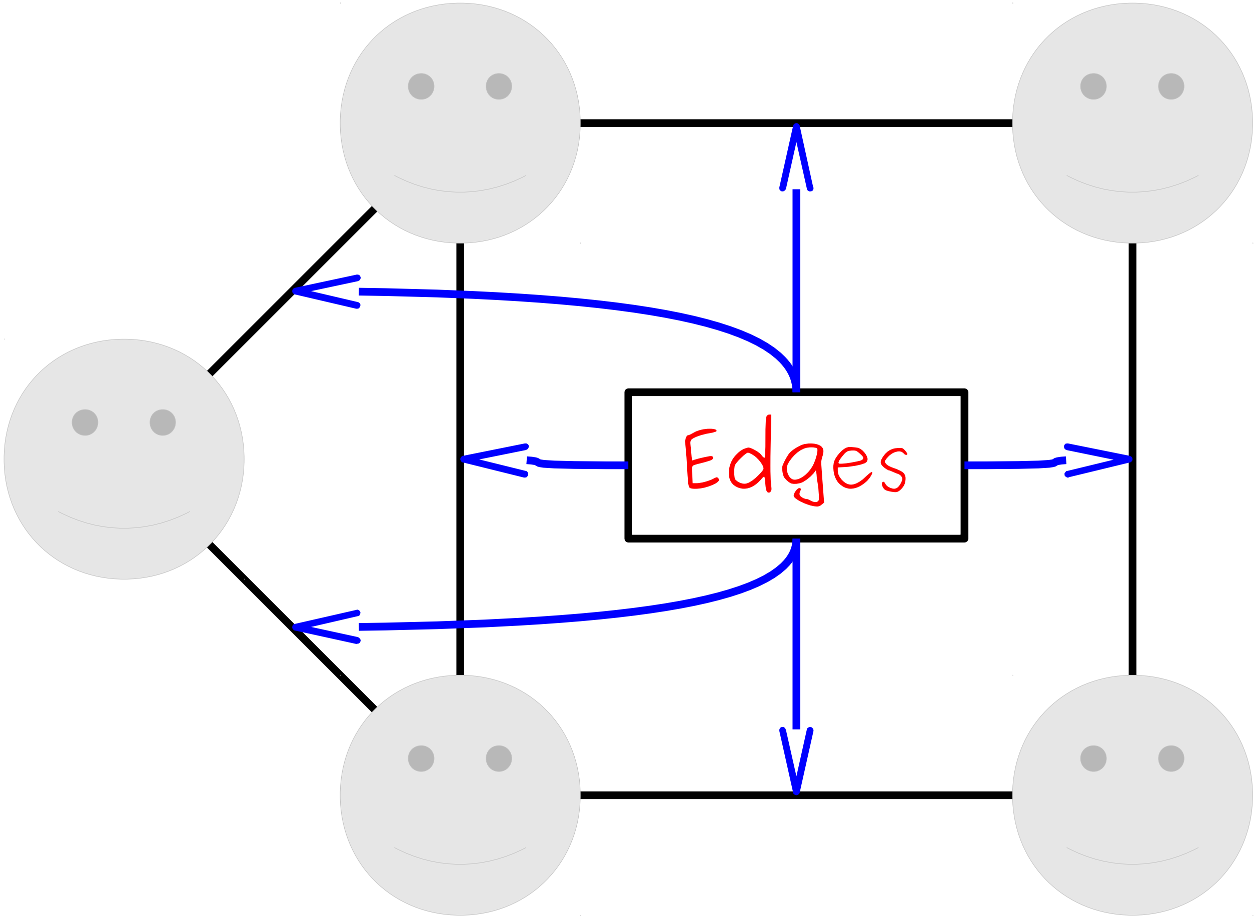

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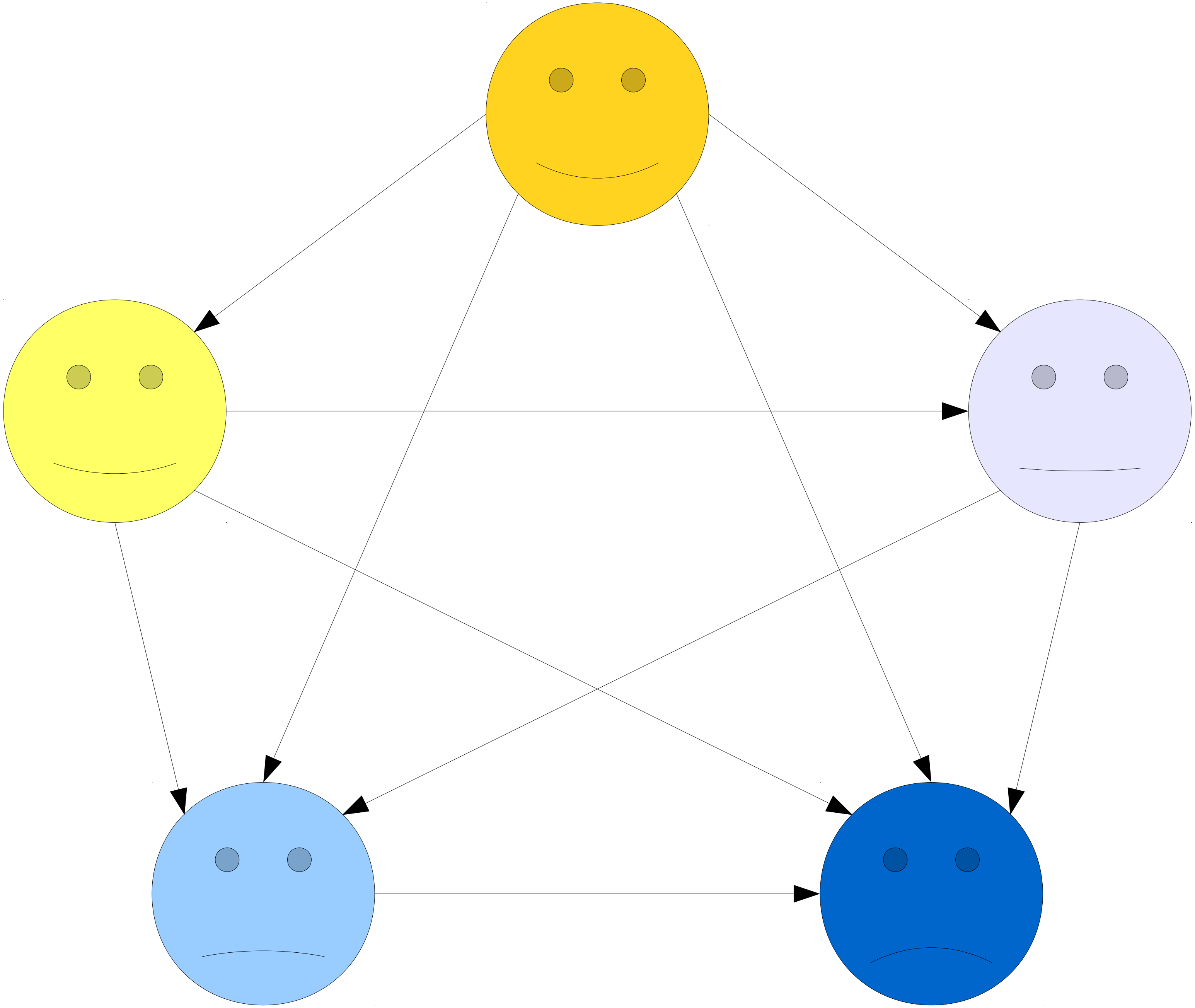

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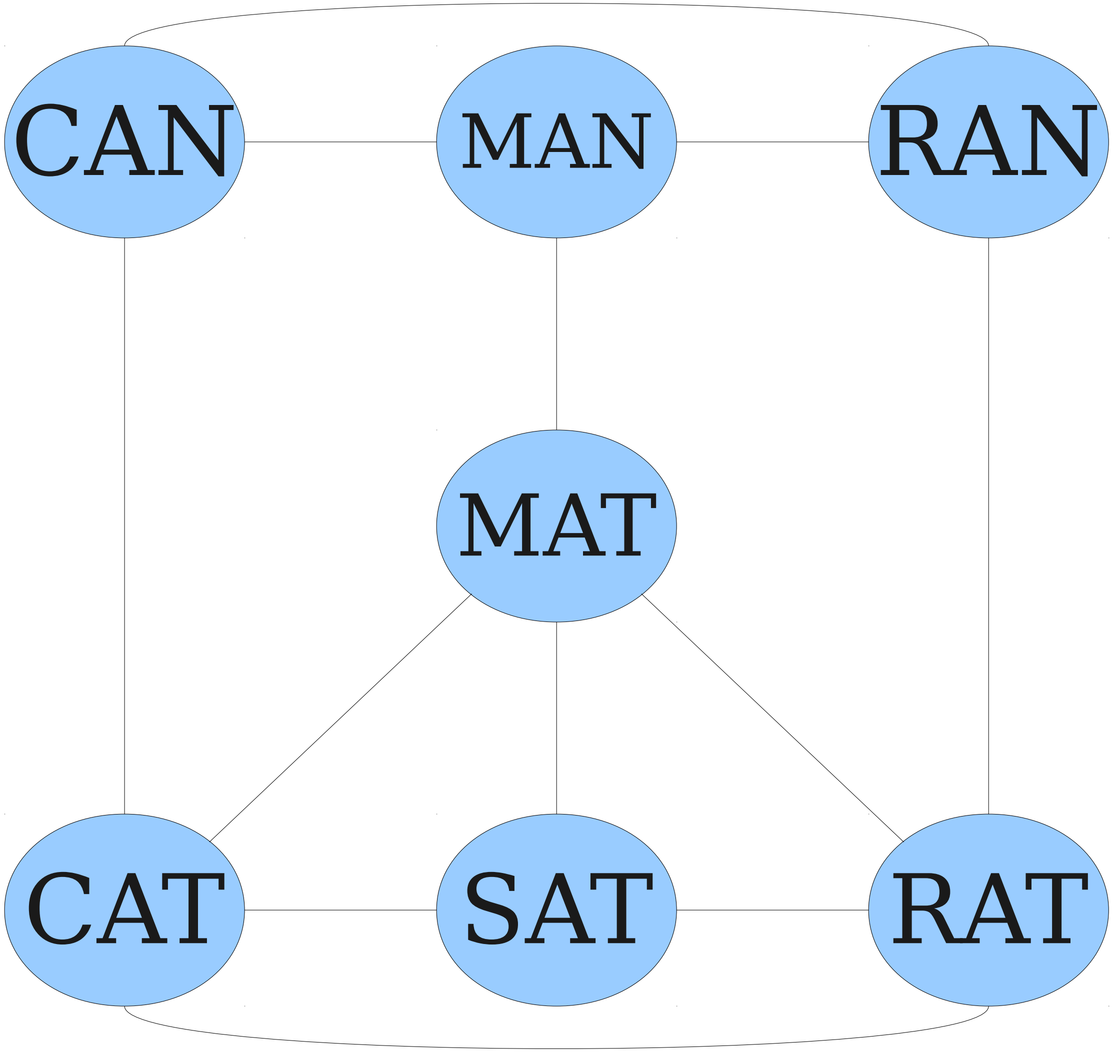

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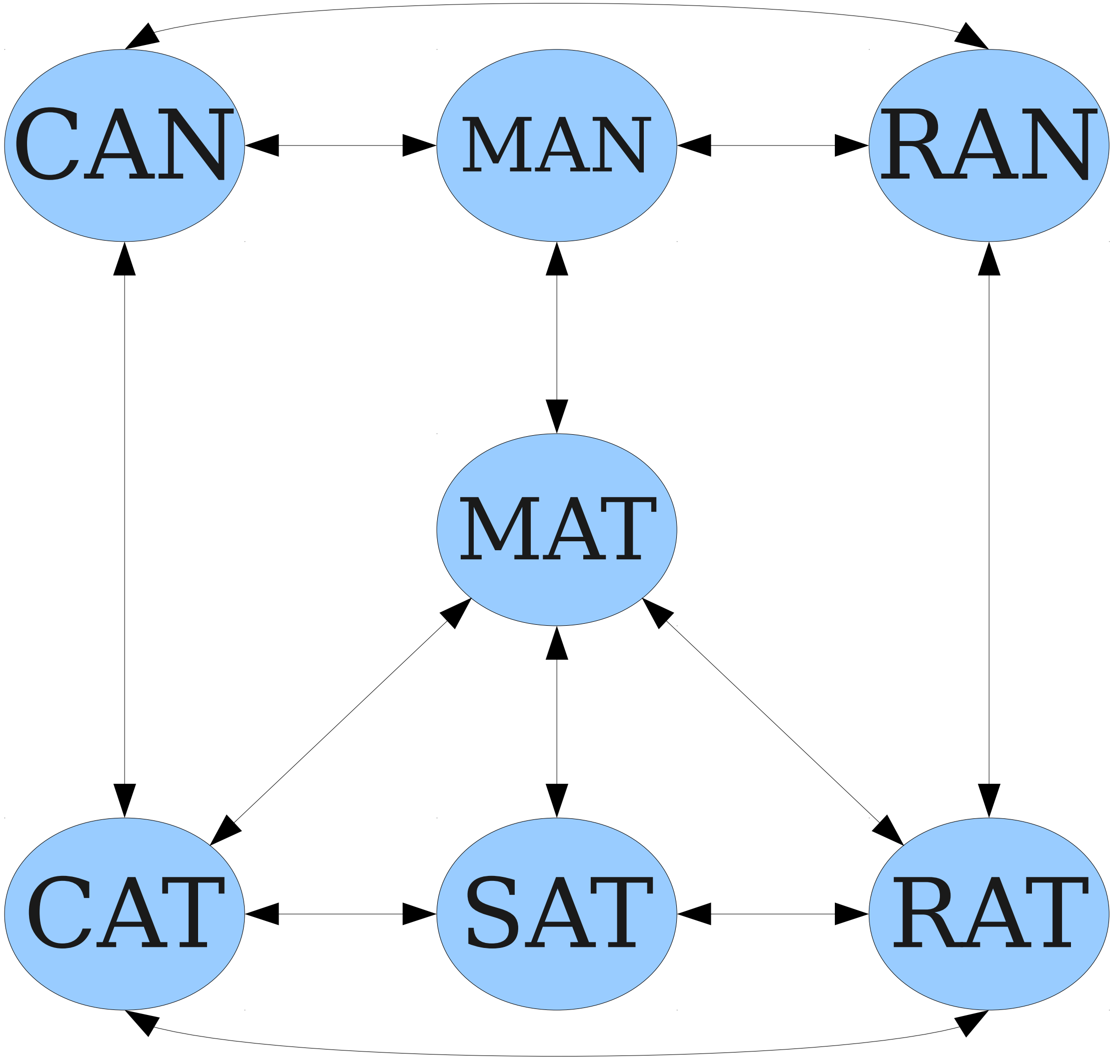


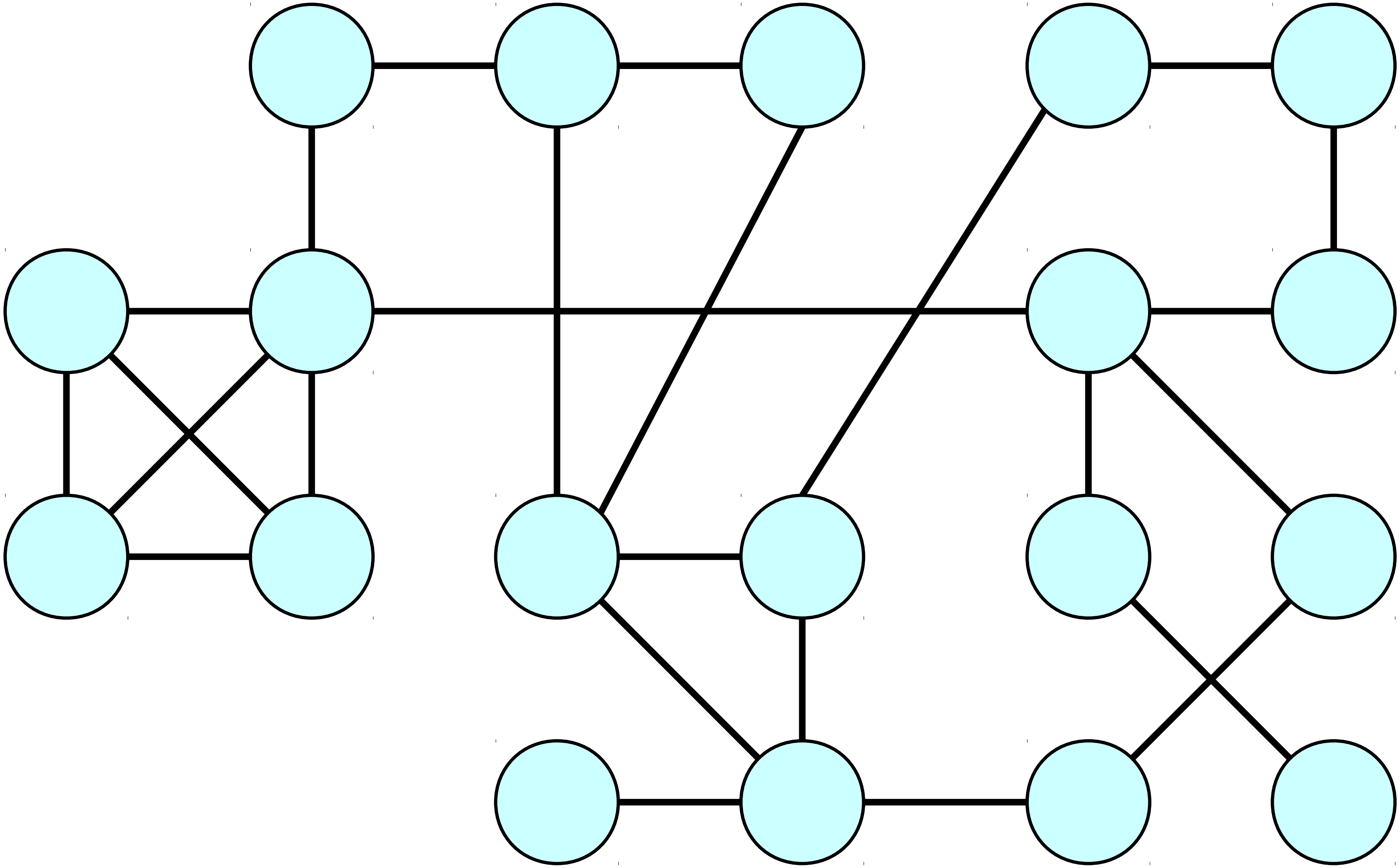
You can think of them as directed graphs with edges both ways.

Formalisms

- A **graph** is an ordered pair $G = (V, E)$ where
  - V is a set of the **vertices** (nodes) of the graph.
  - E is a set of the **edges** (arcs) of the graph.
- E can be a set of ordered pairs or unordered pairs.
  - If E consists of ordered pairs, G is **directed**
  - If E consists of unordered pairs, G is **undirected**.
- In an *undirected* graph, the **degree** of node v (denoted **deg(v)**) is the number of edges incident to v.
- In a *directed* graph, the **indegree** of a node v (denoted **deg^{-}(v)**) is the number of edges entering v and the **outdegree** of a node v (denoted (**deg^{+}(v)**) is the number of edges leaving v.

An Application: Six Degrees of Separation

A Social Network



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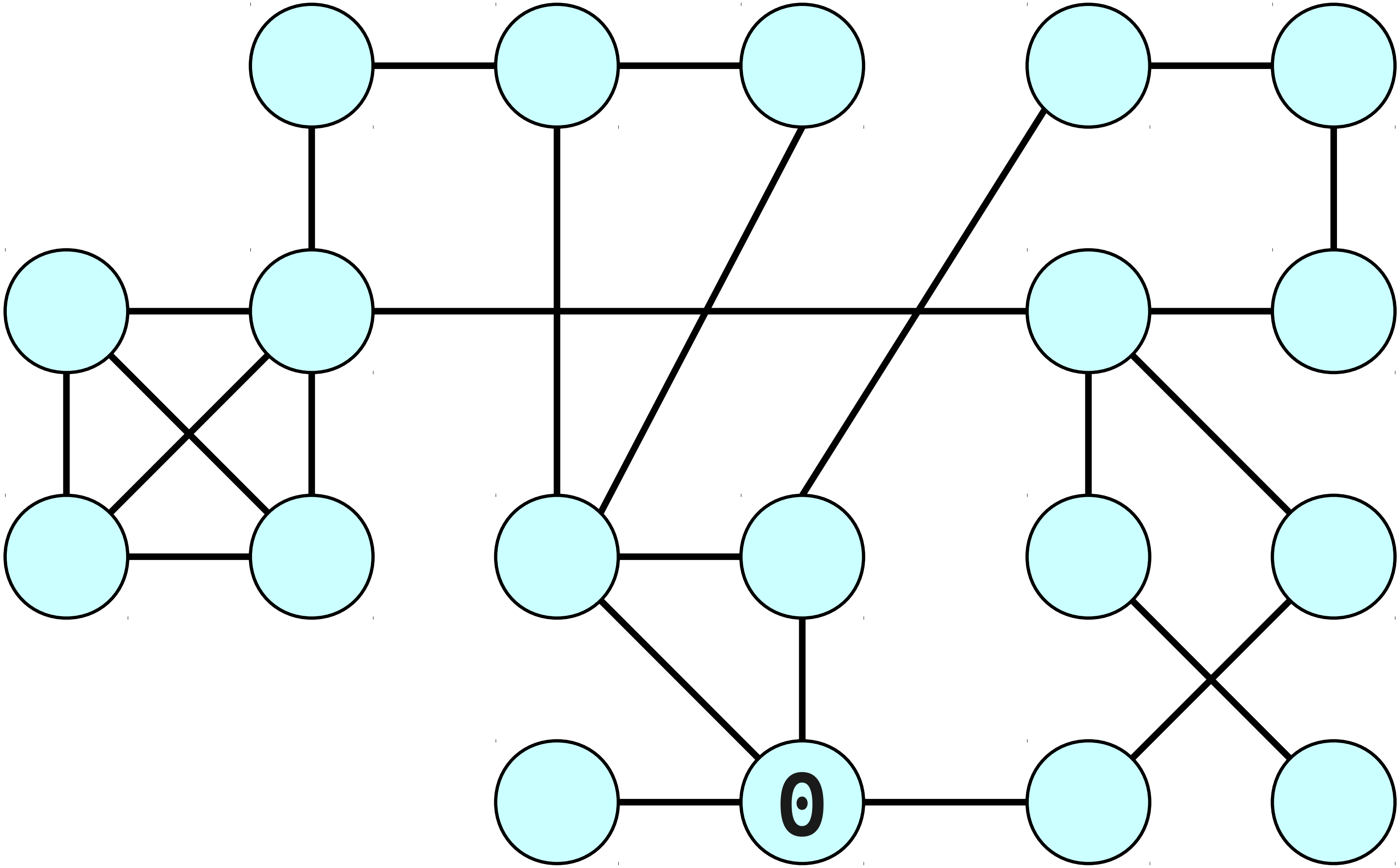

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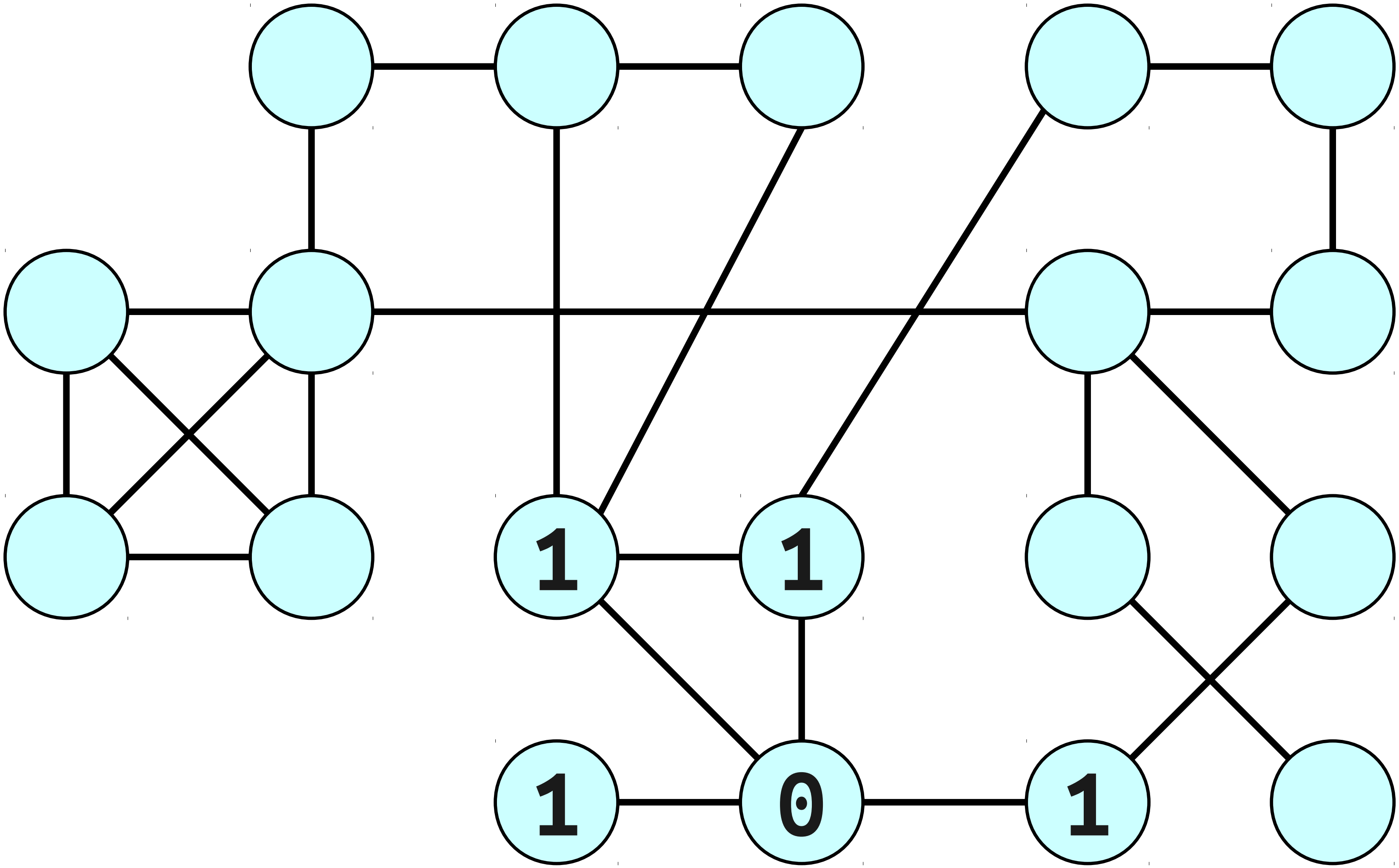

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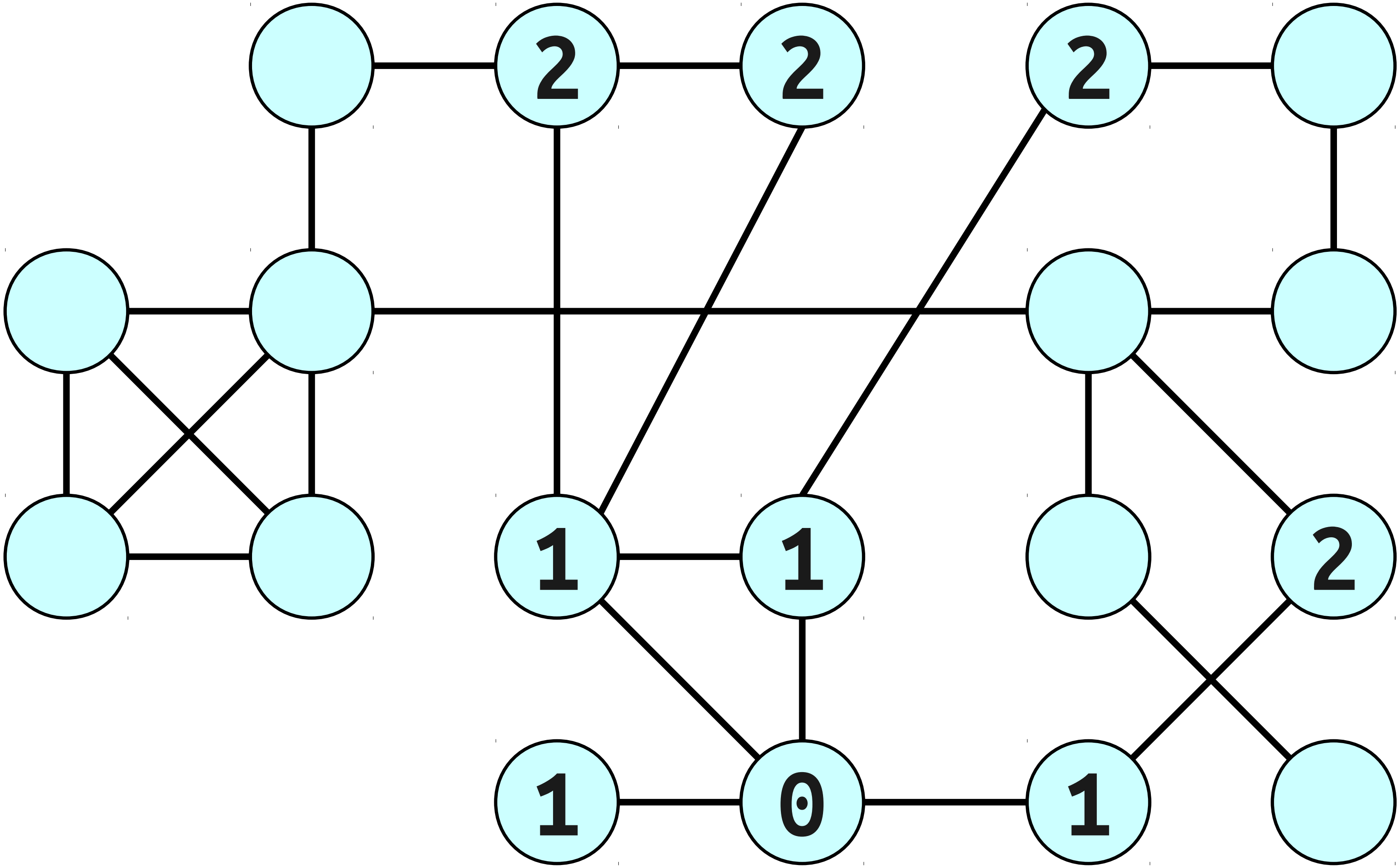

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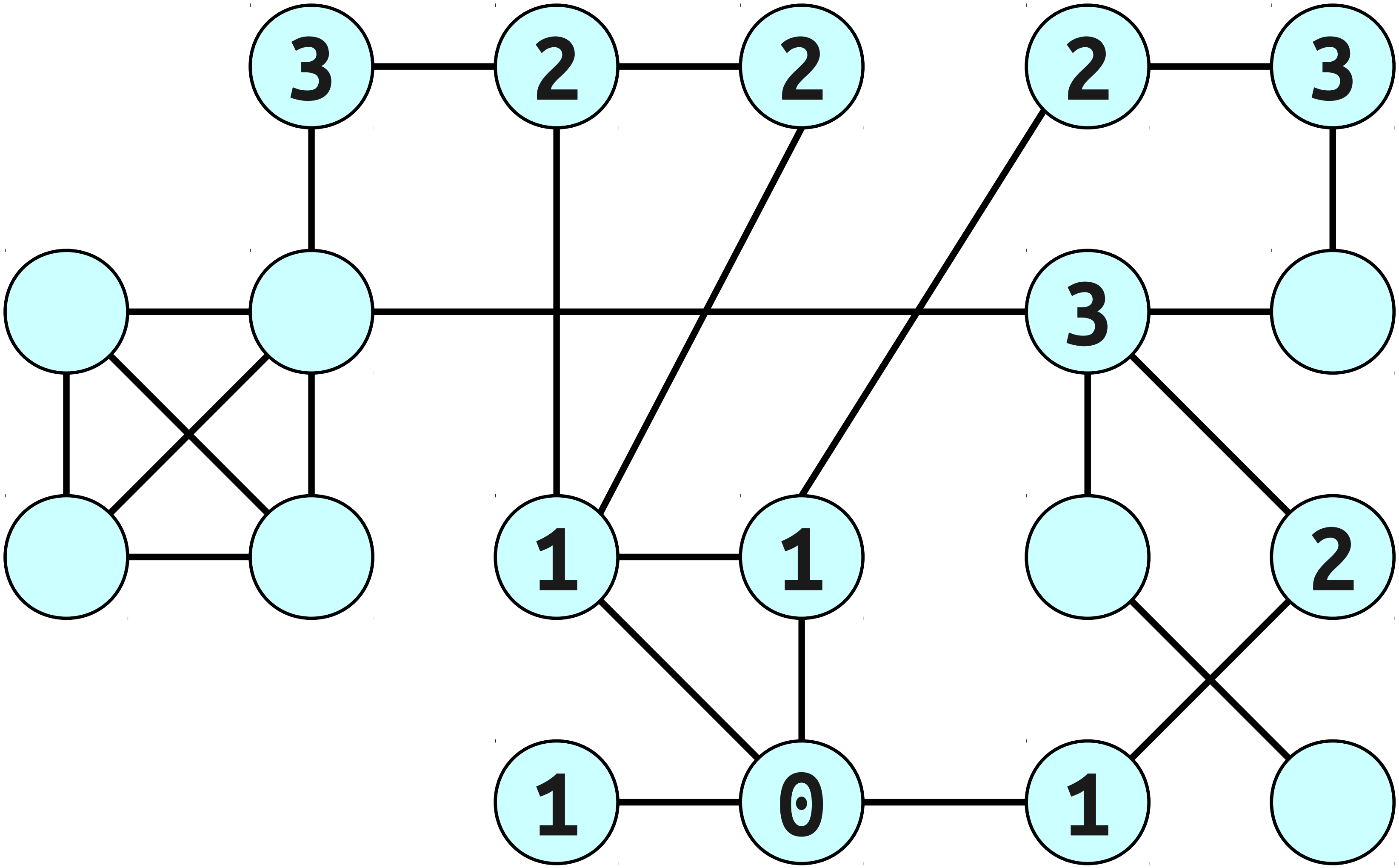

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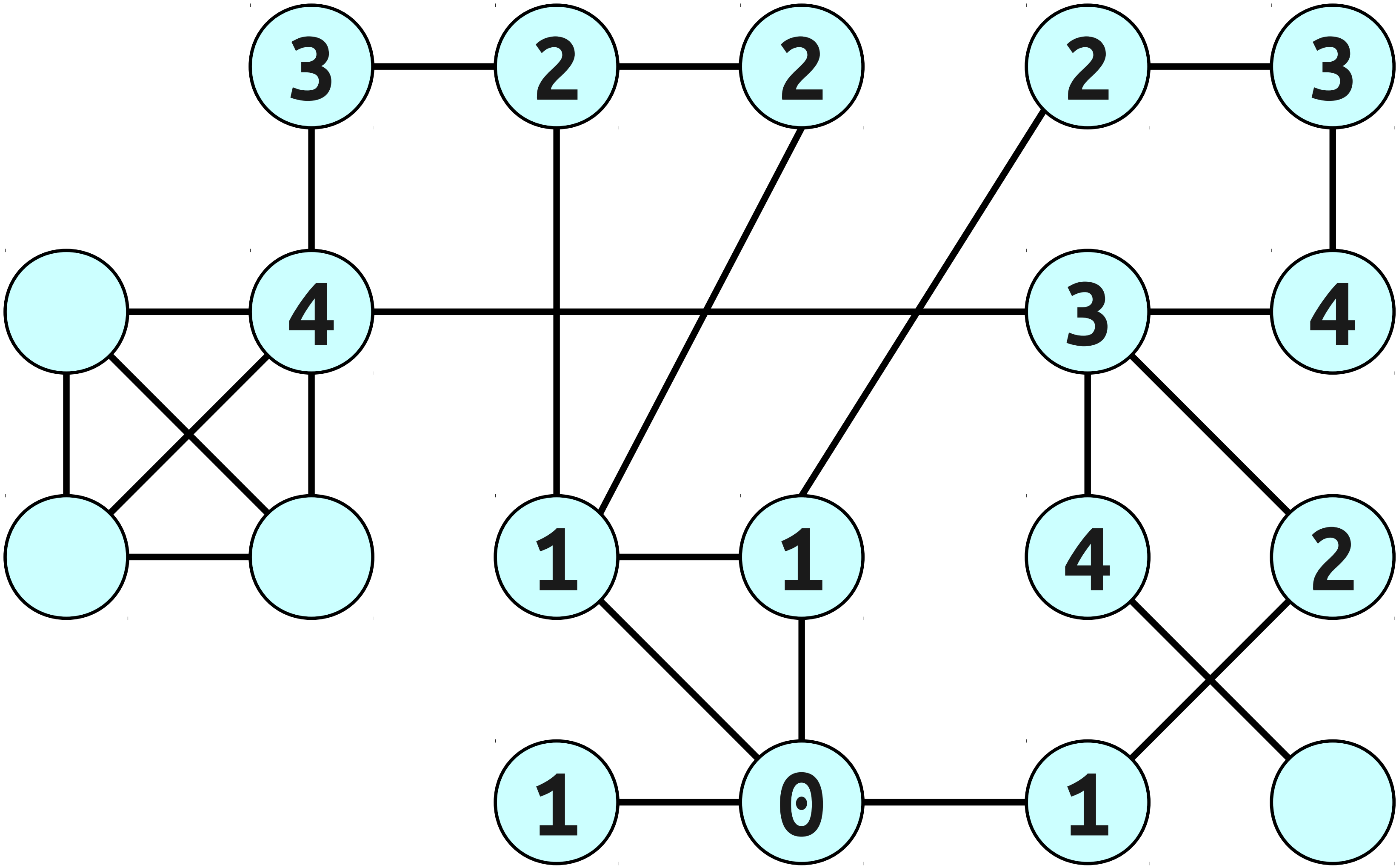

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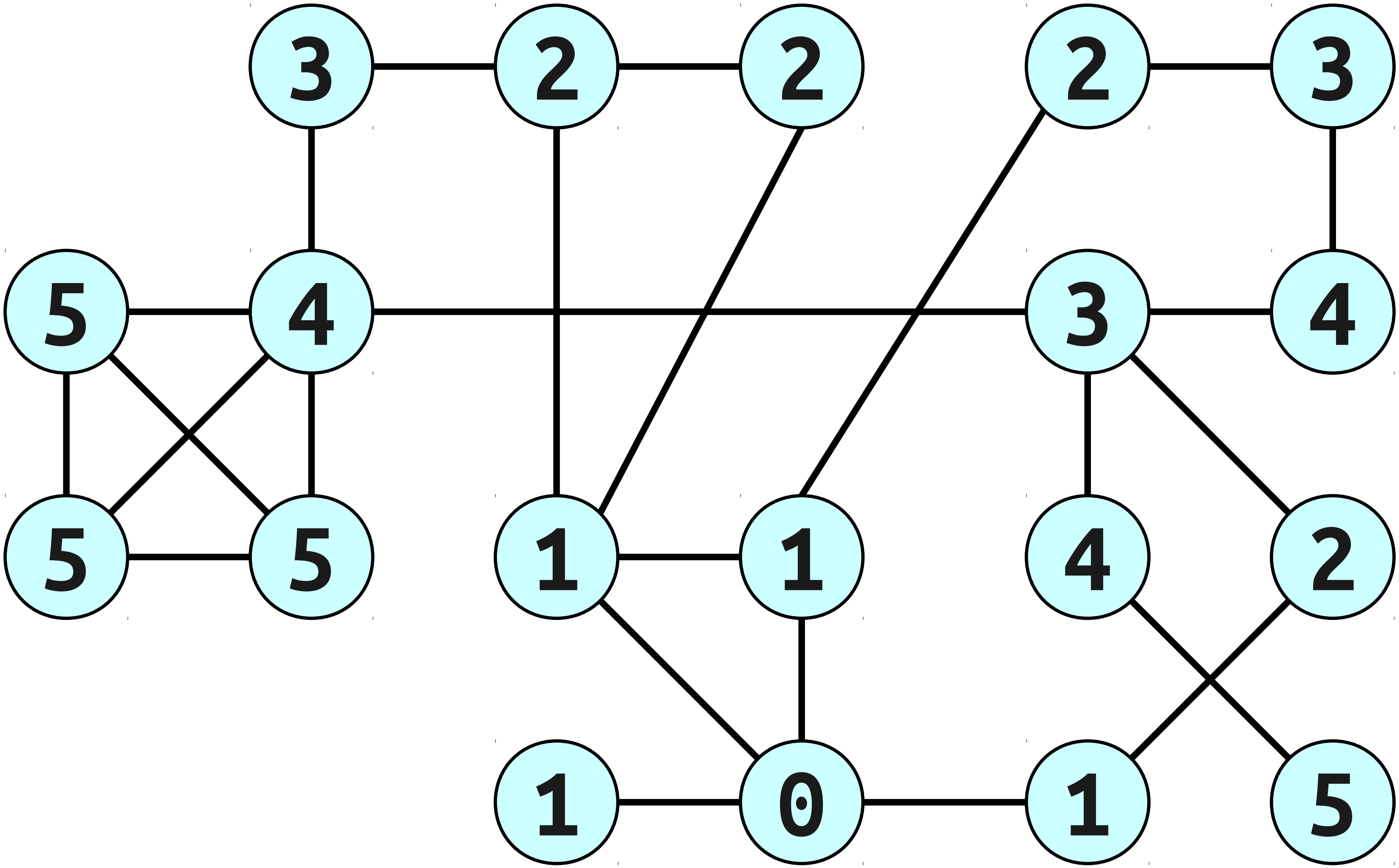

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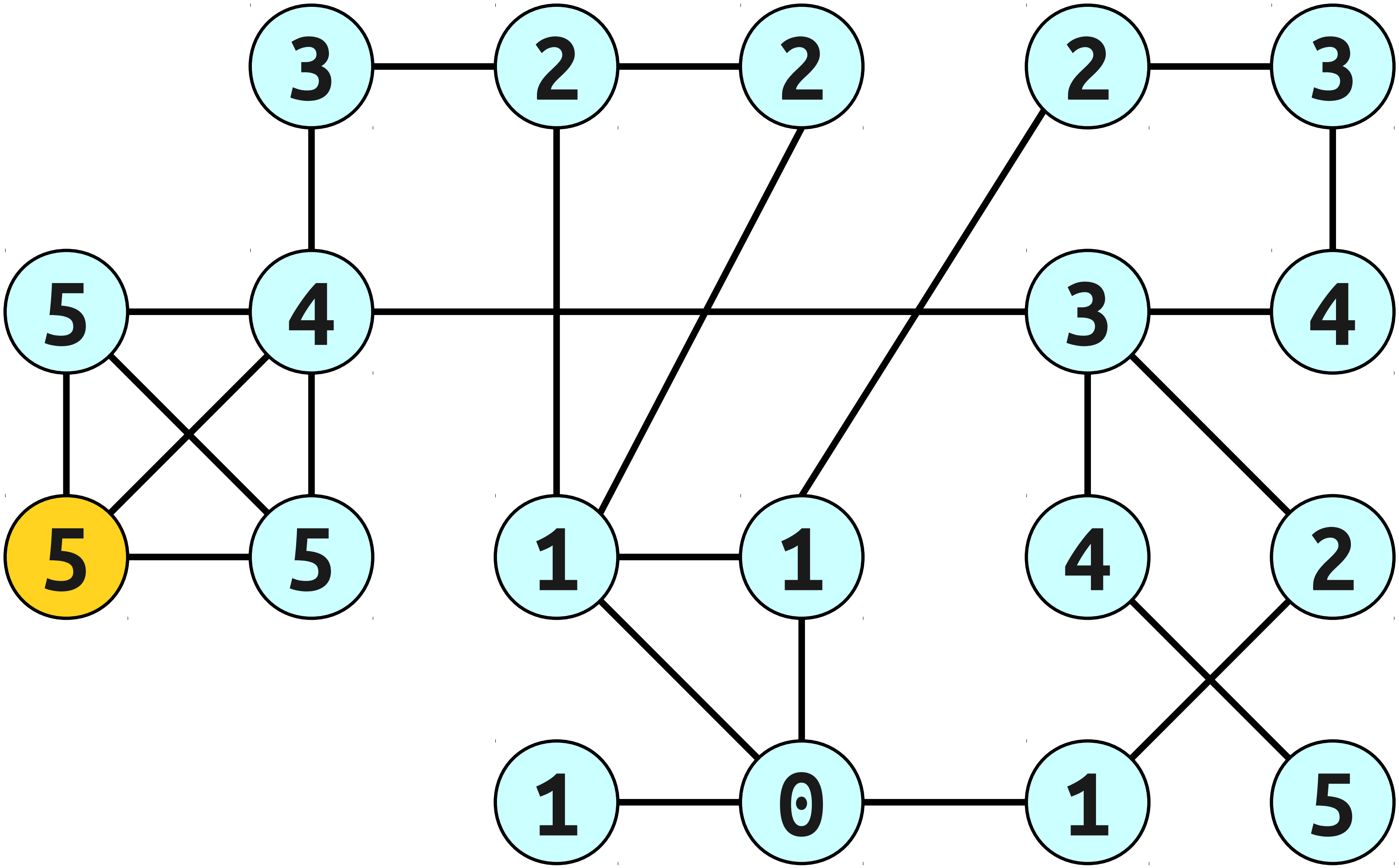

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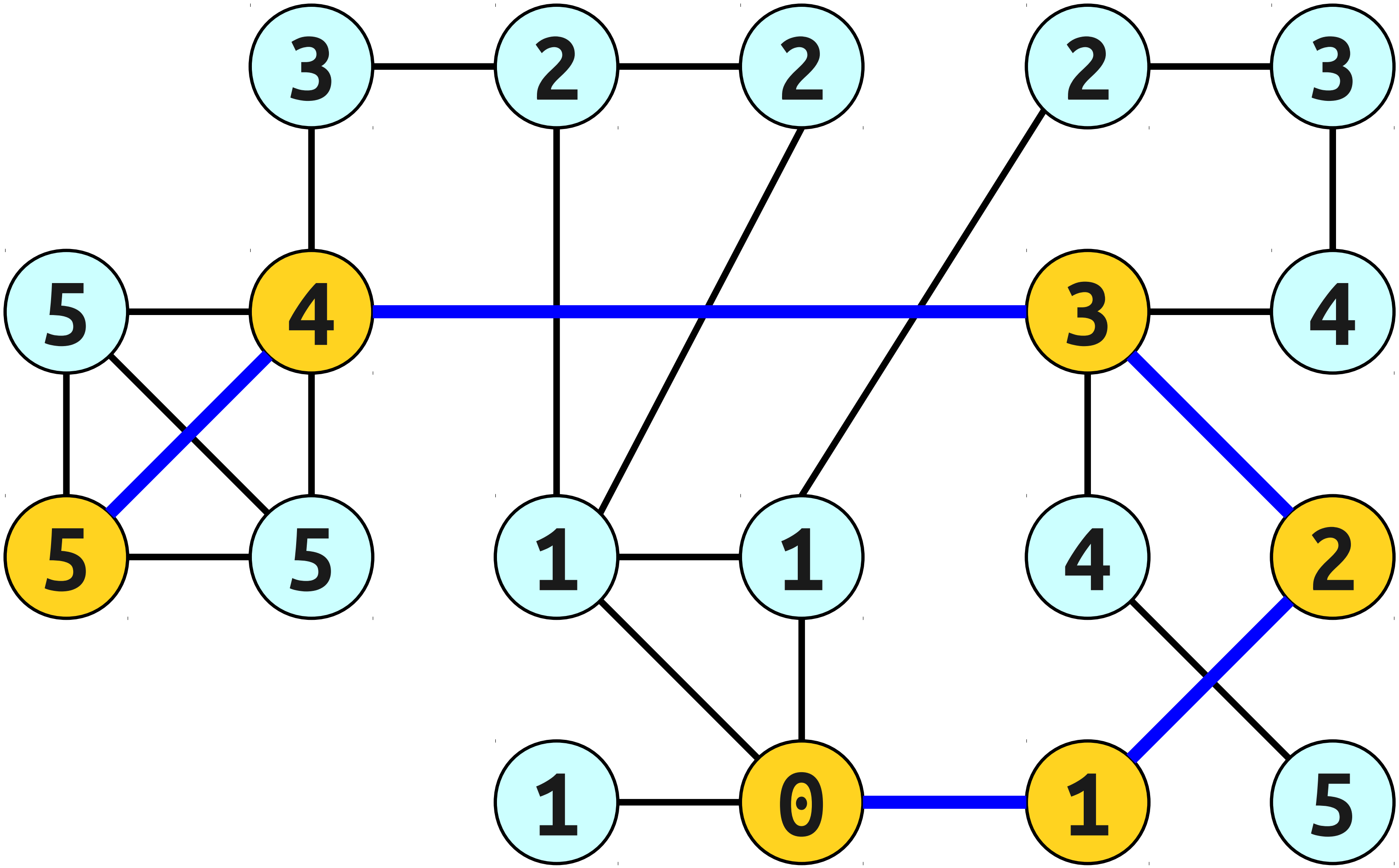

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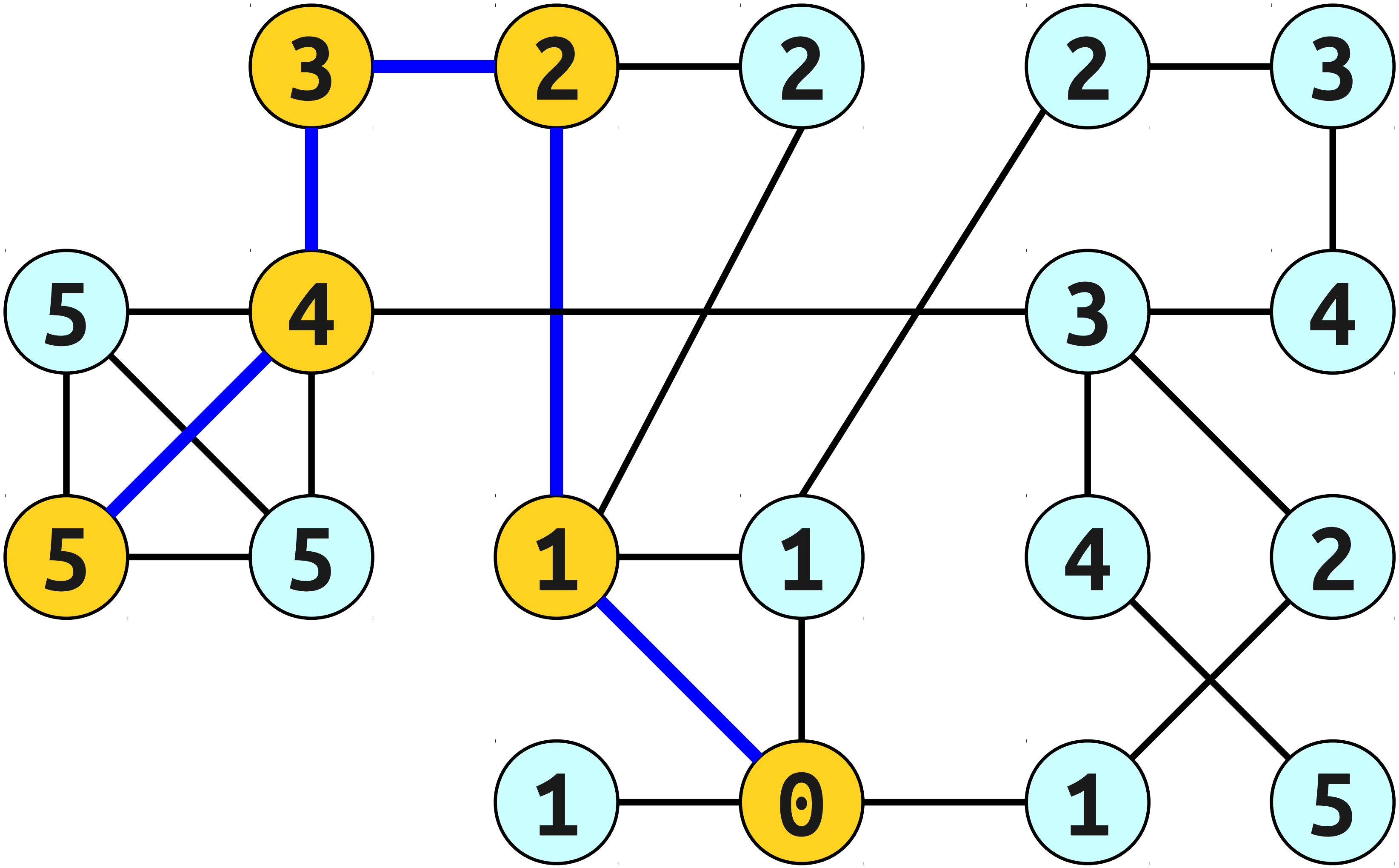

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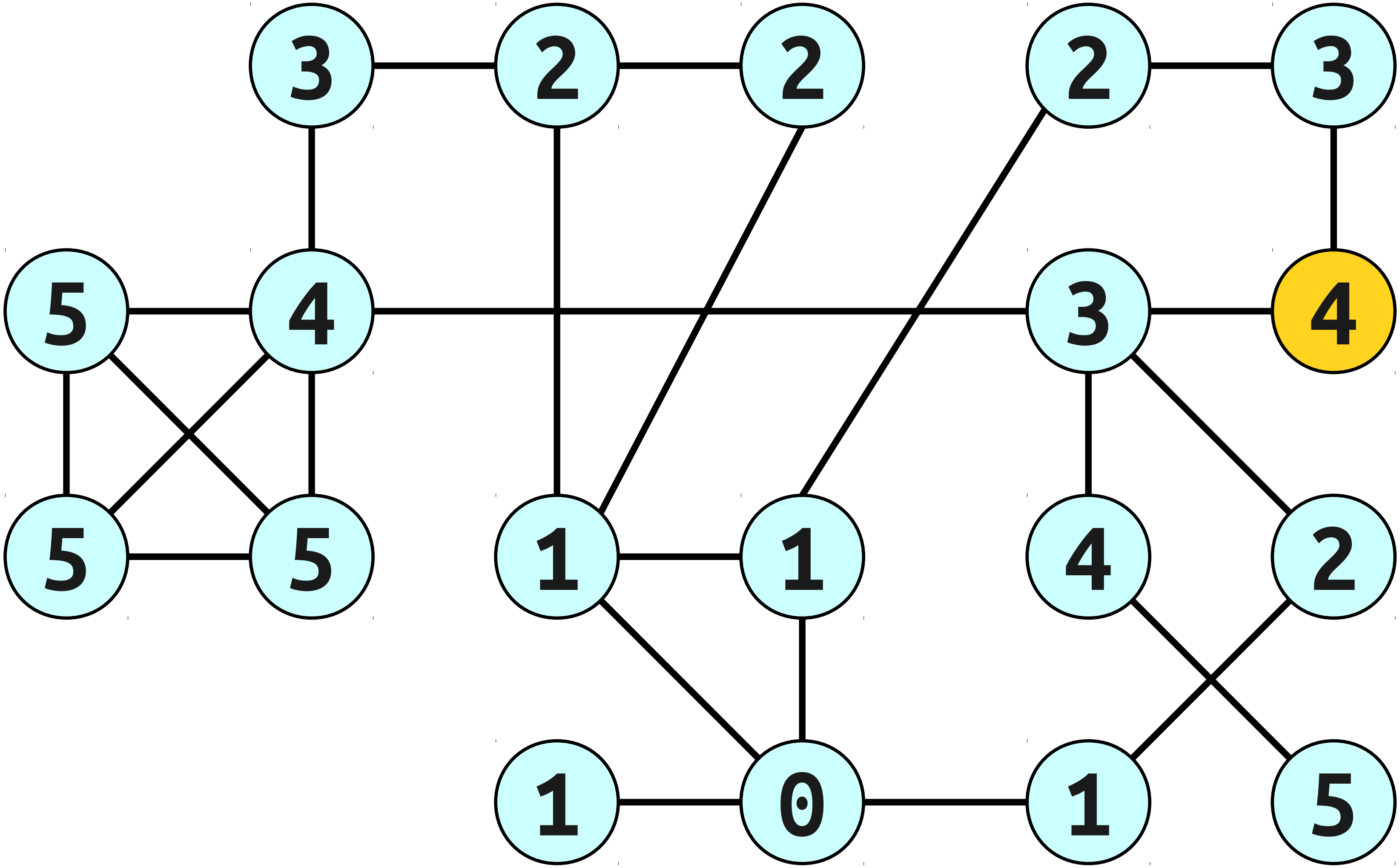

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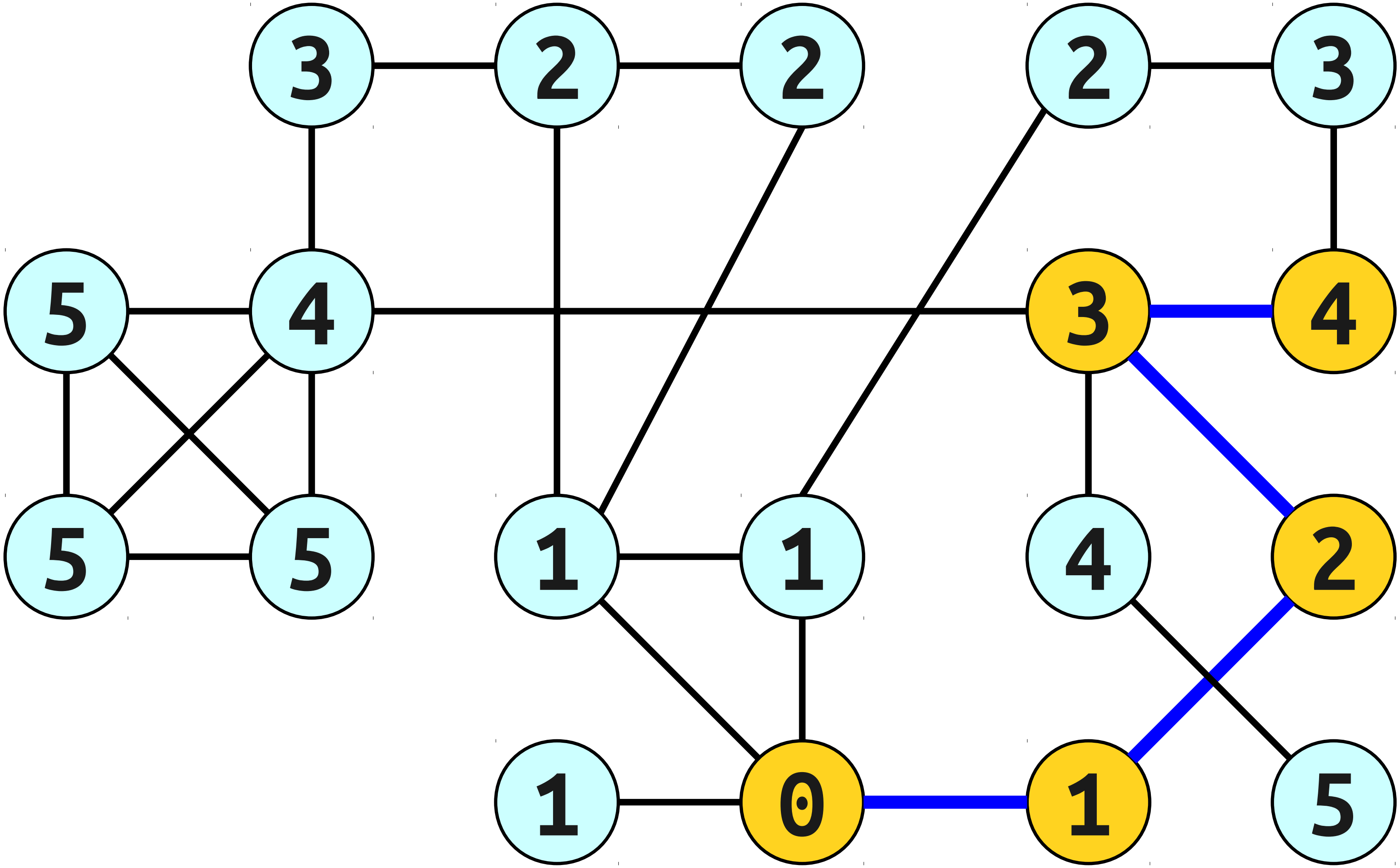

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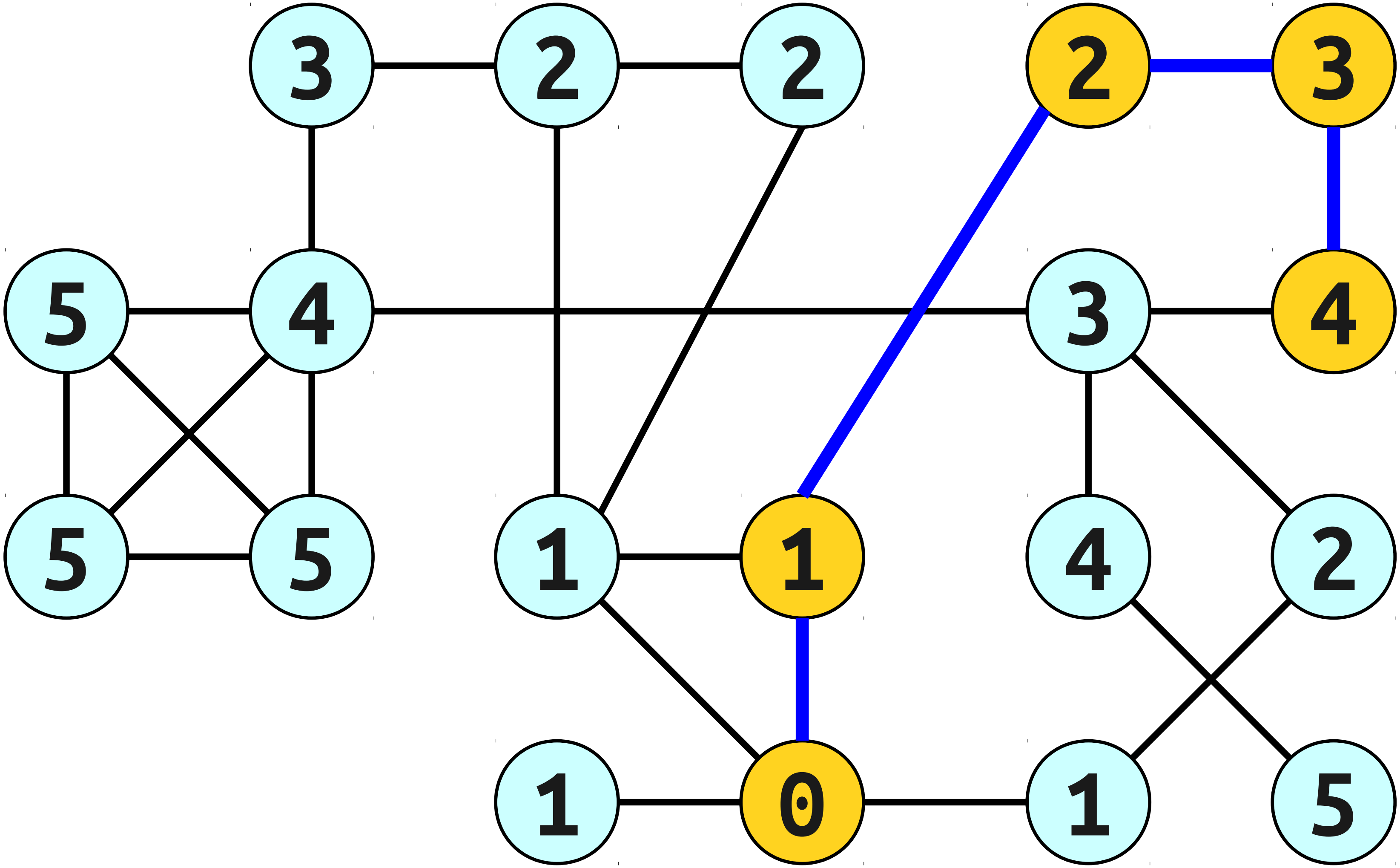


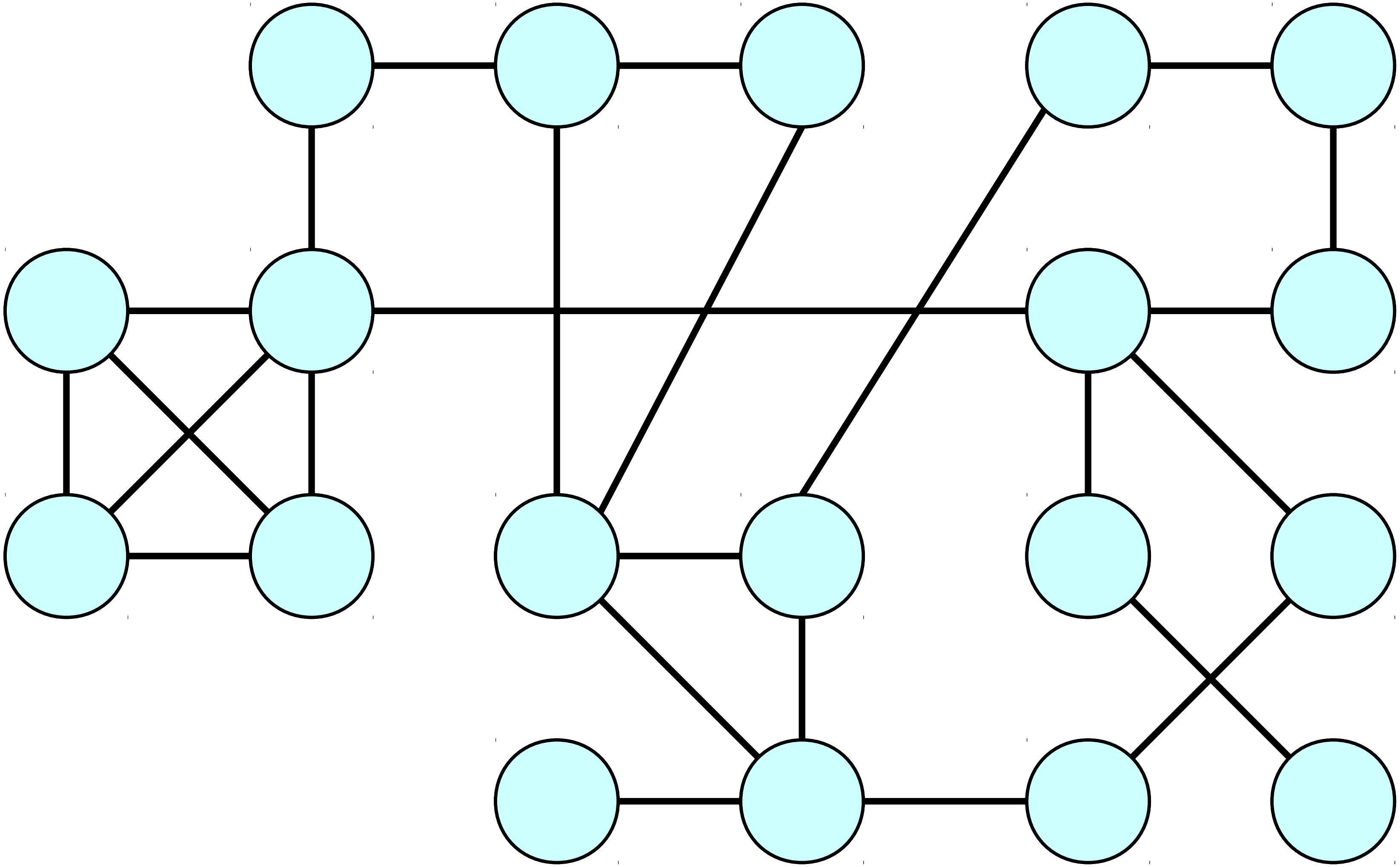
Shortest Paths

- The **length** of a path P (denoted $|P|$) in a graph is the number of edges it contains.
- A **shortest path** between u and v is a path P where $|P| \leq |P'|$ for any path P' from u to v.
- For any nodes u and v, define **d(*u*, *v*)** to be the length of the shortest path from u to v, or ∞ if no such path exists.
- What is d(v, v) for any $v \in V$?

The Shortest Path Problem

- **Input:**
  - A graph $G = (V, E)$, which may be directed or undirected.
  - A start node $s \in V$.
- **Output:**
  - A table dist[v], where dist[v] = d(s, v) for any $v \in V$.

An Inefficient Algorithm



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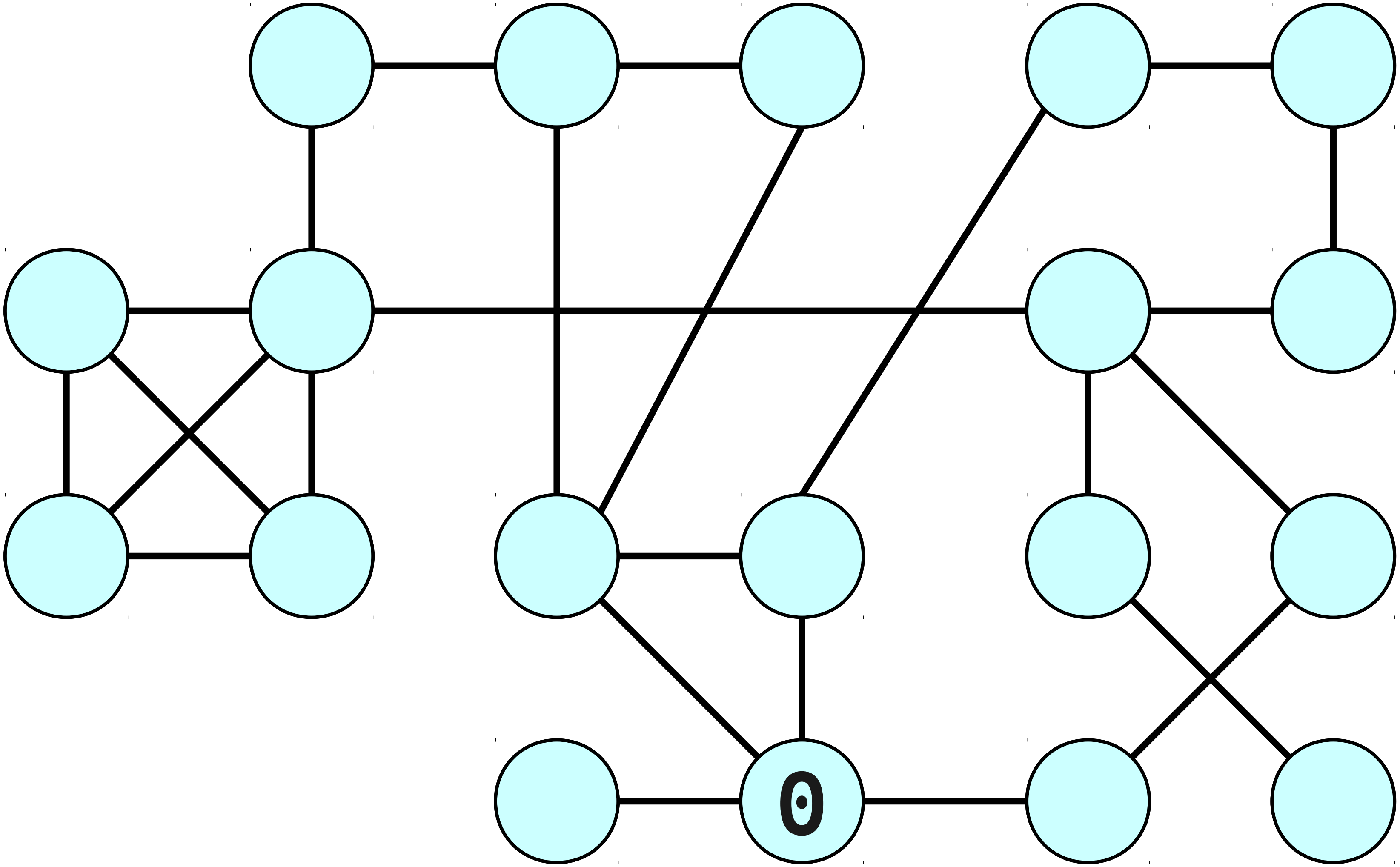

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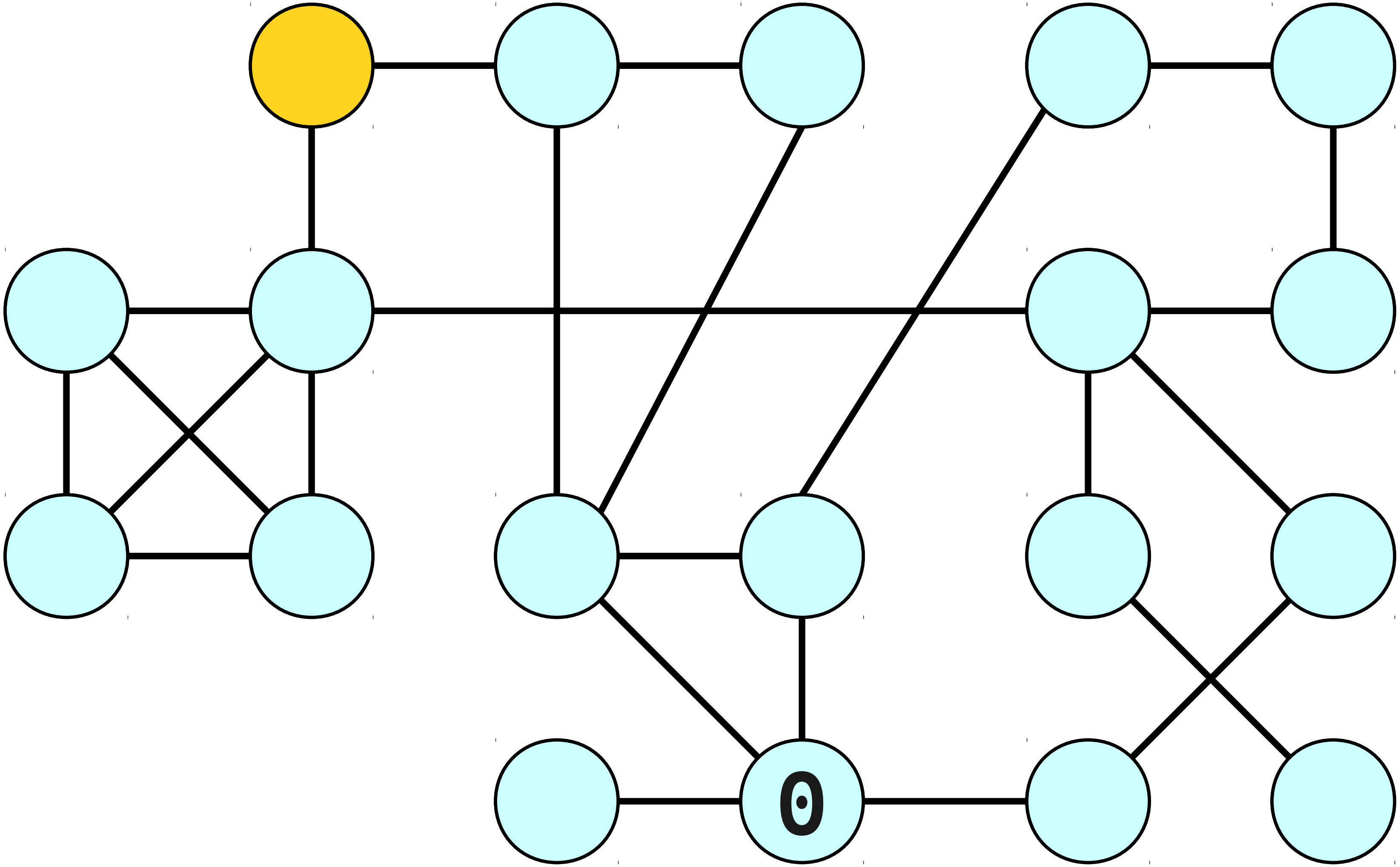

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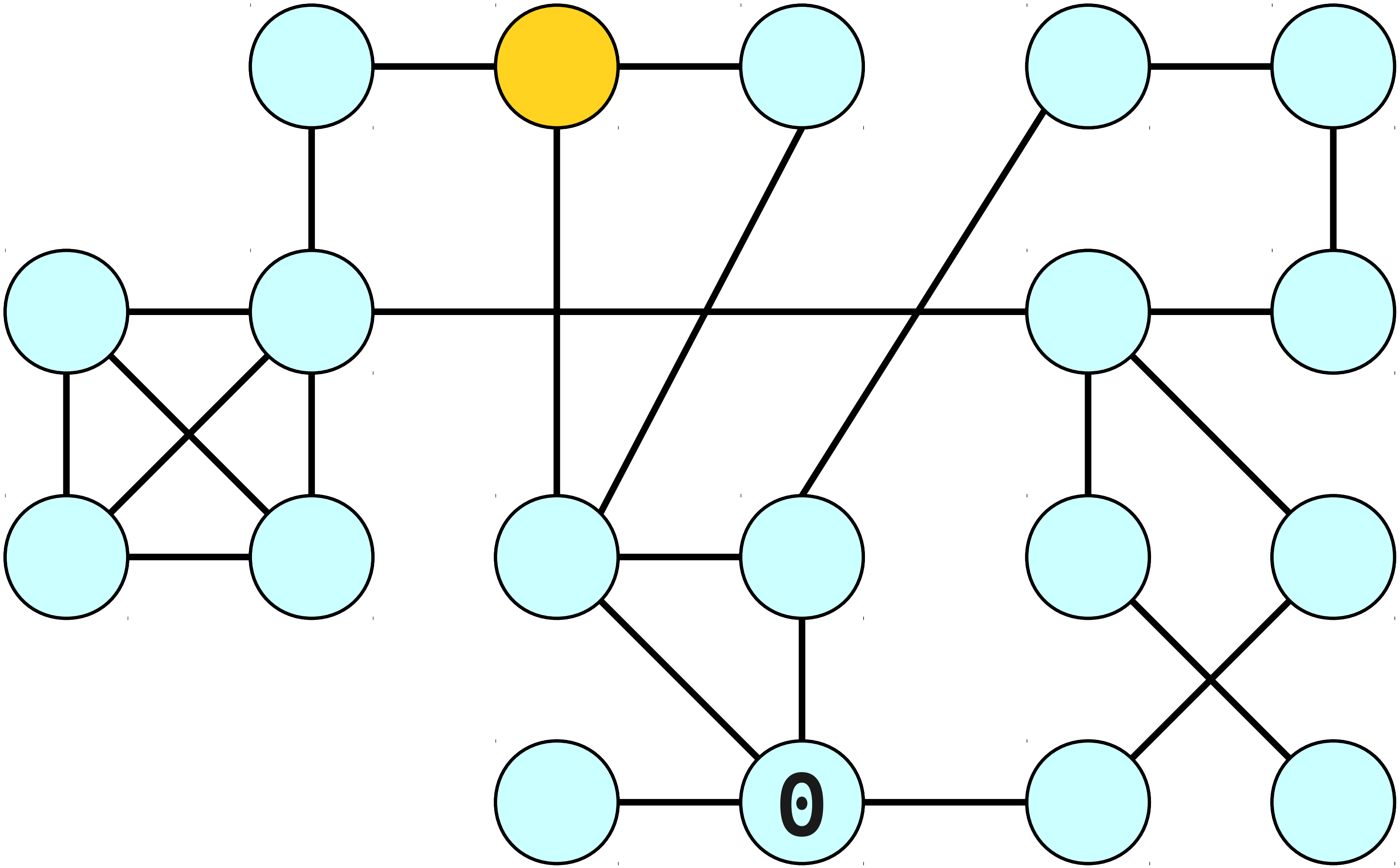

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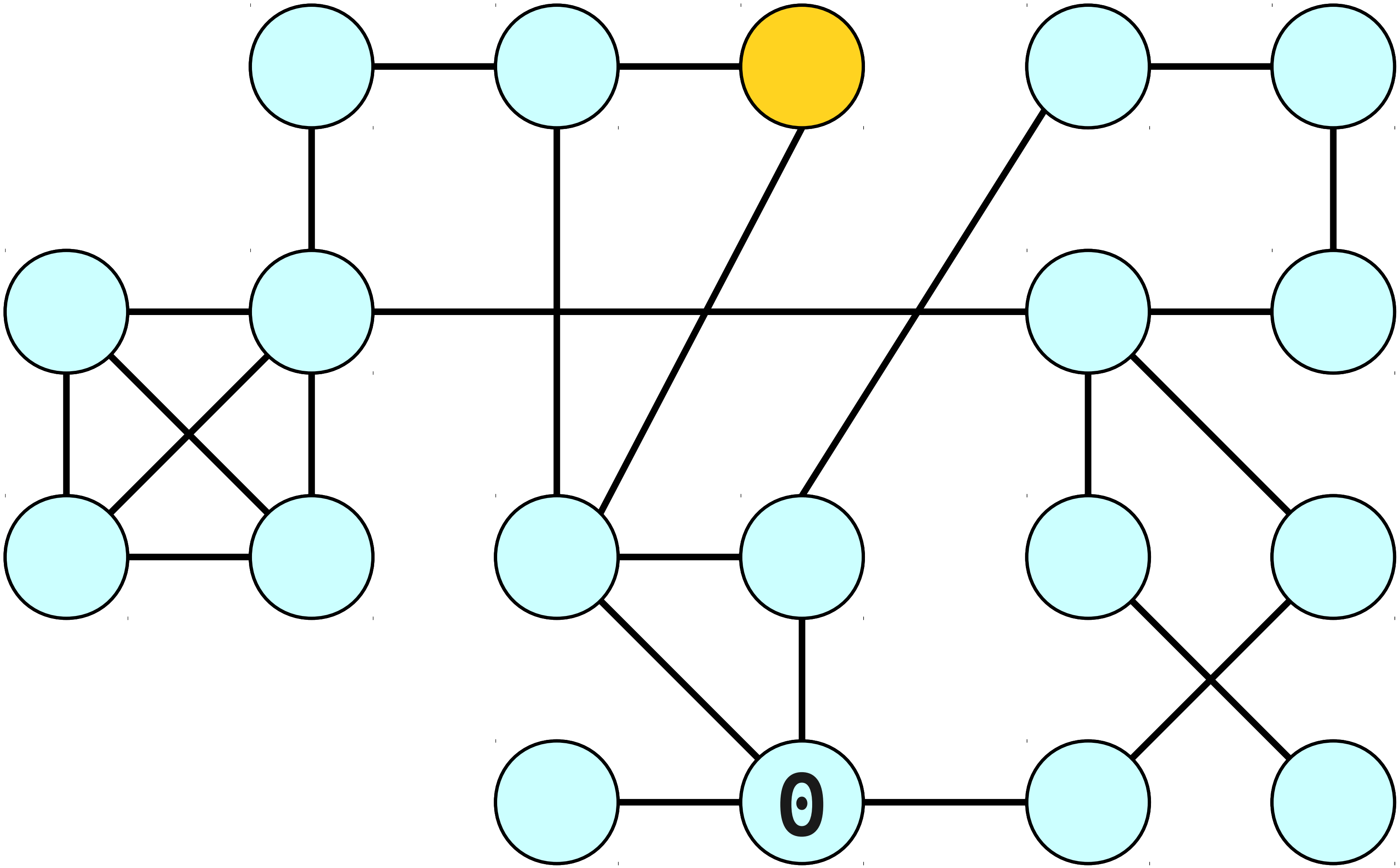

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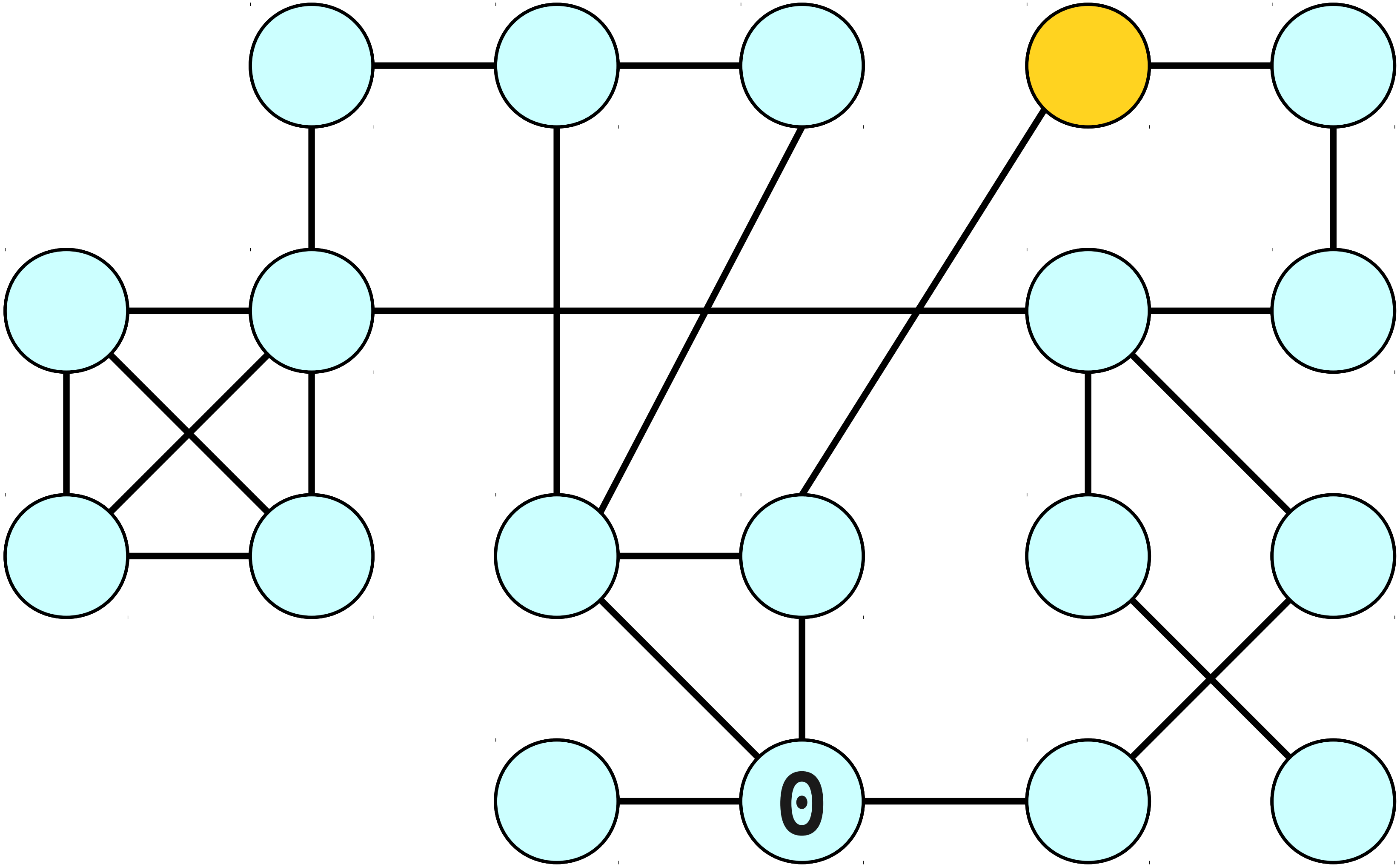

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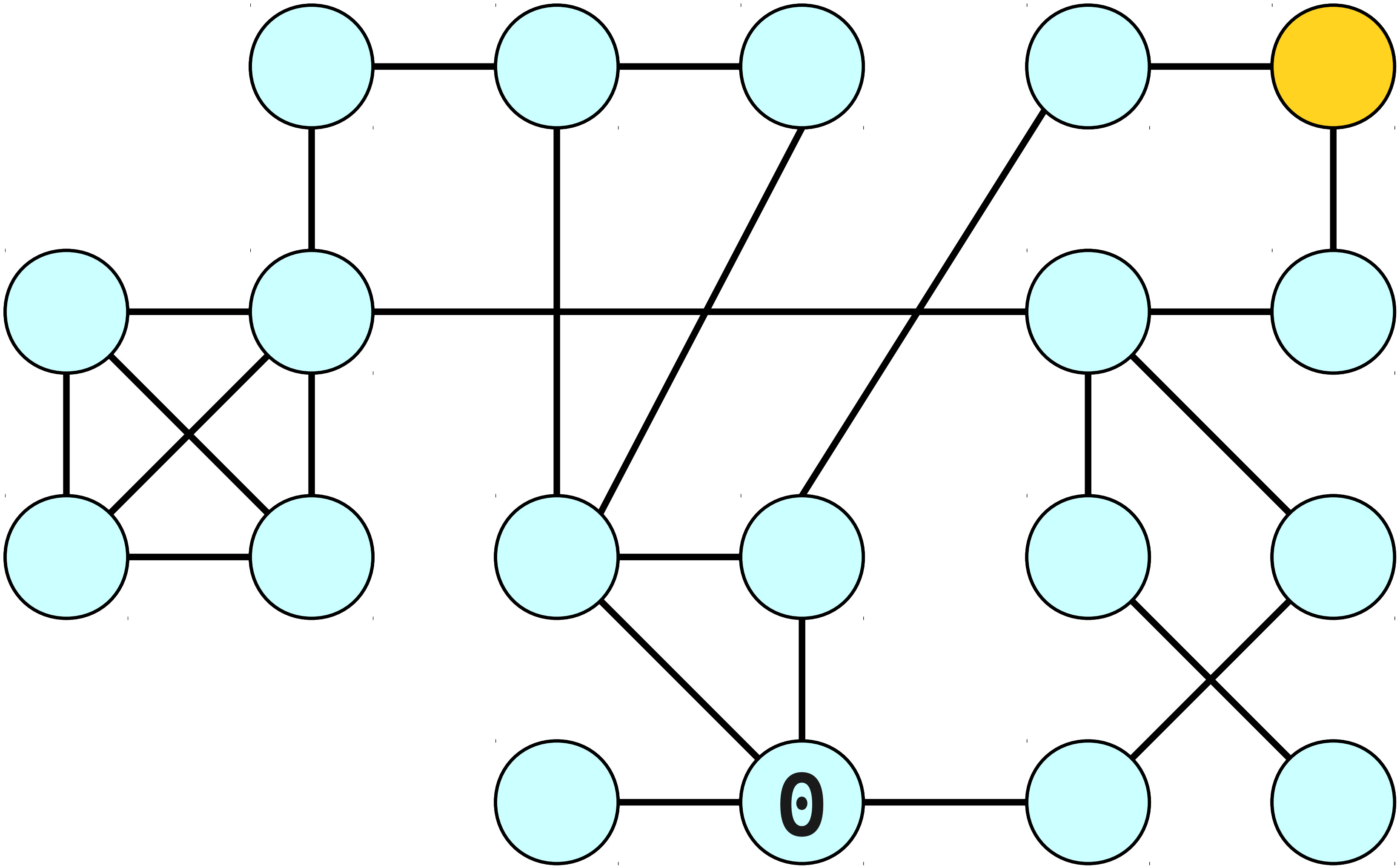

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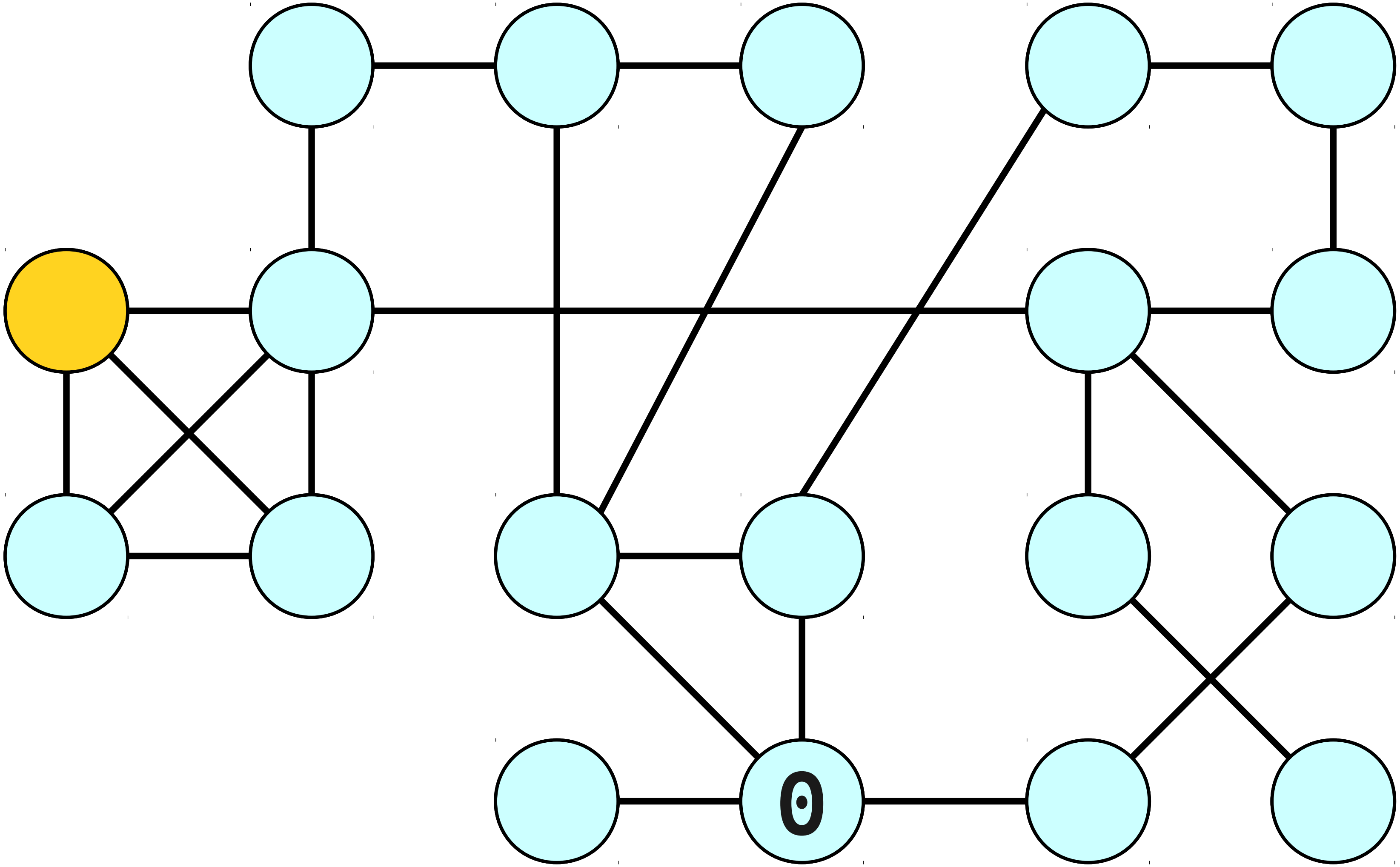

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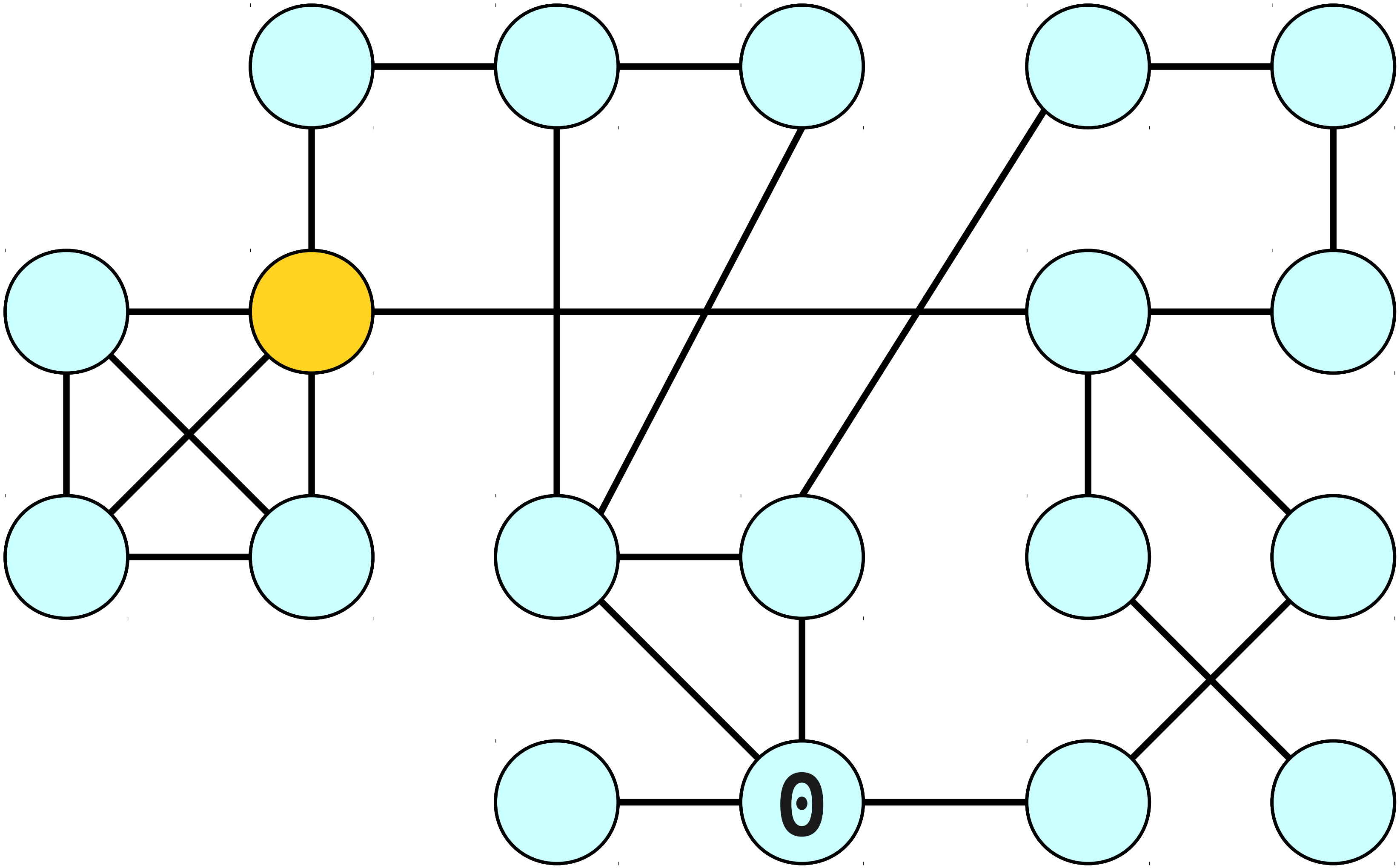

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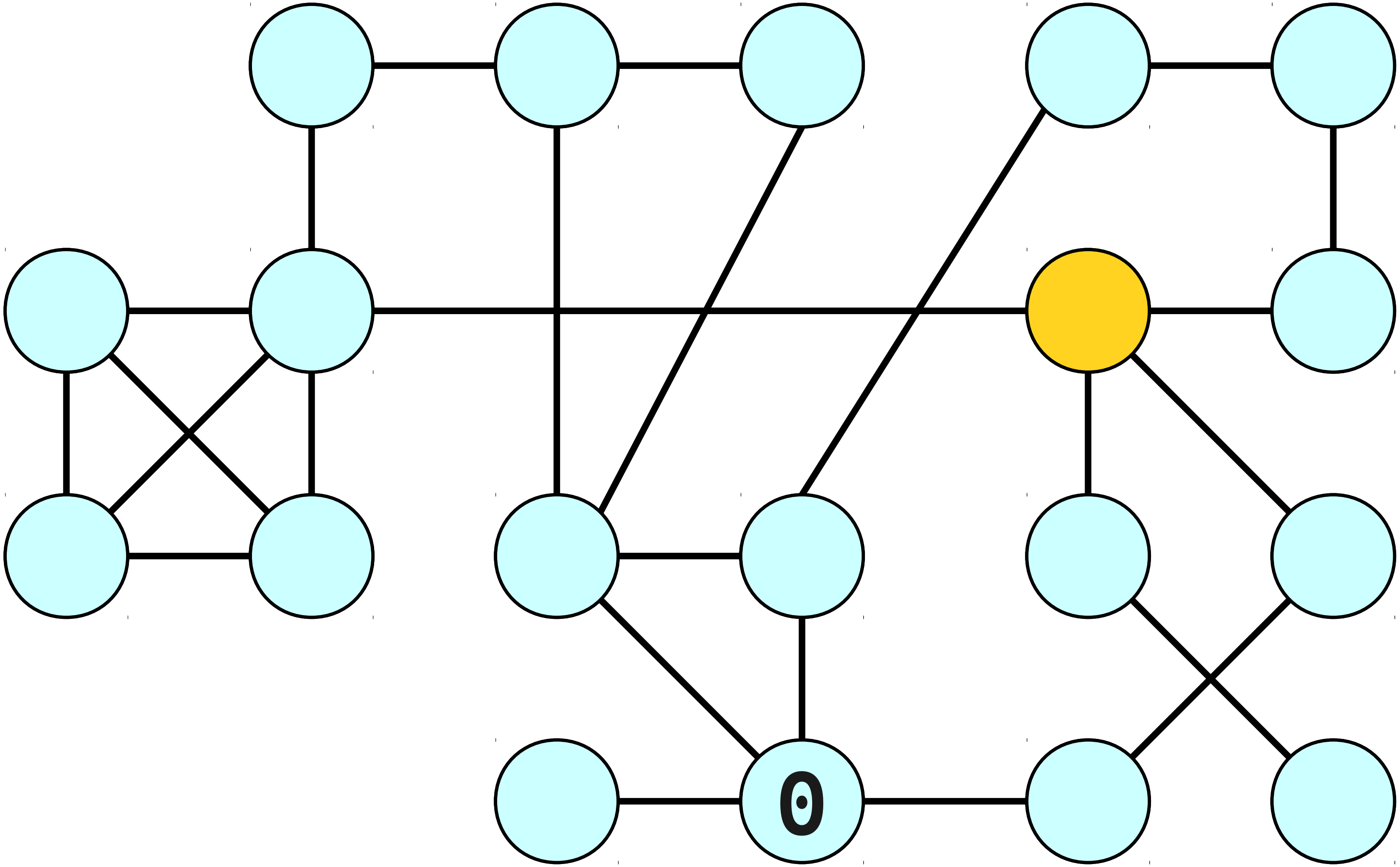

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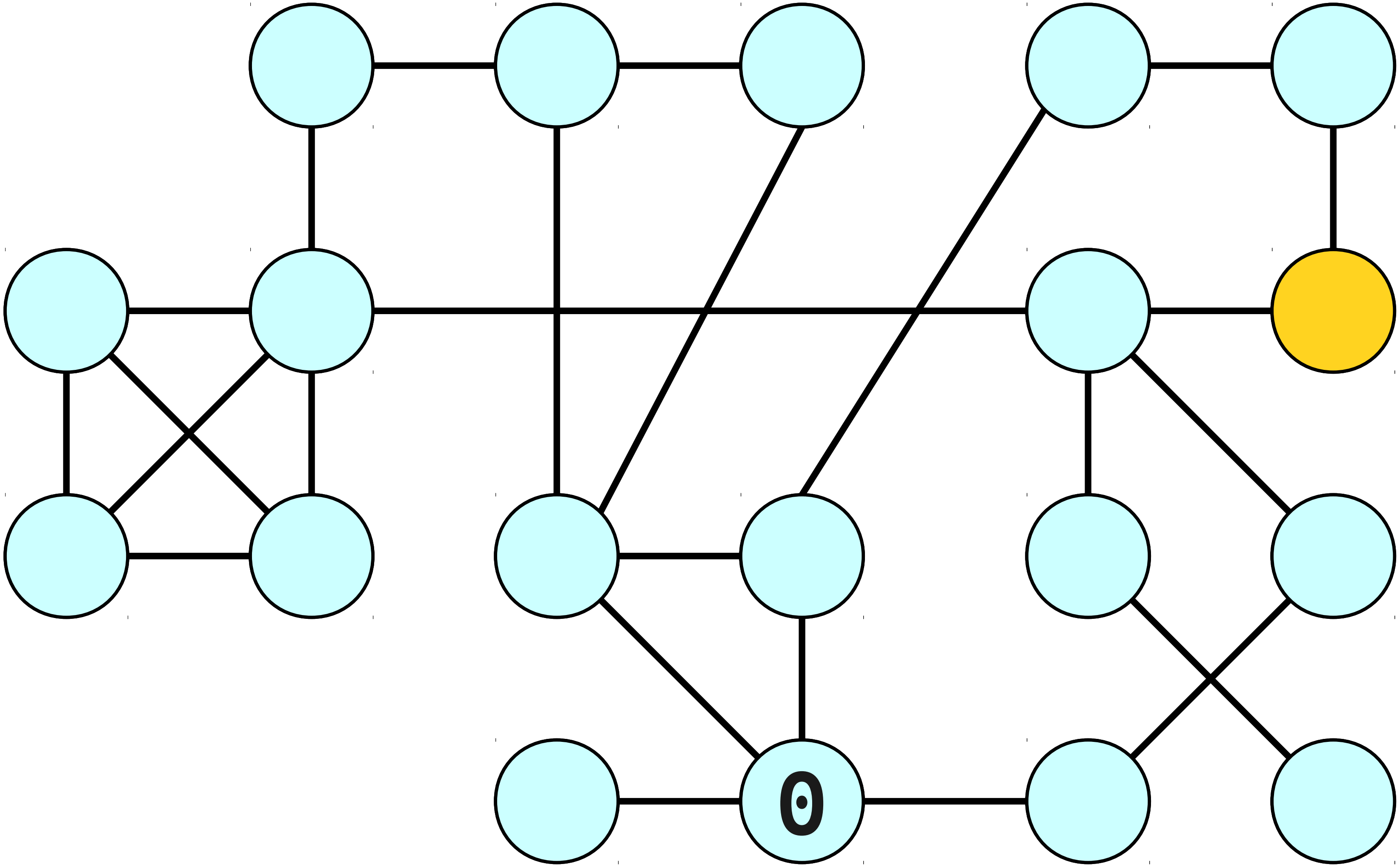

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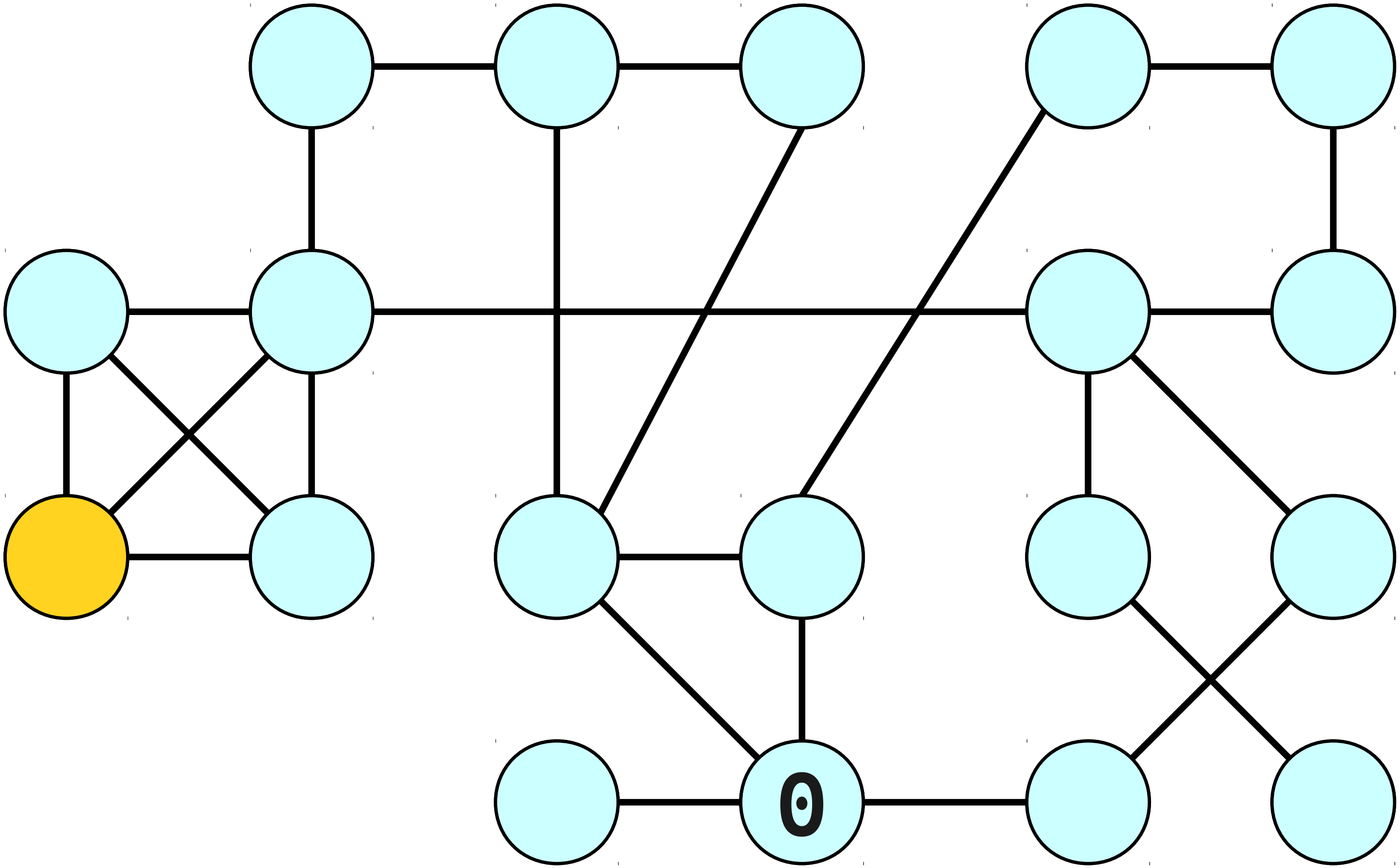

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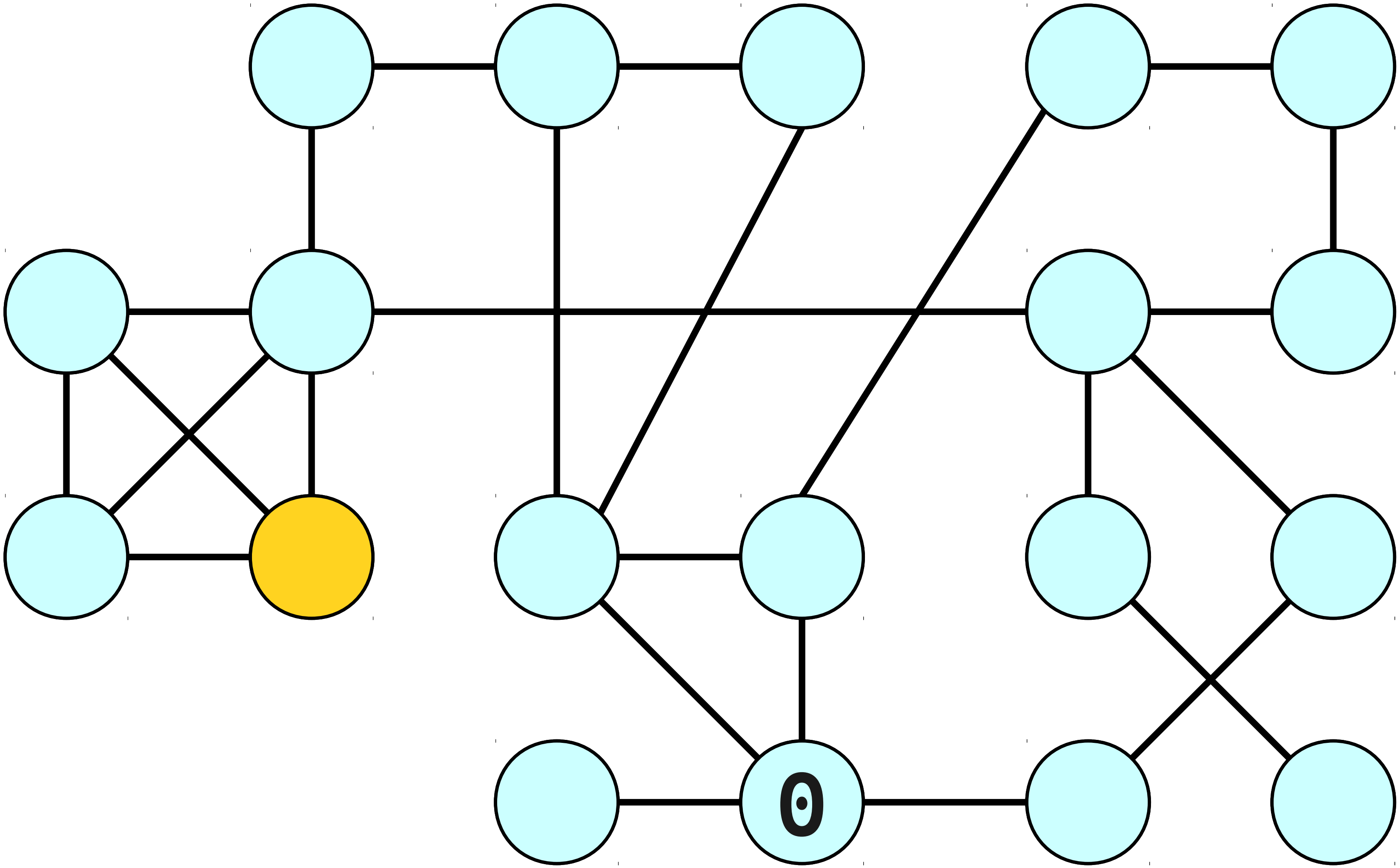

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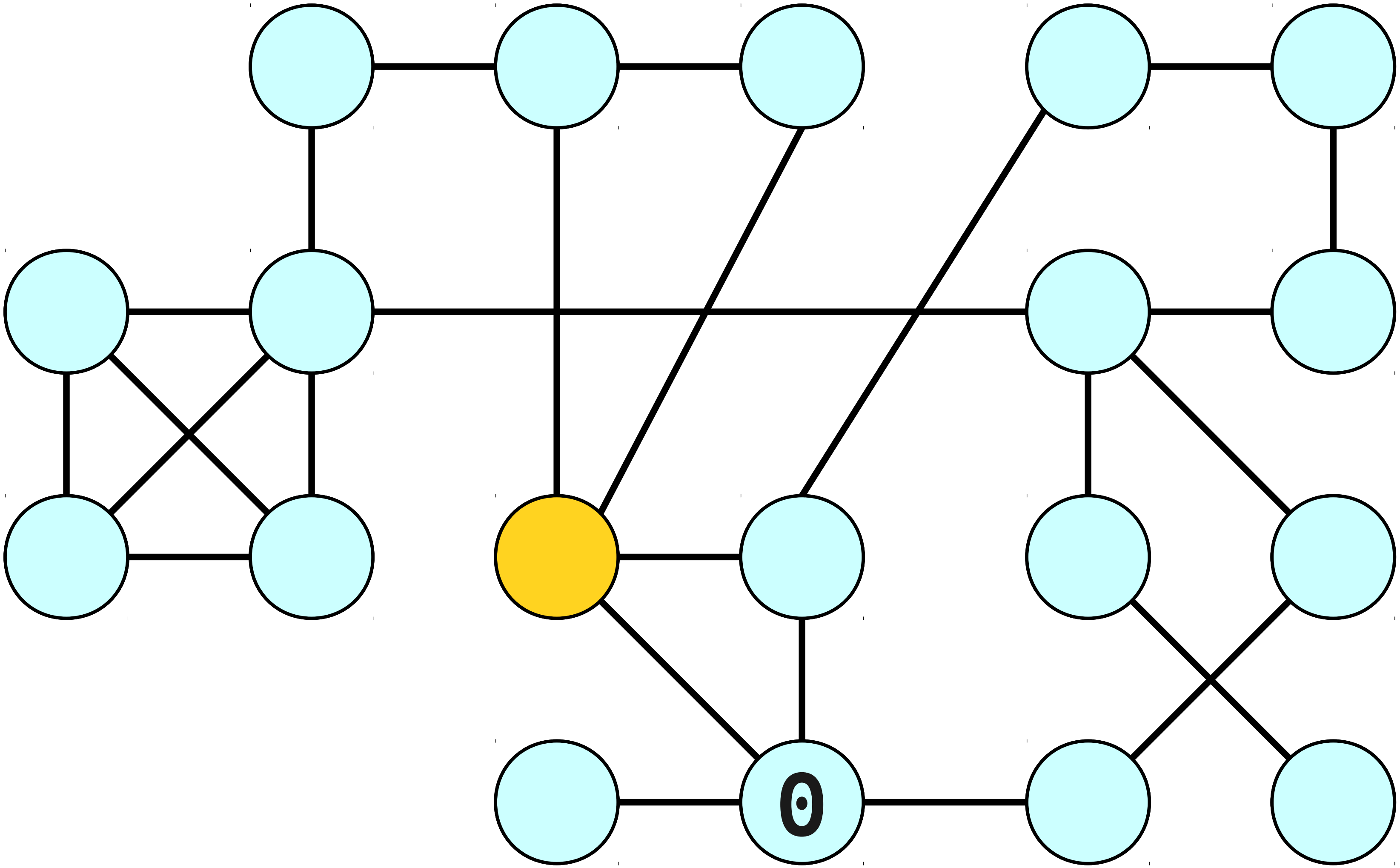

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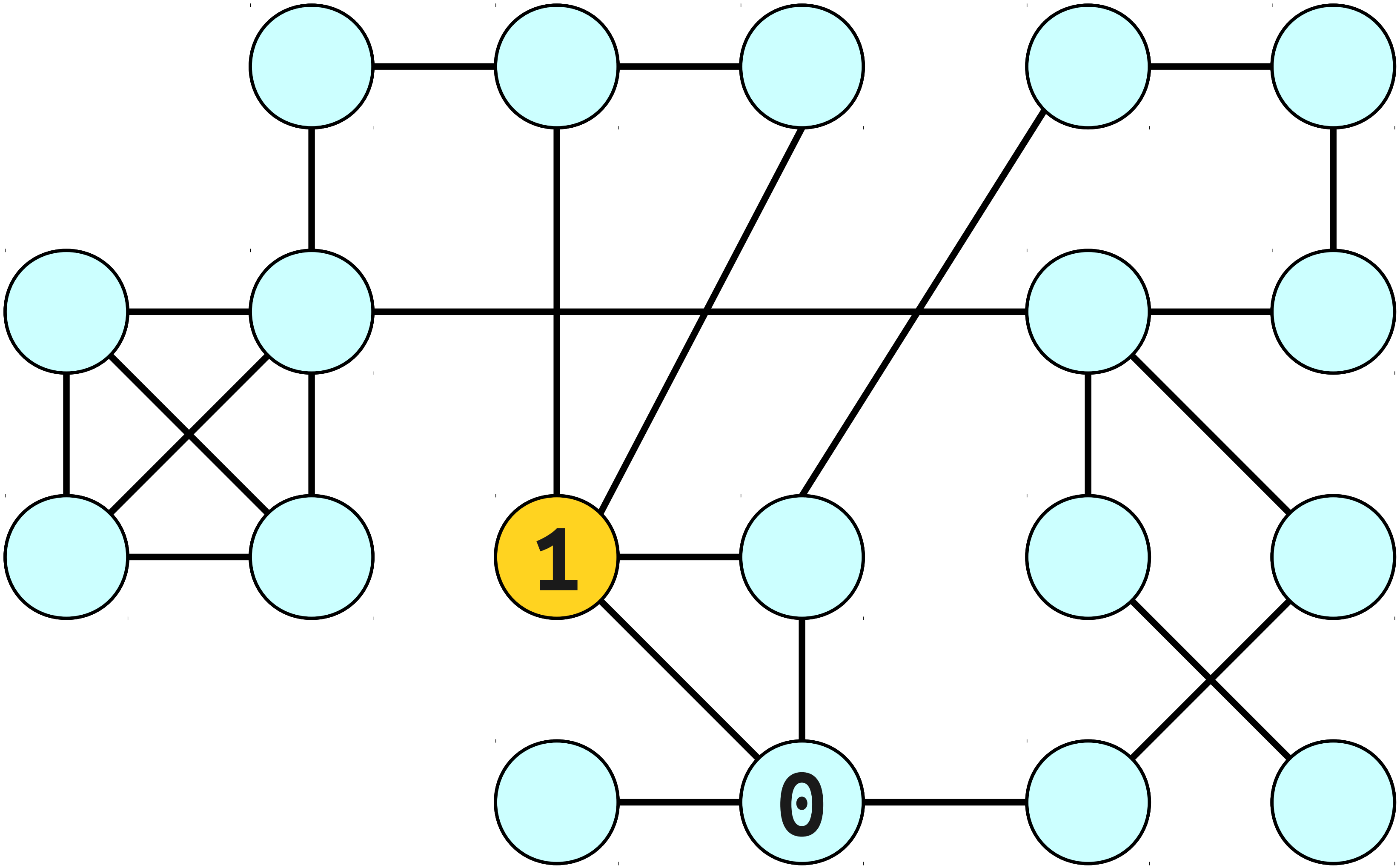

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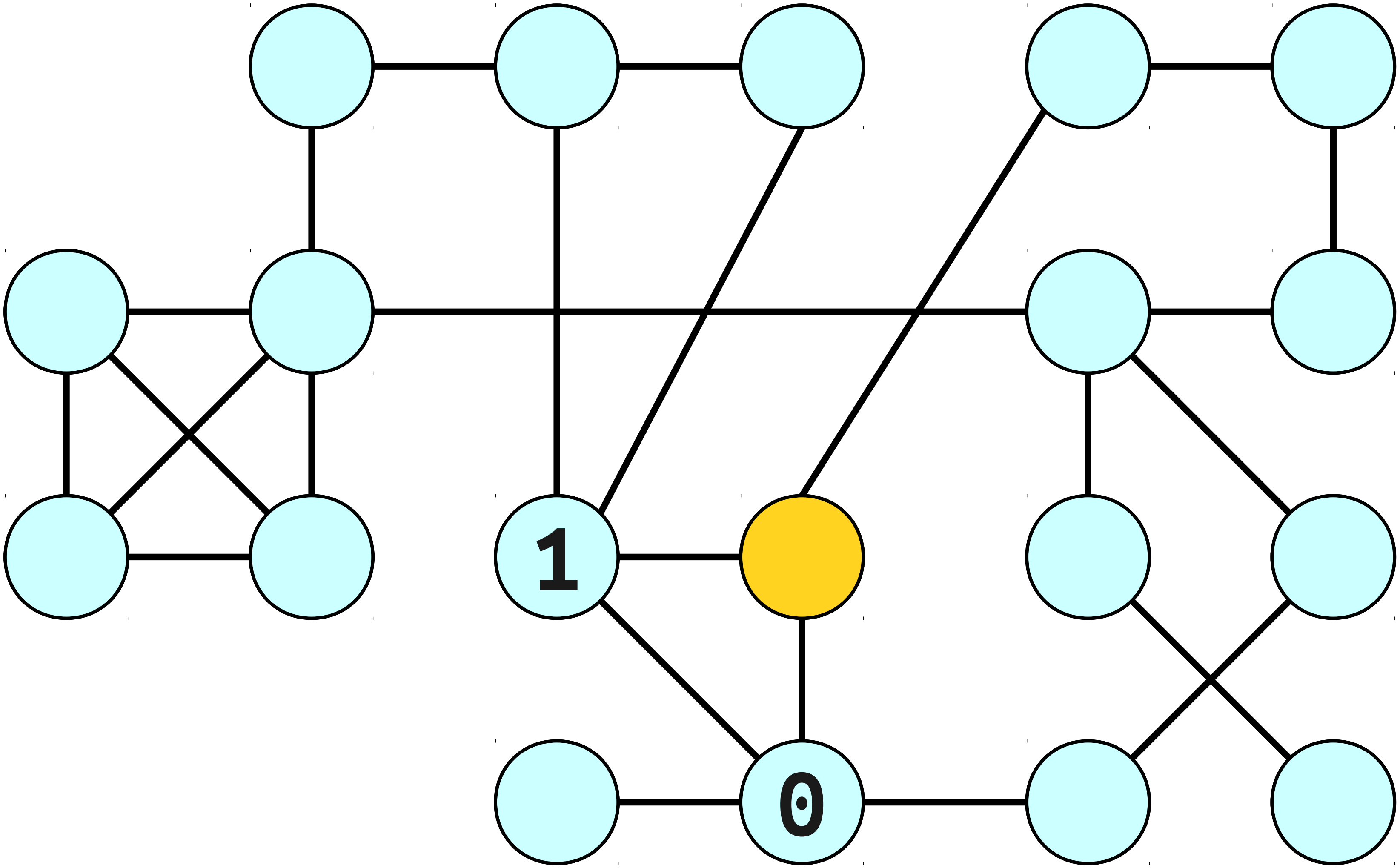

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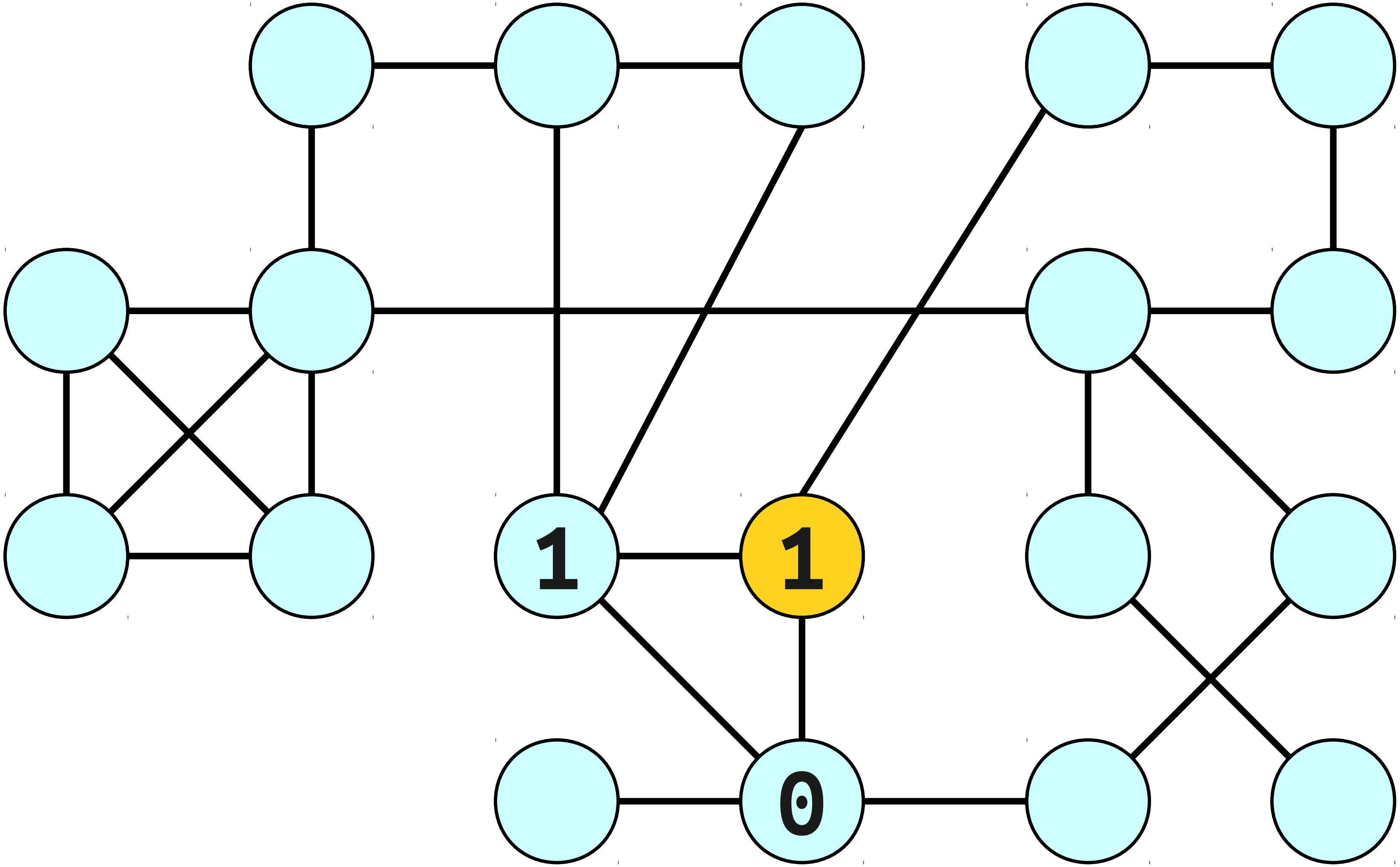

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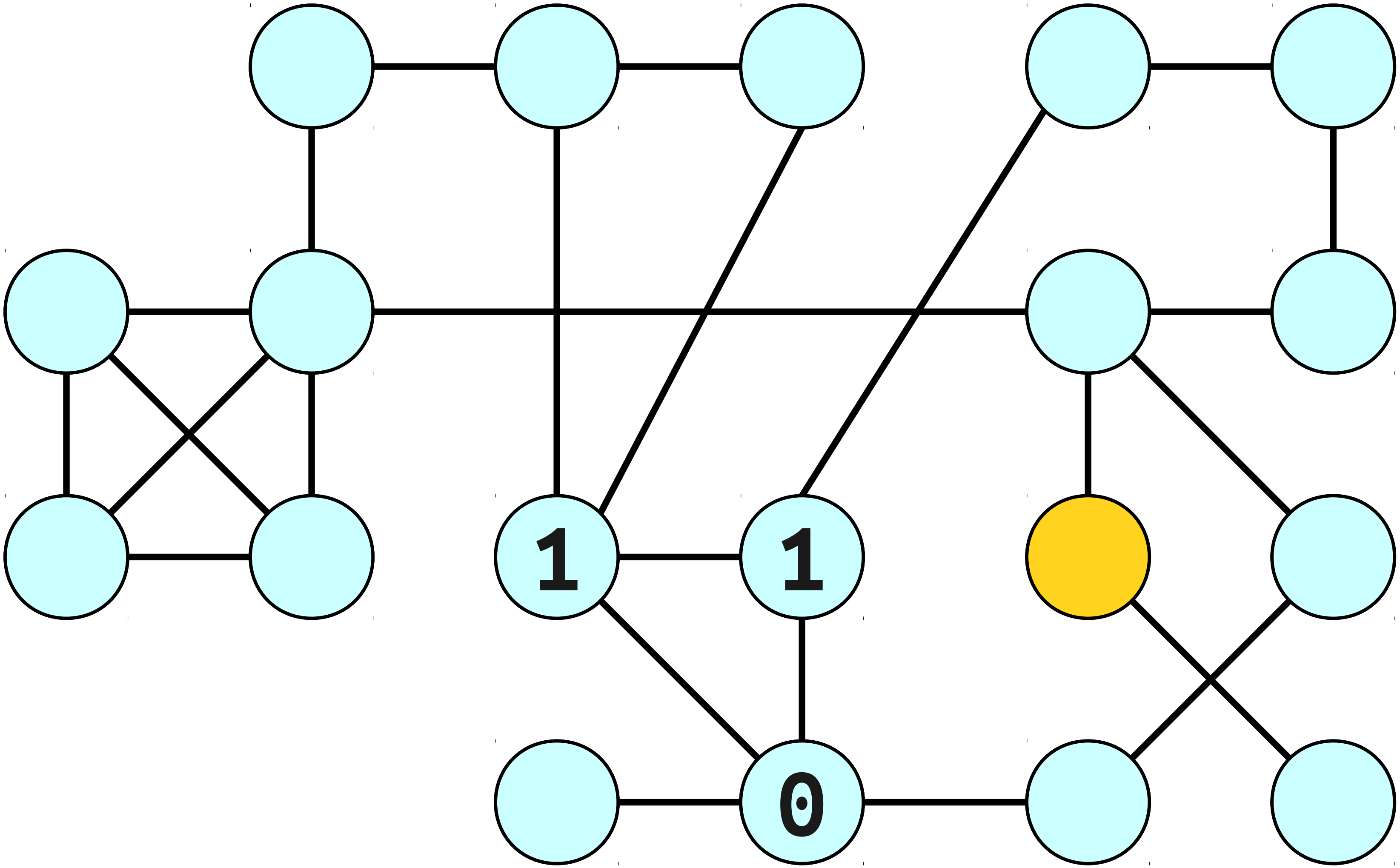

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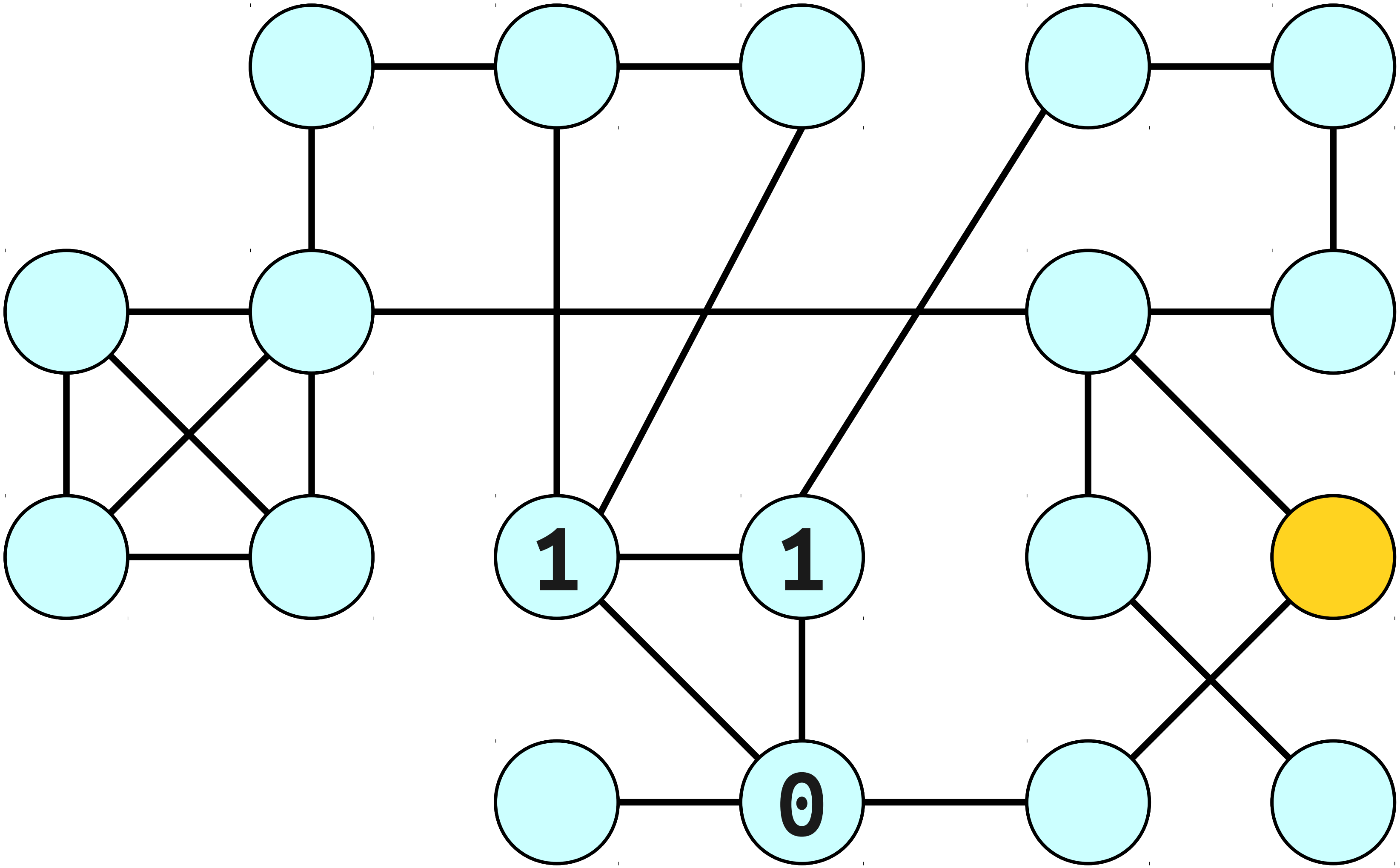

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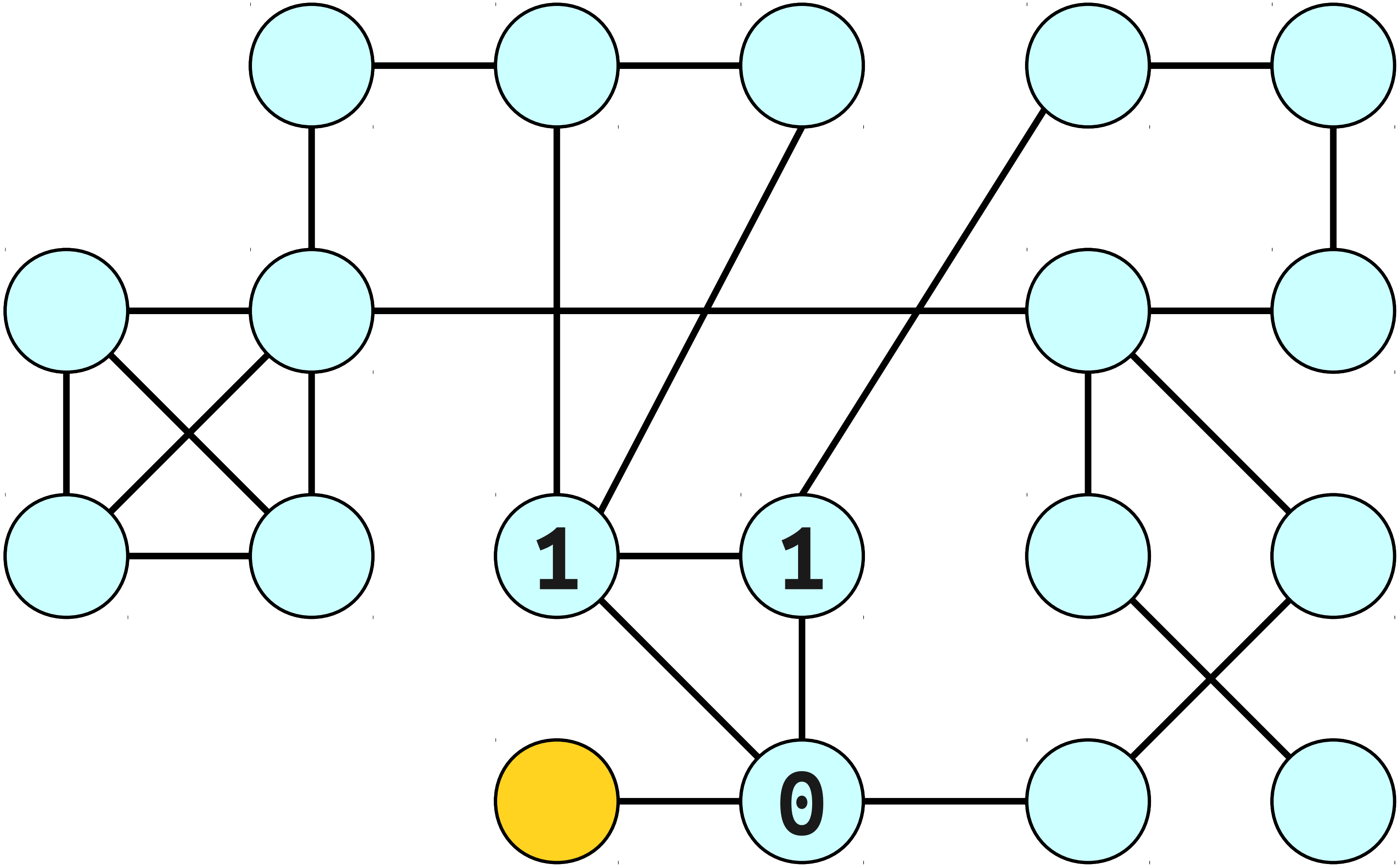

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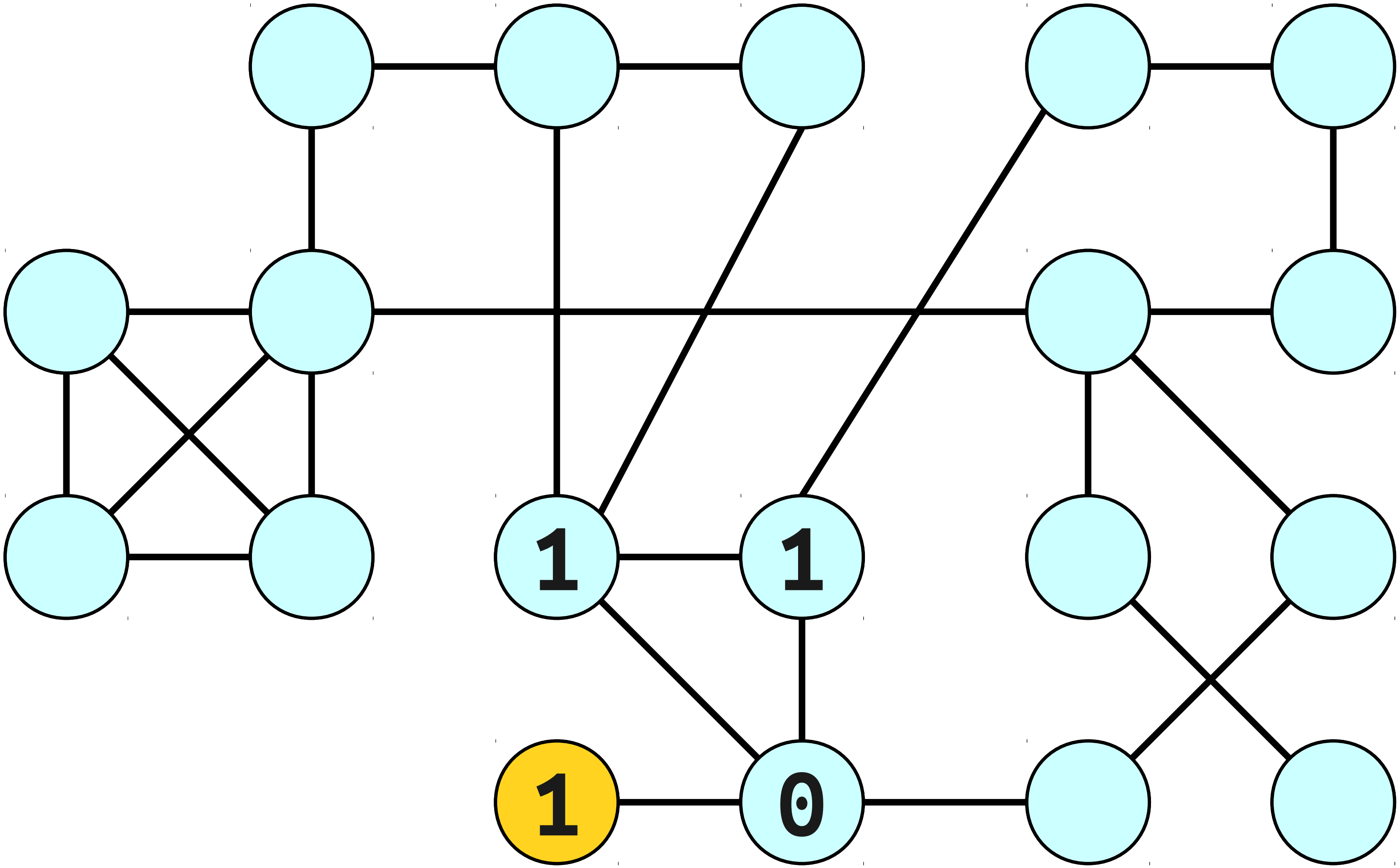

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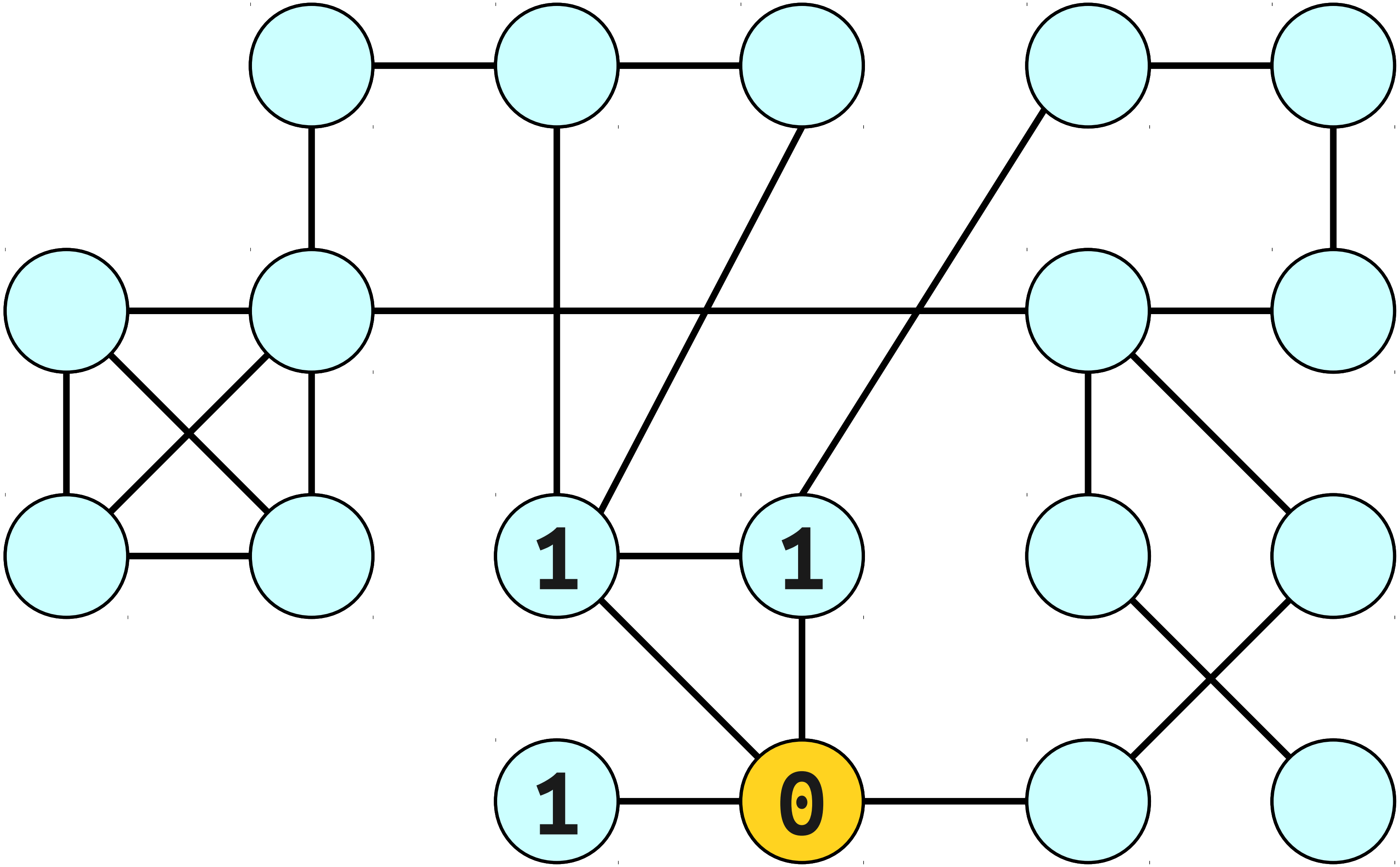

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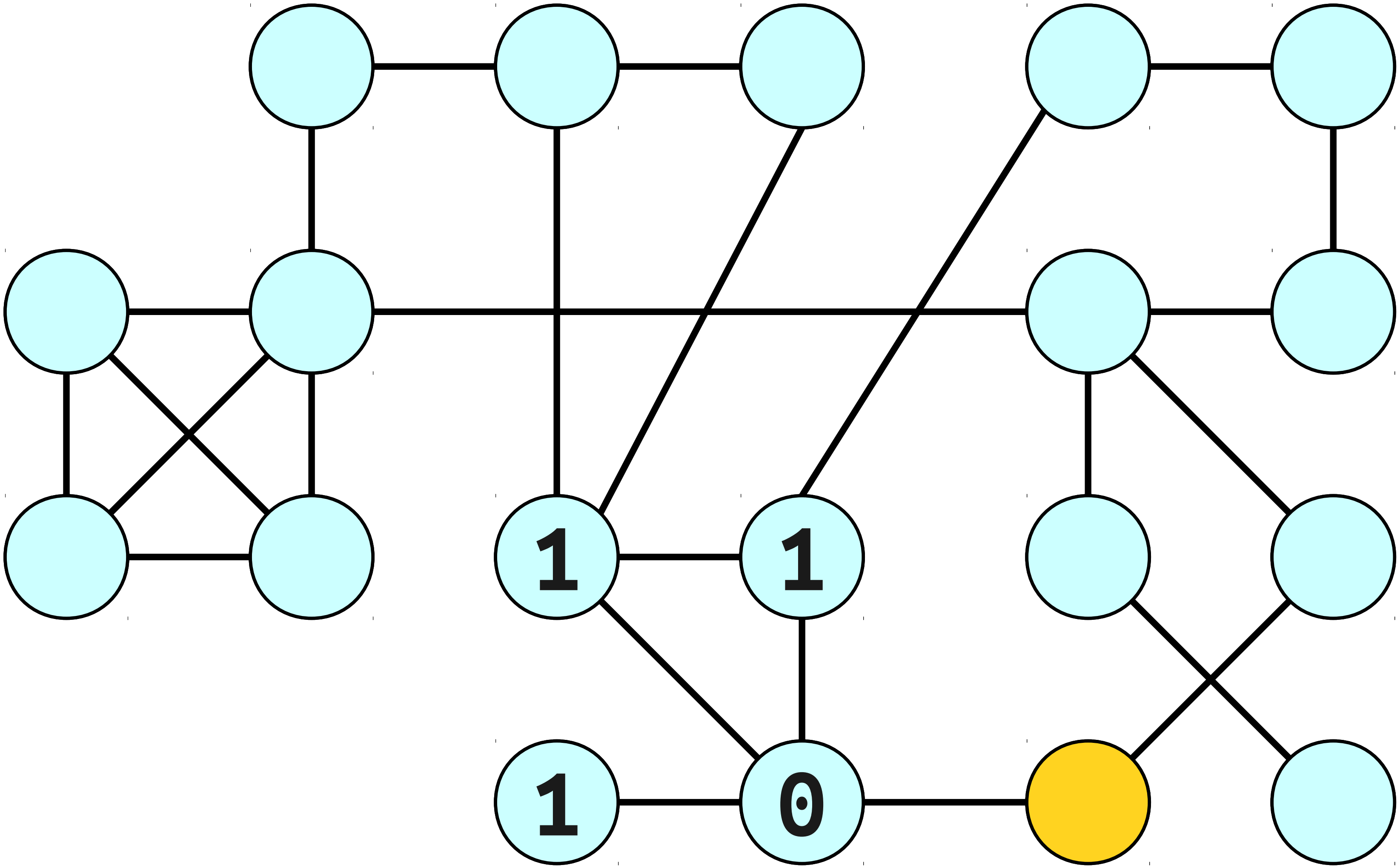

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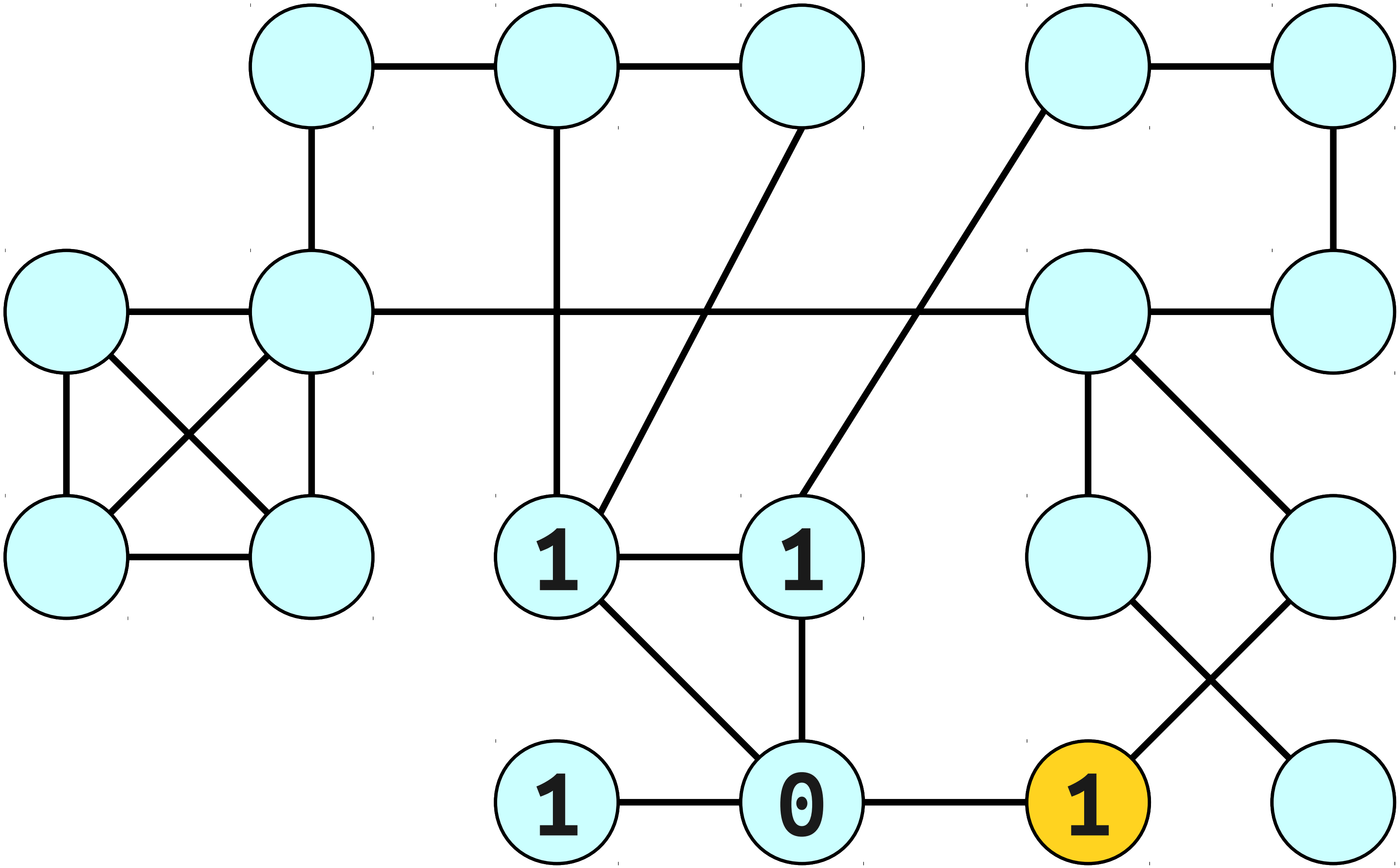

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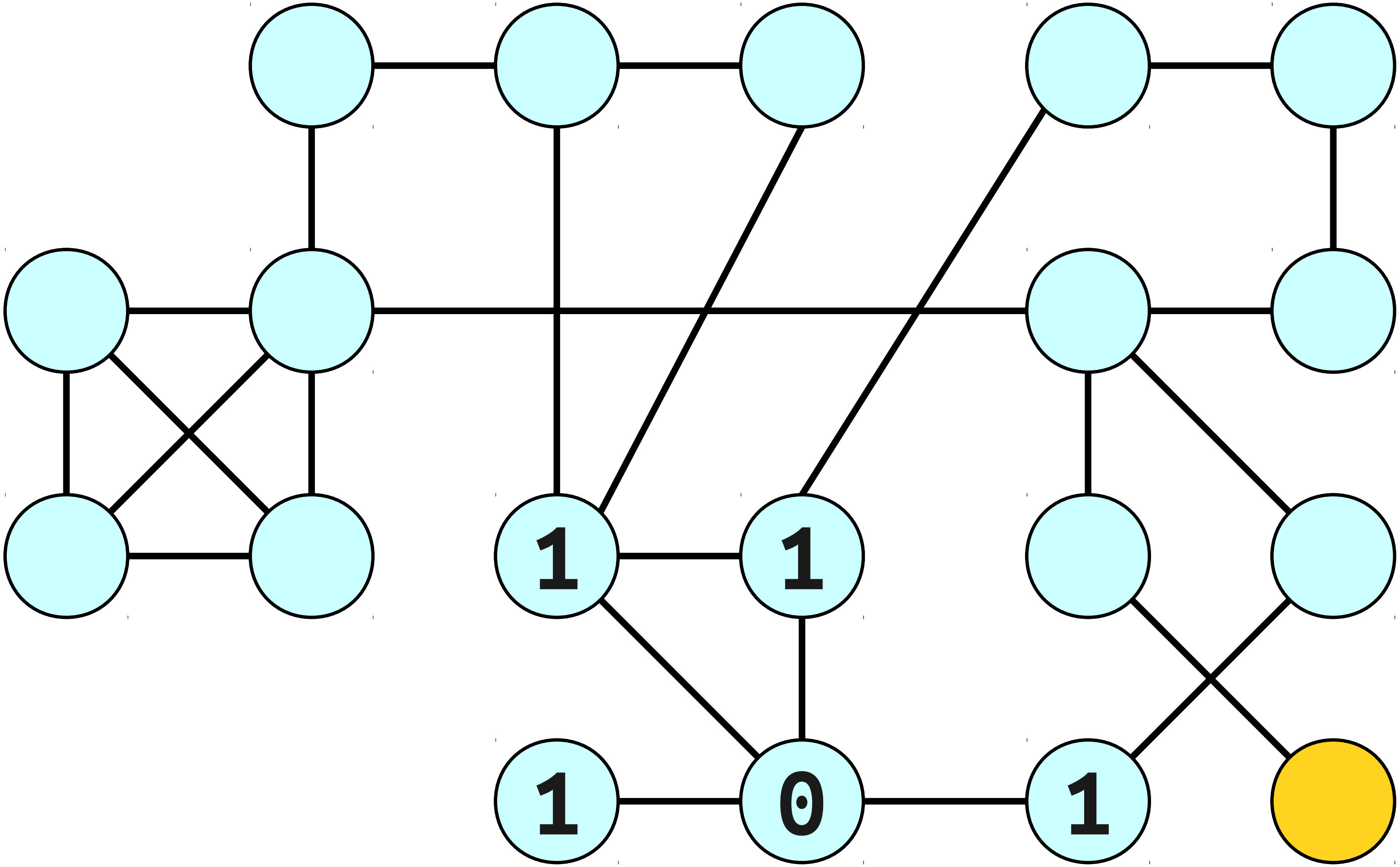

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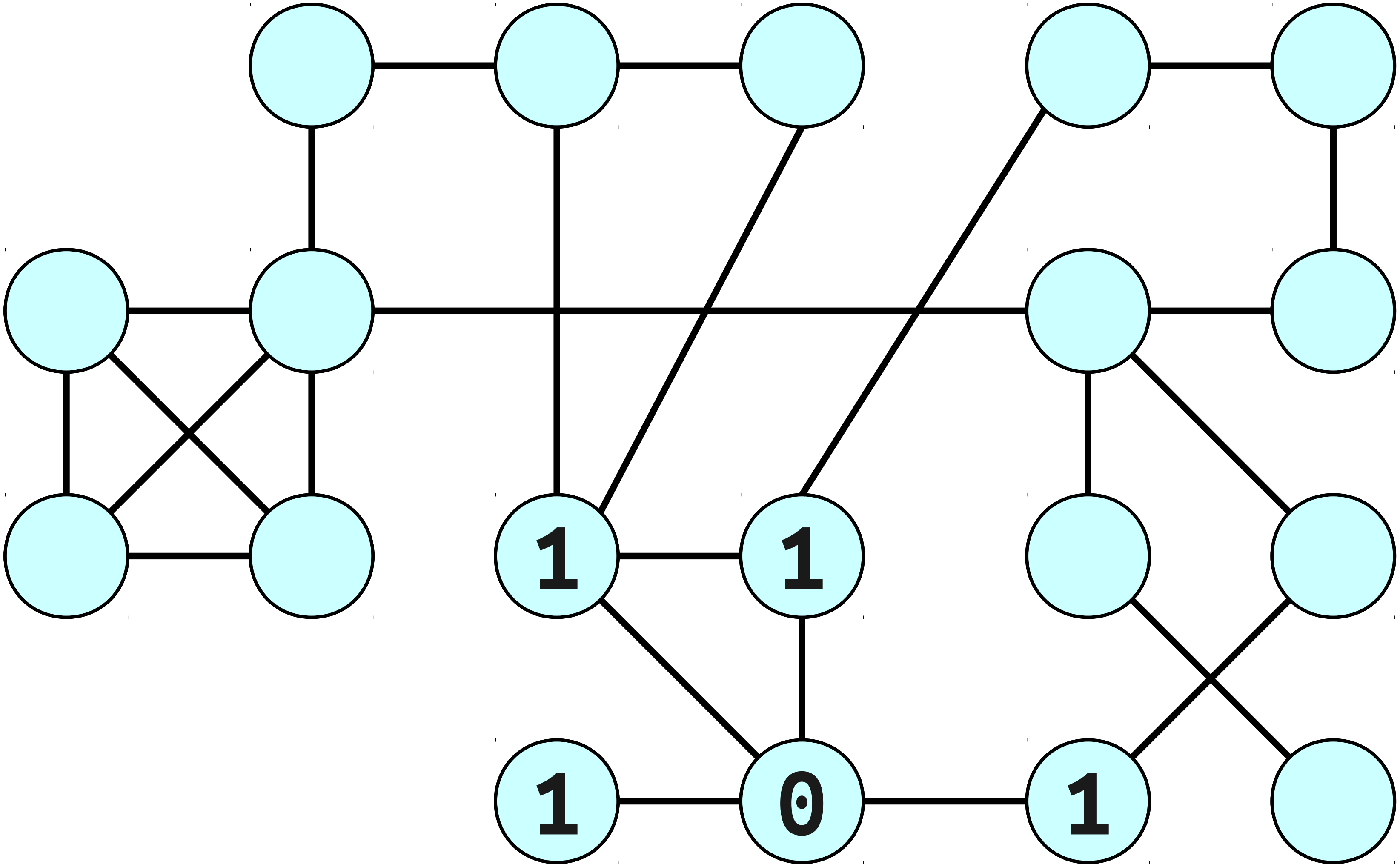

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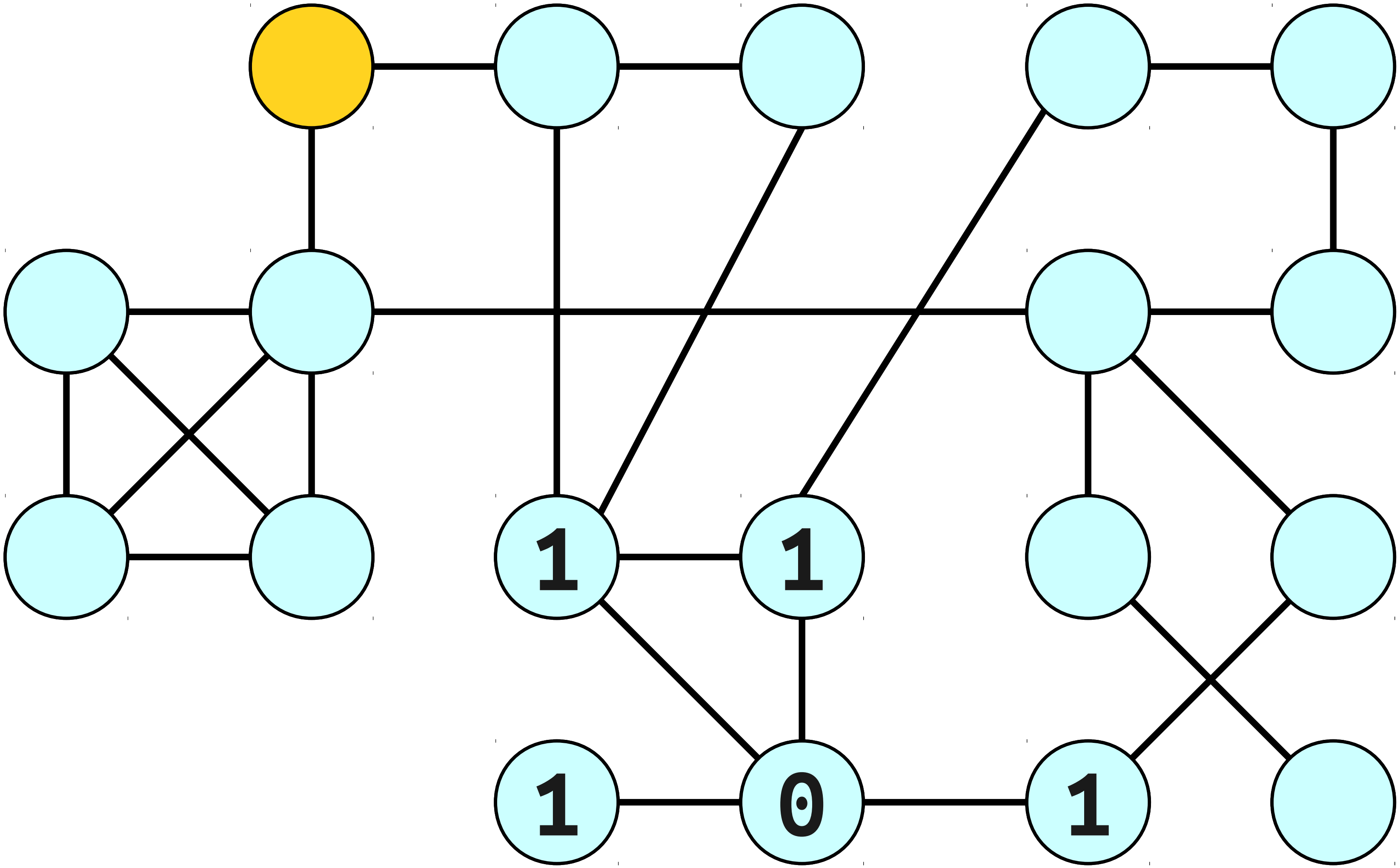

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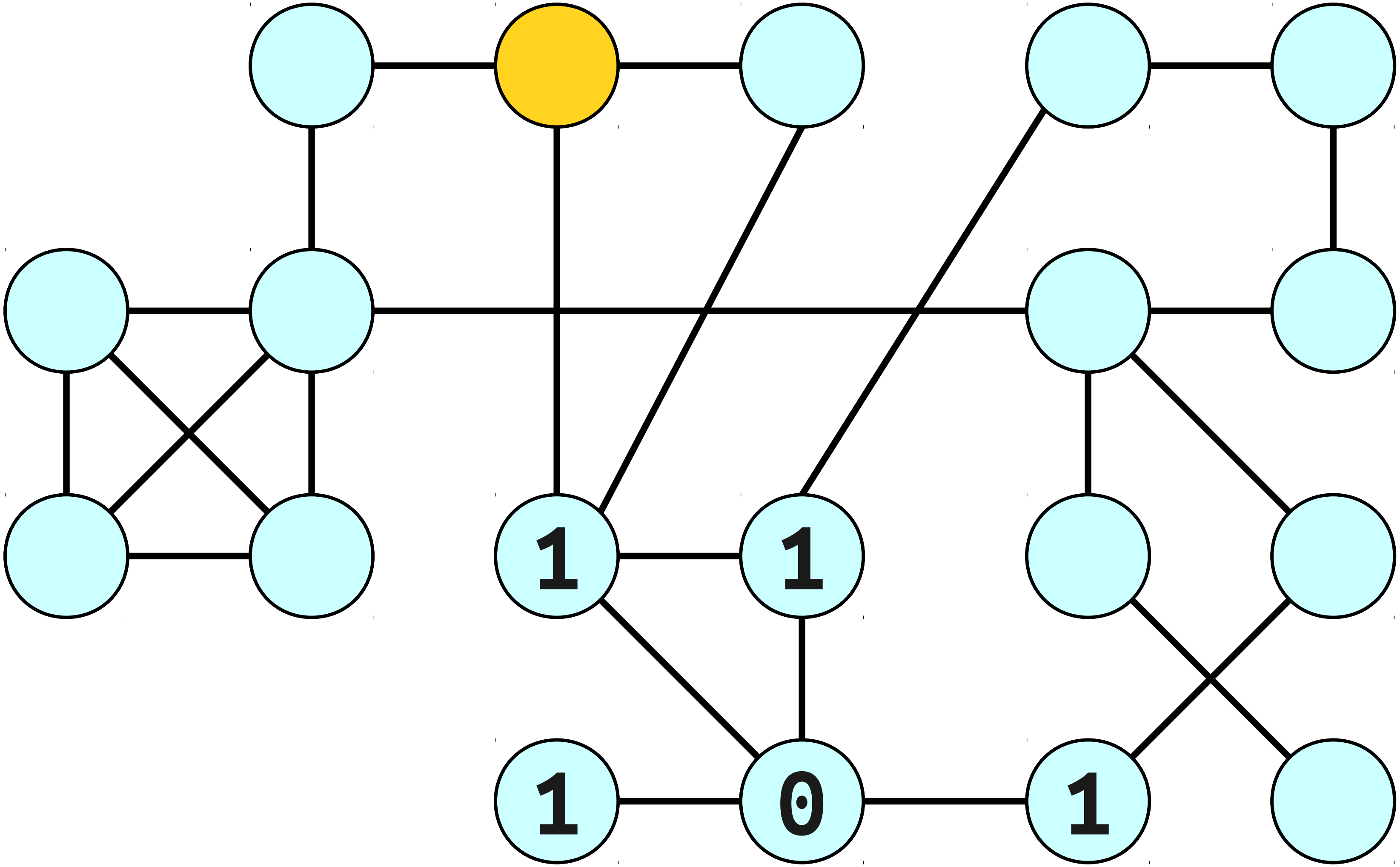

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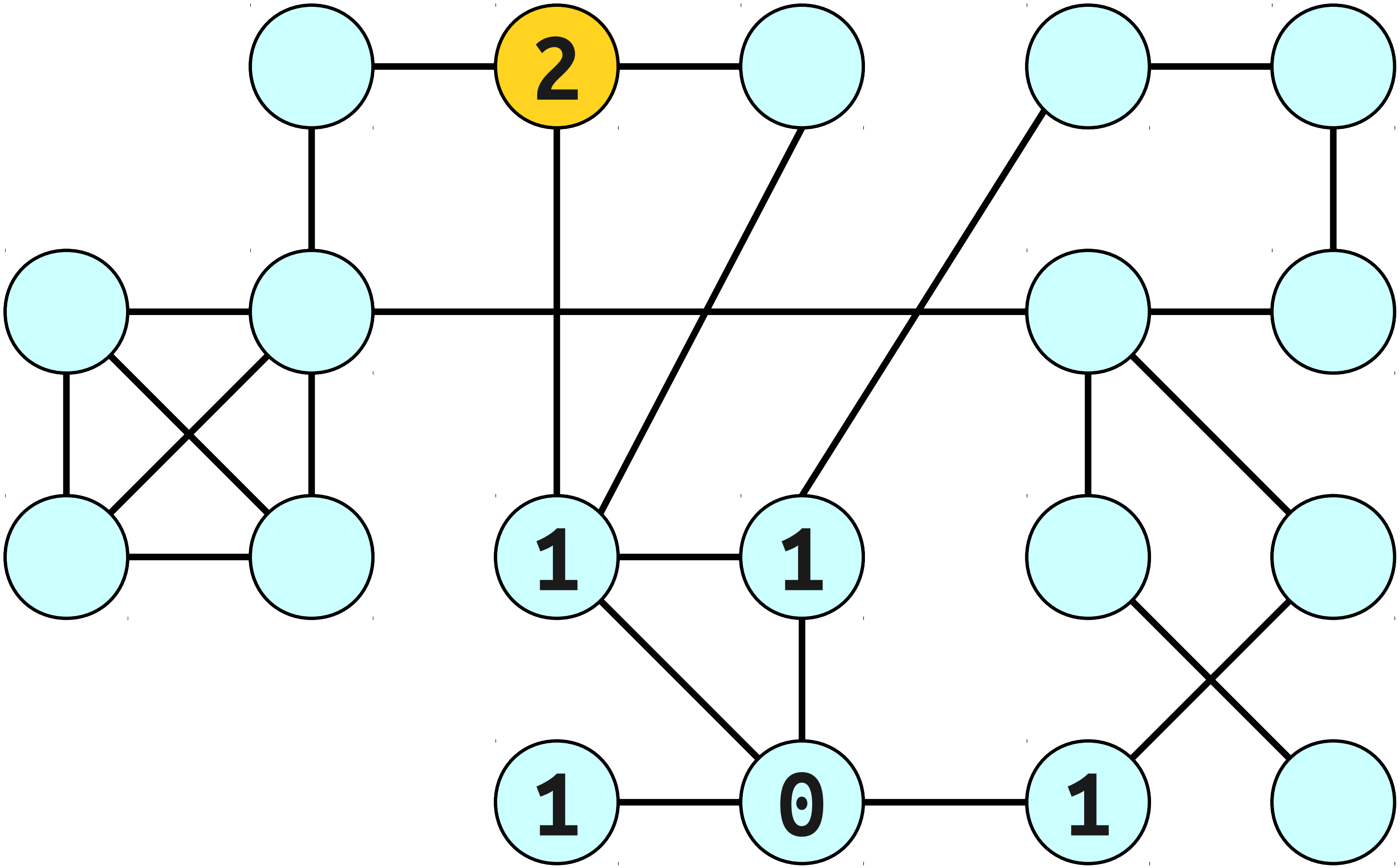

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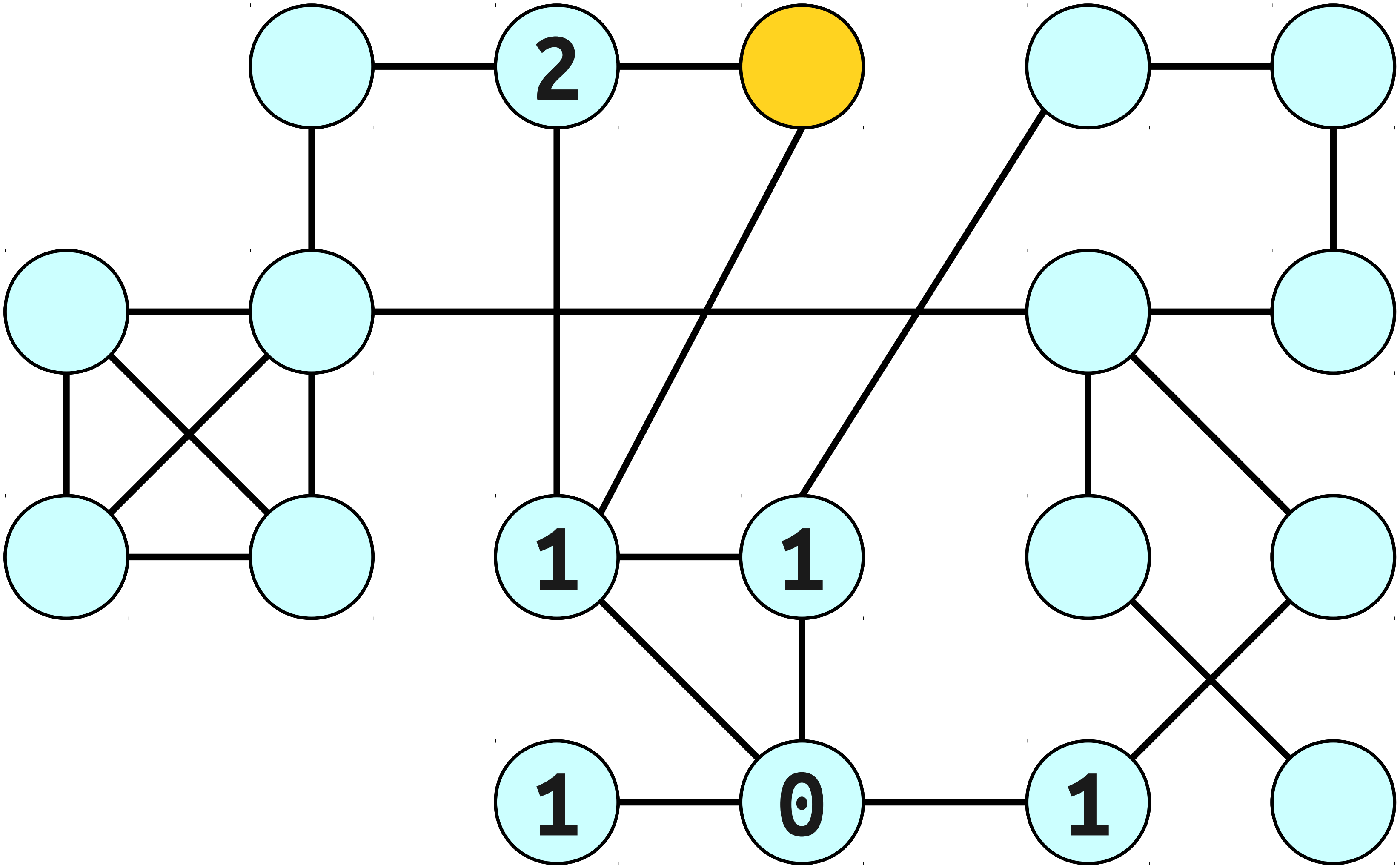

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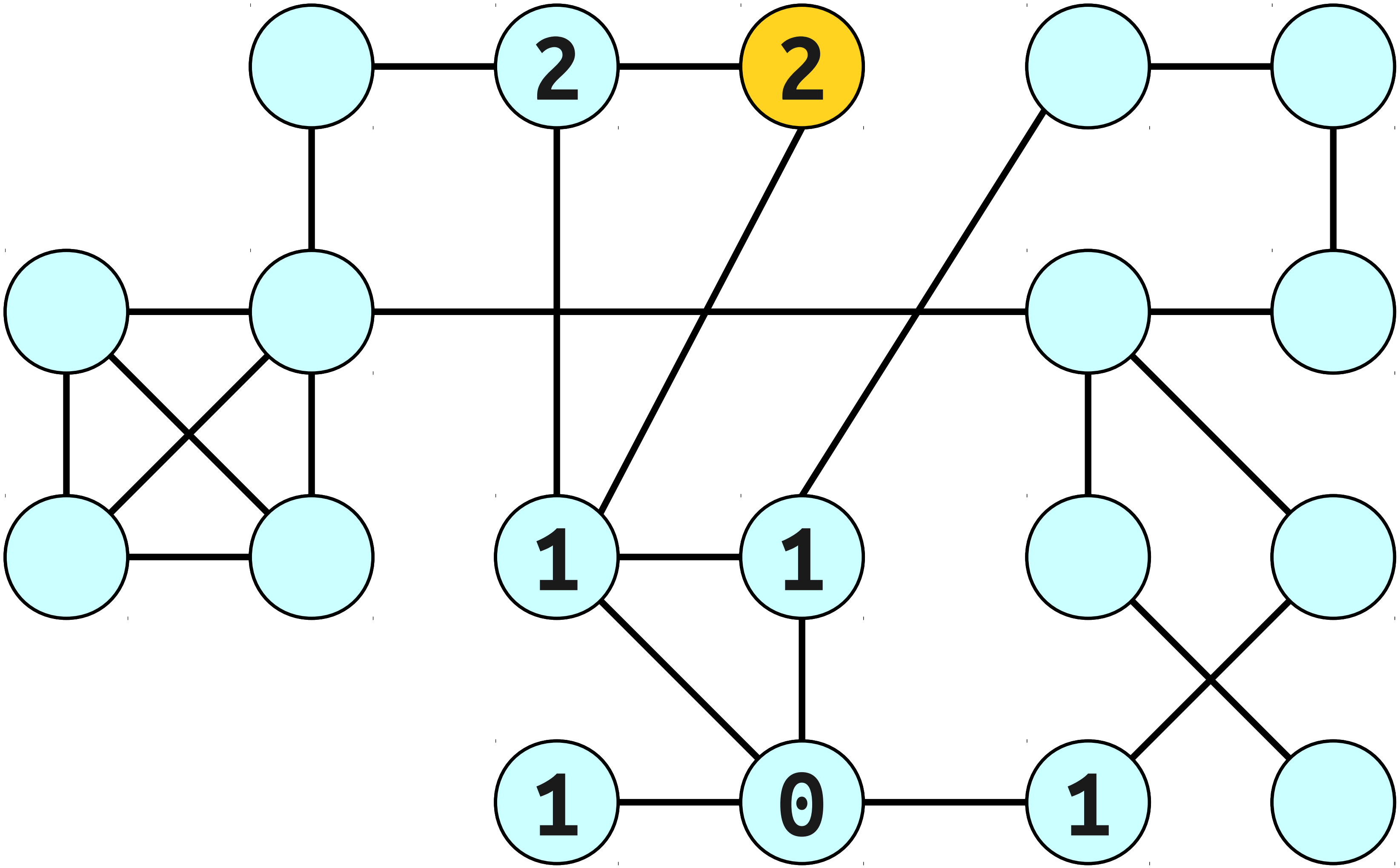

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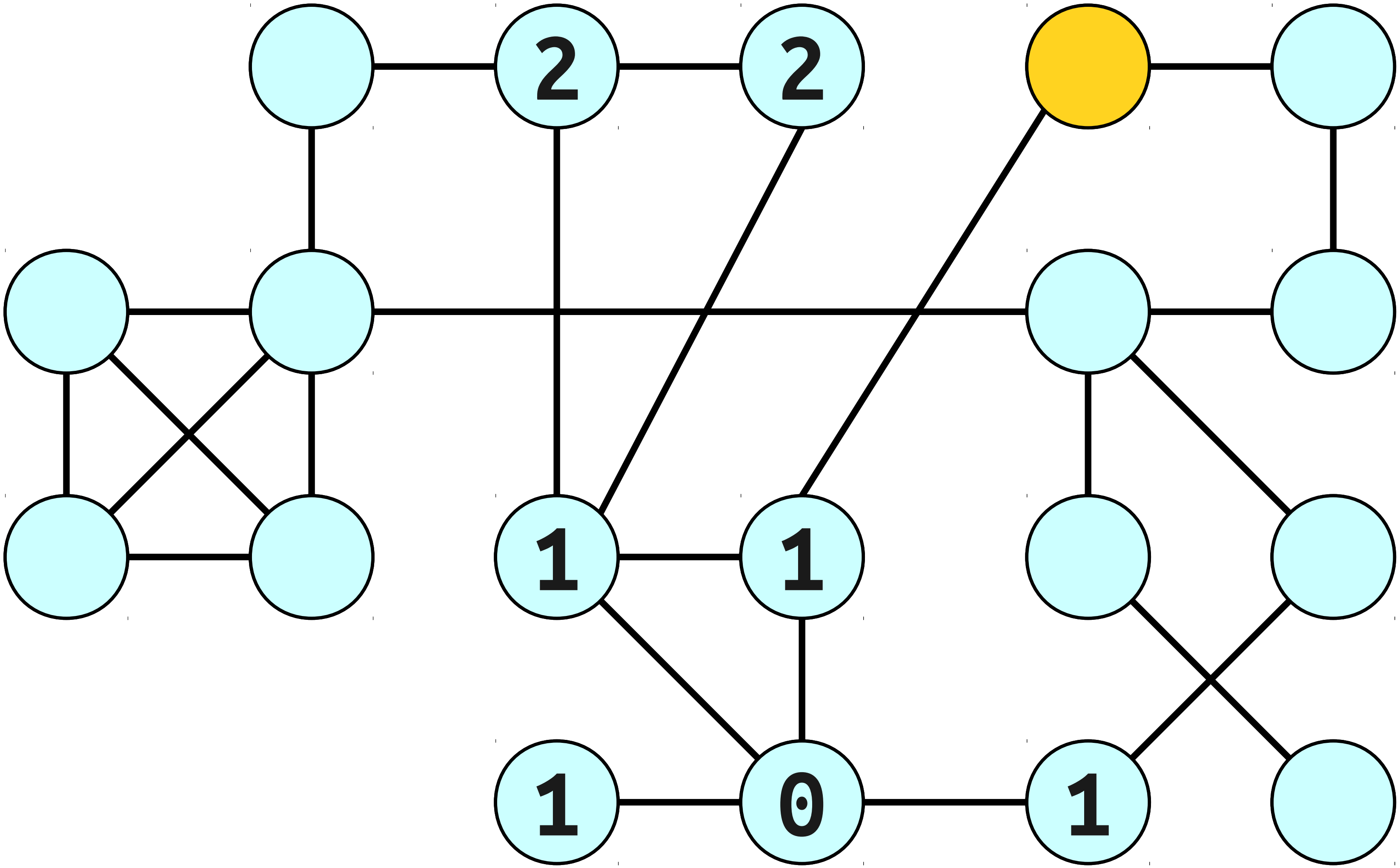

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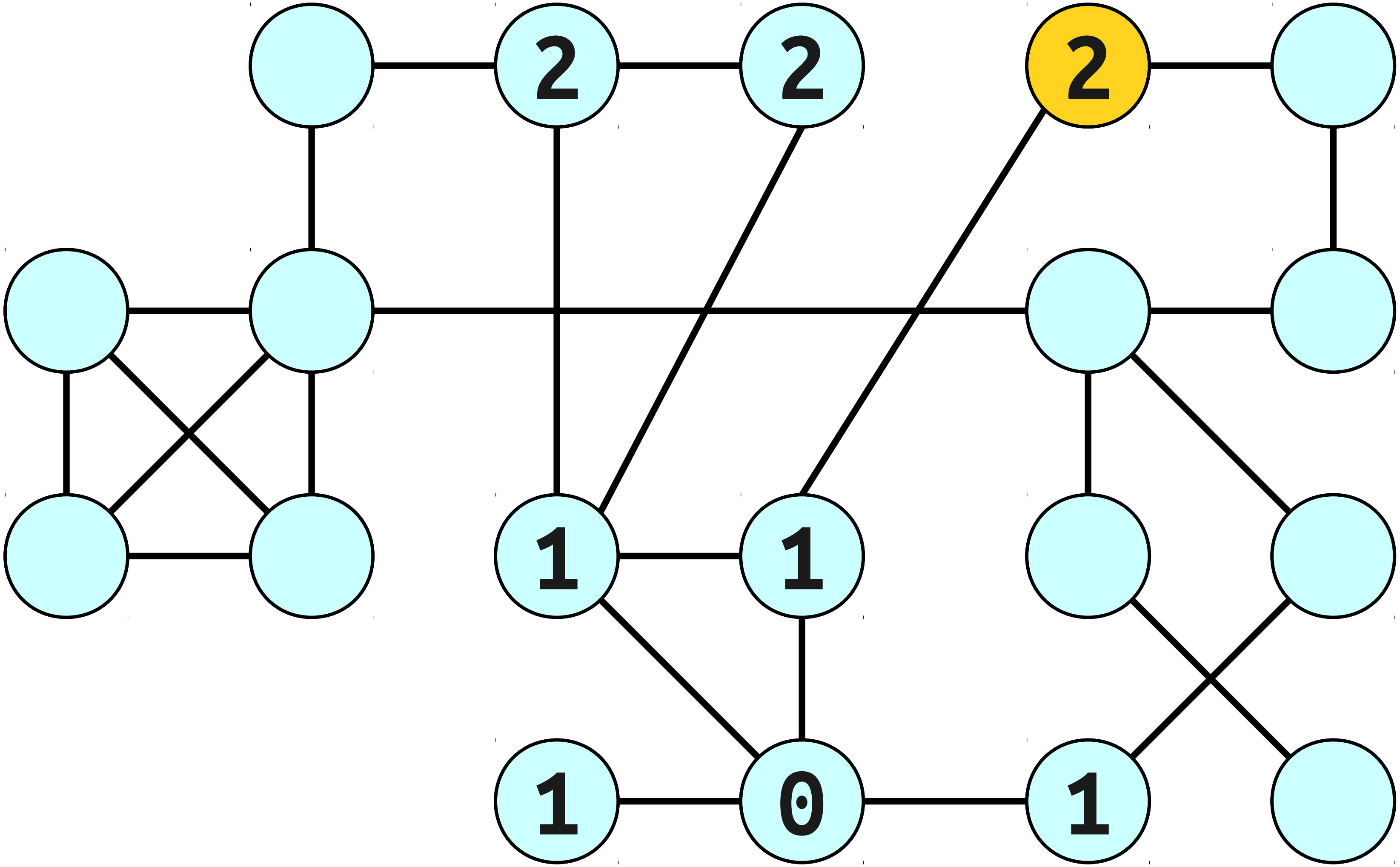

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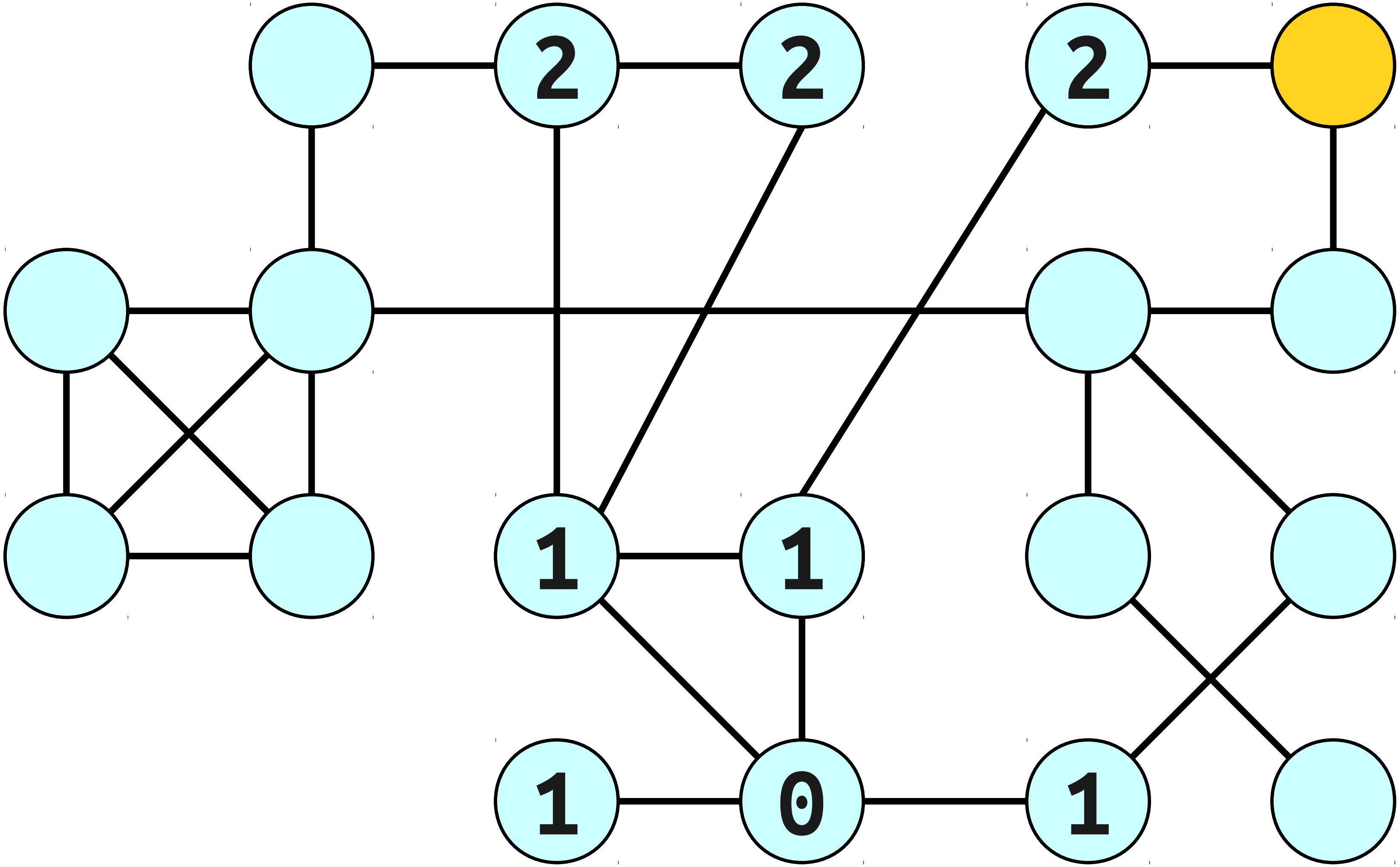

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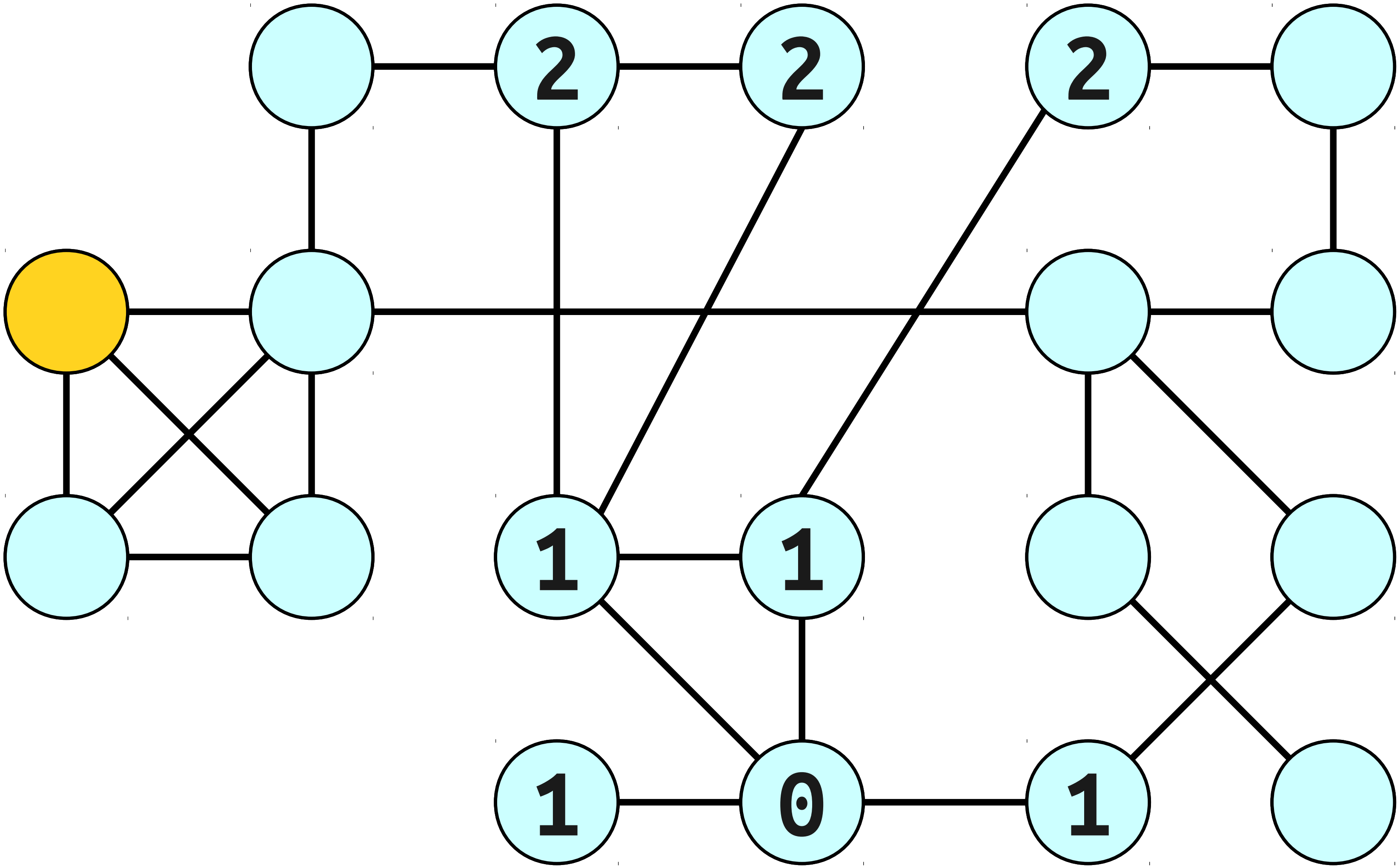

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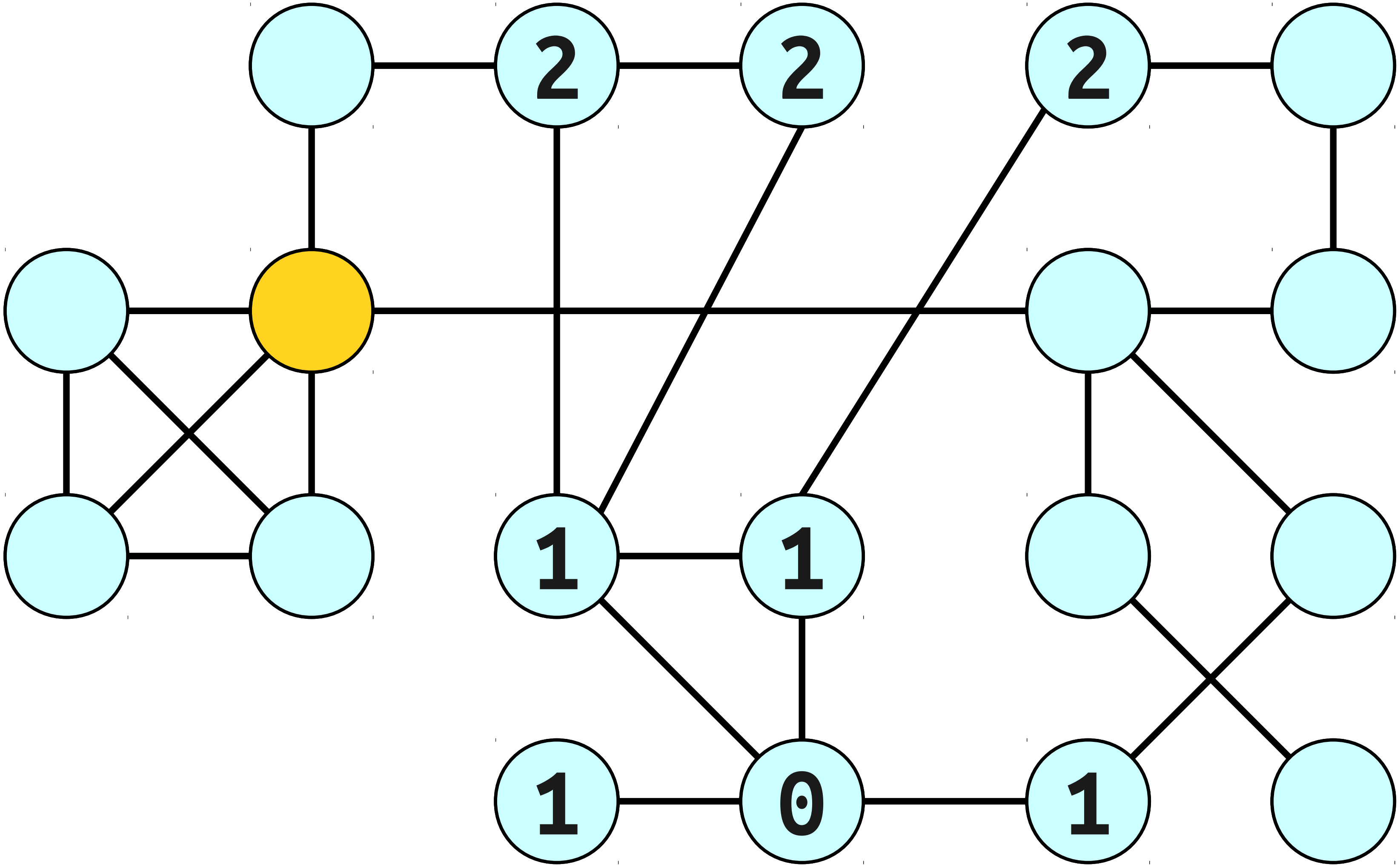

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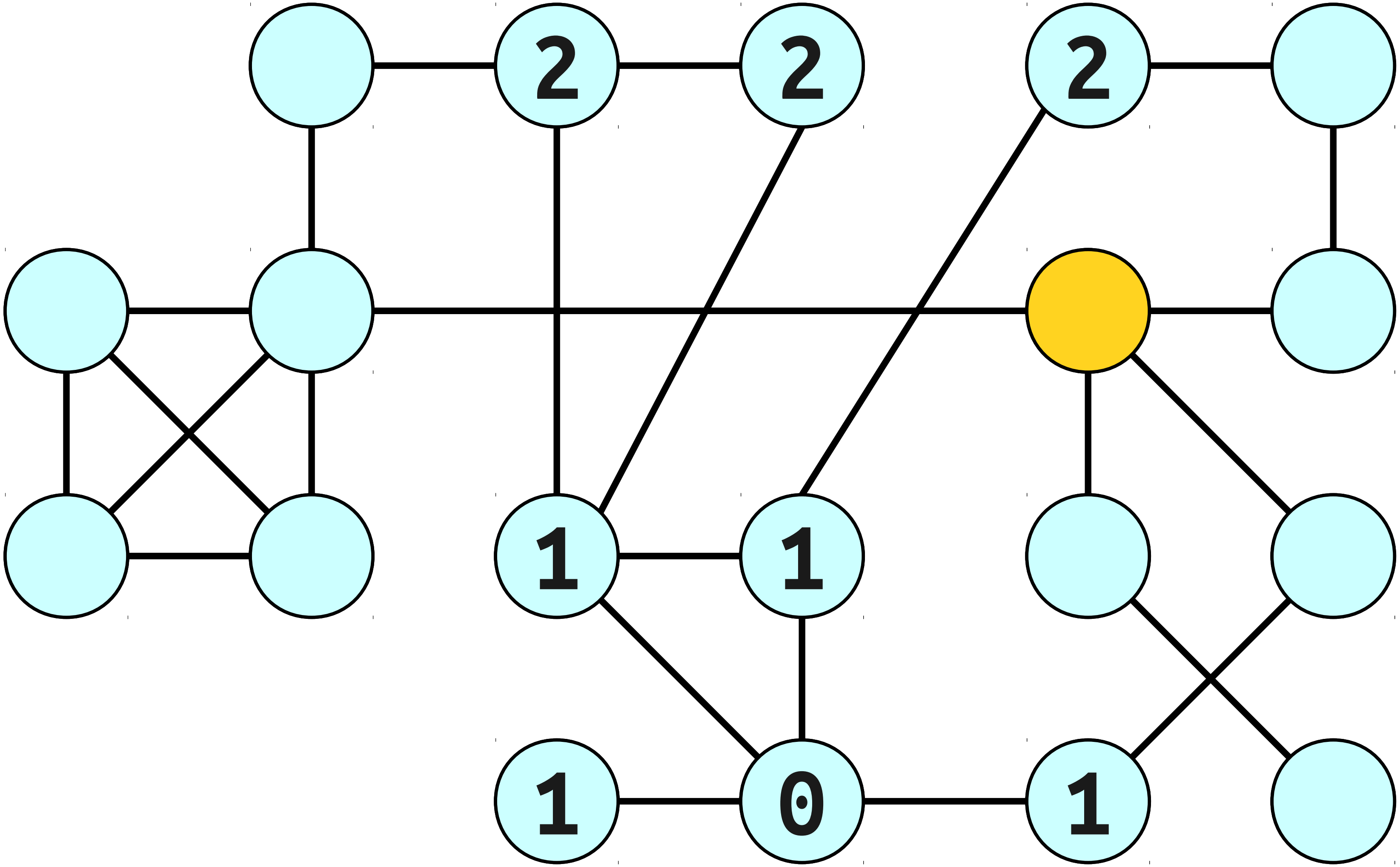

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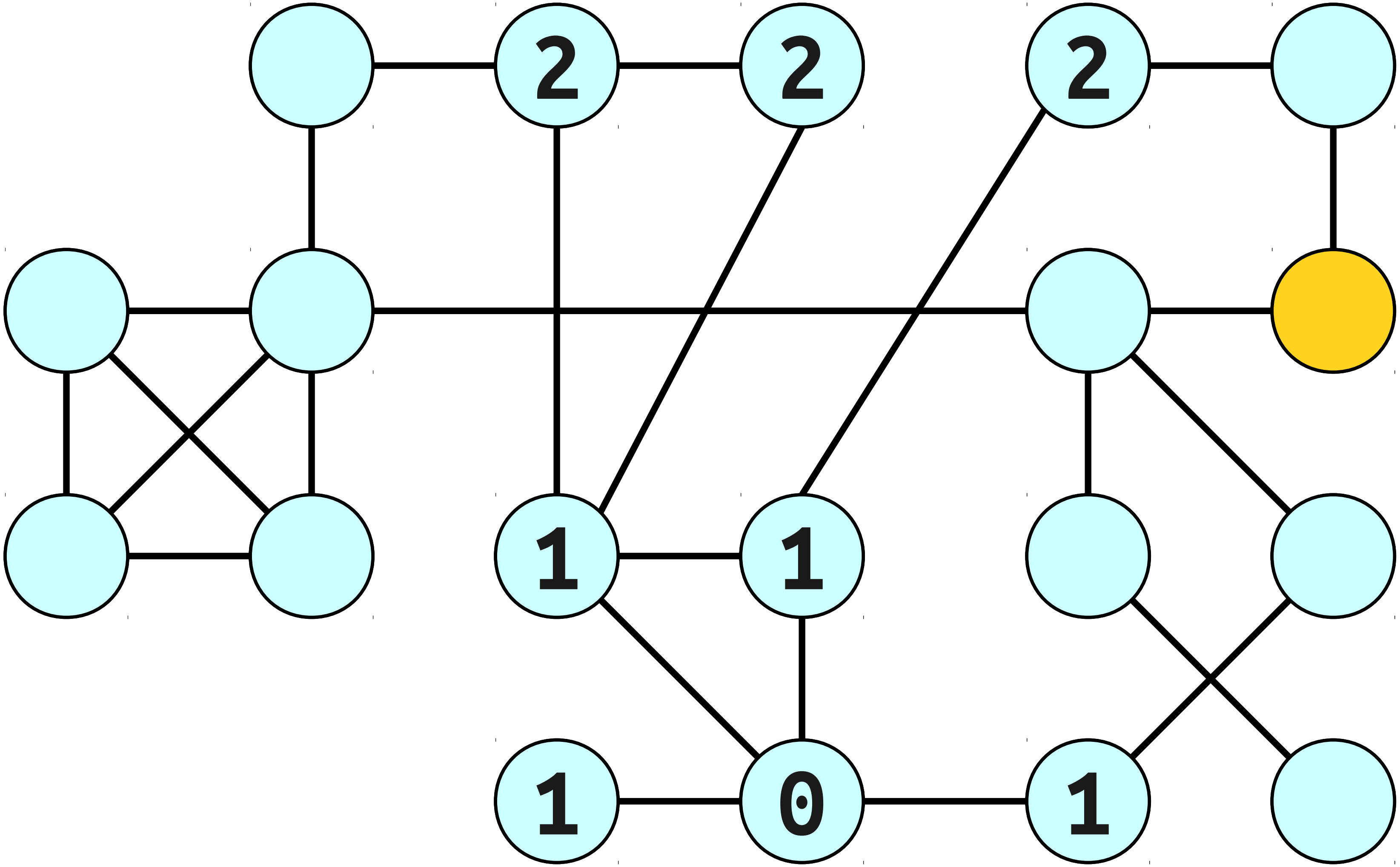

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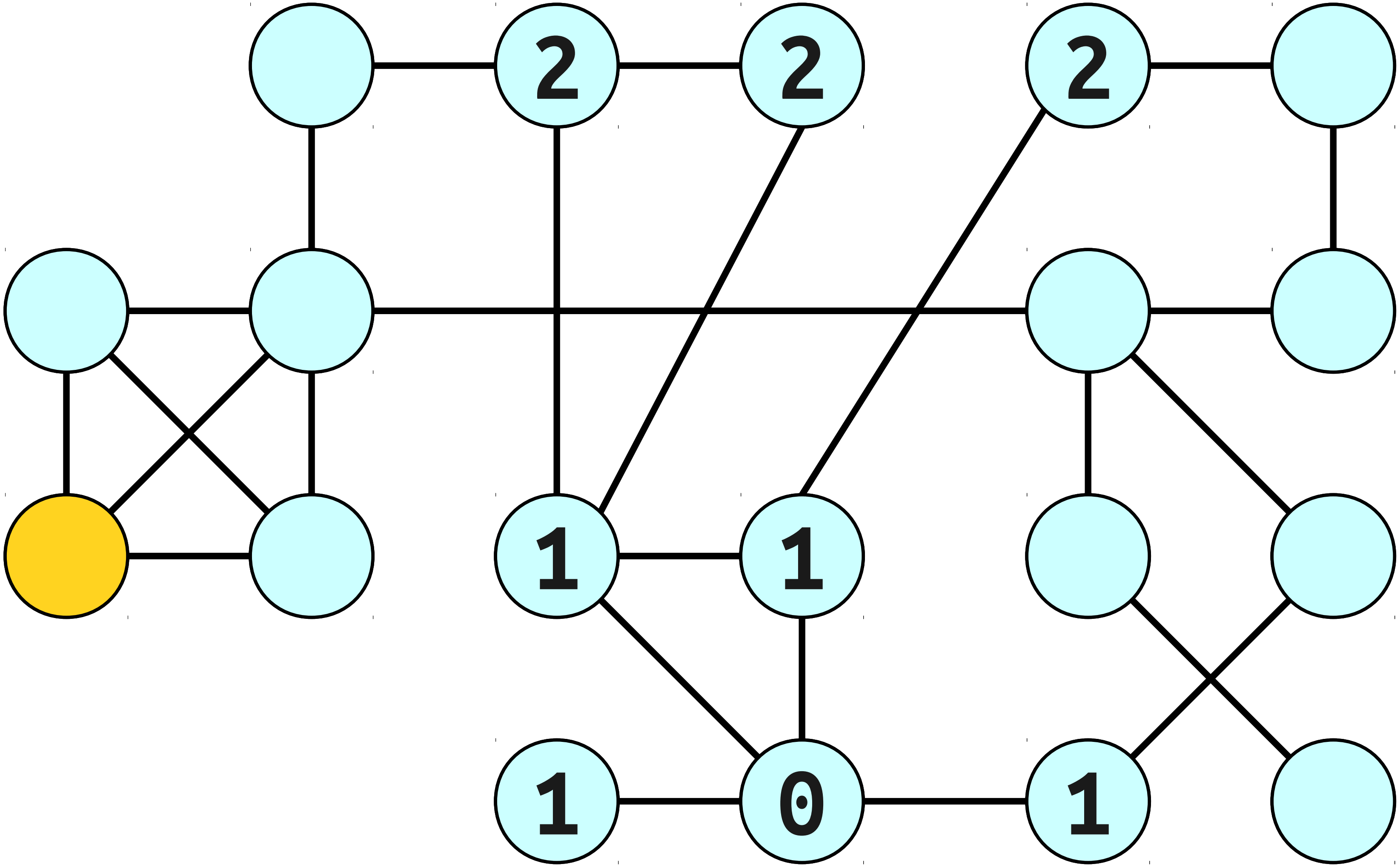

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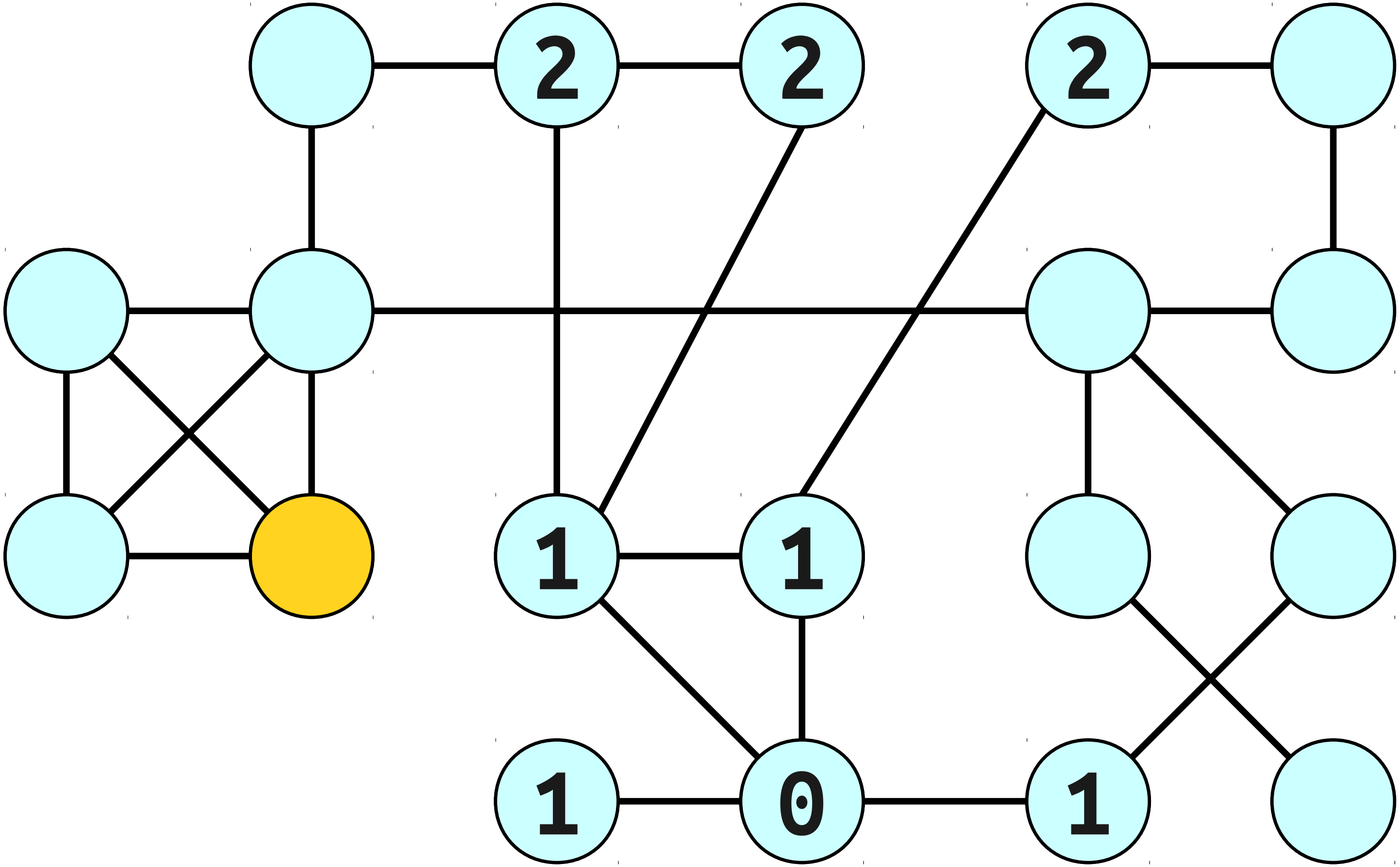
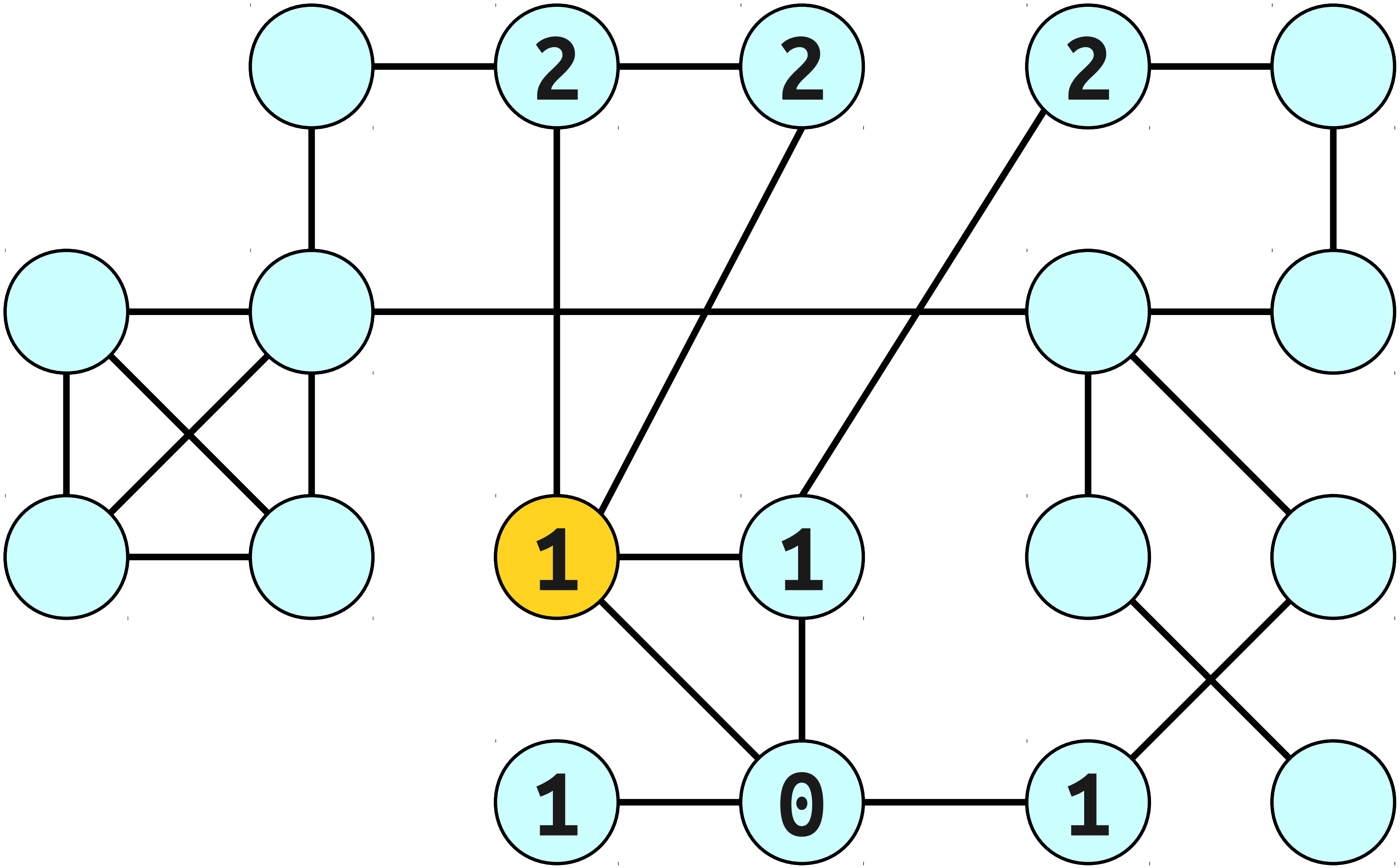


An Inefficient Algorithm



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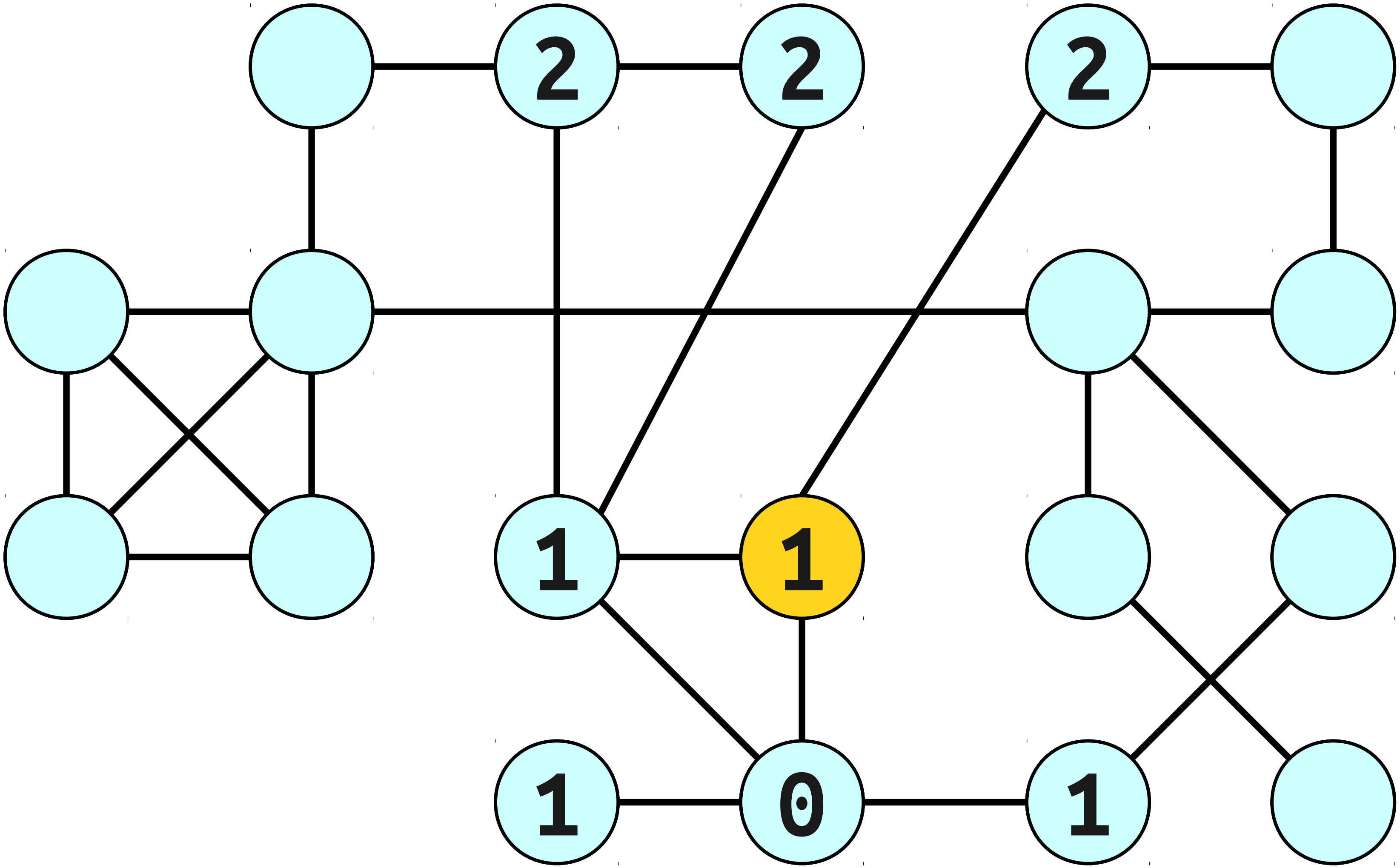
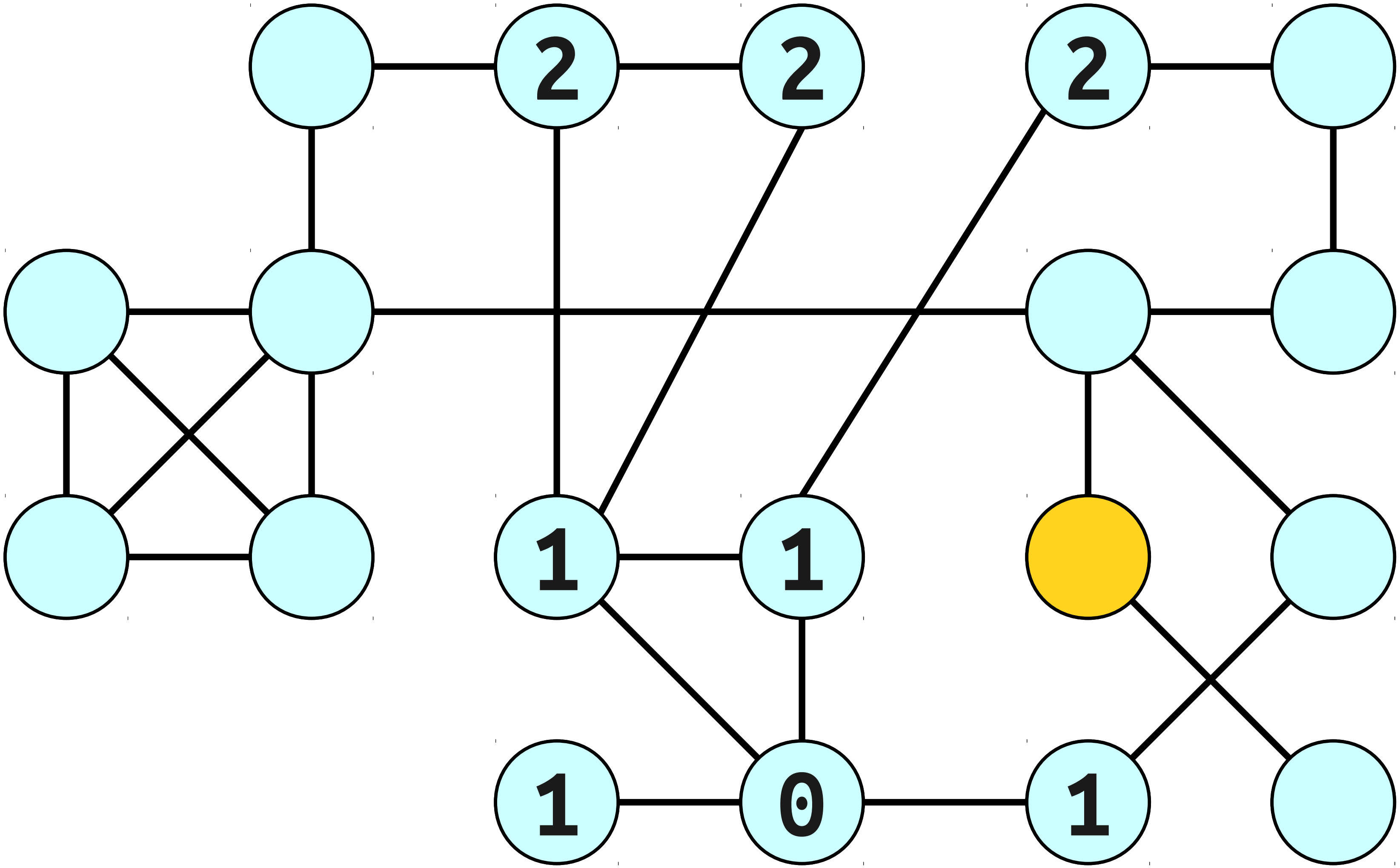


An Inefficient Algorithm



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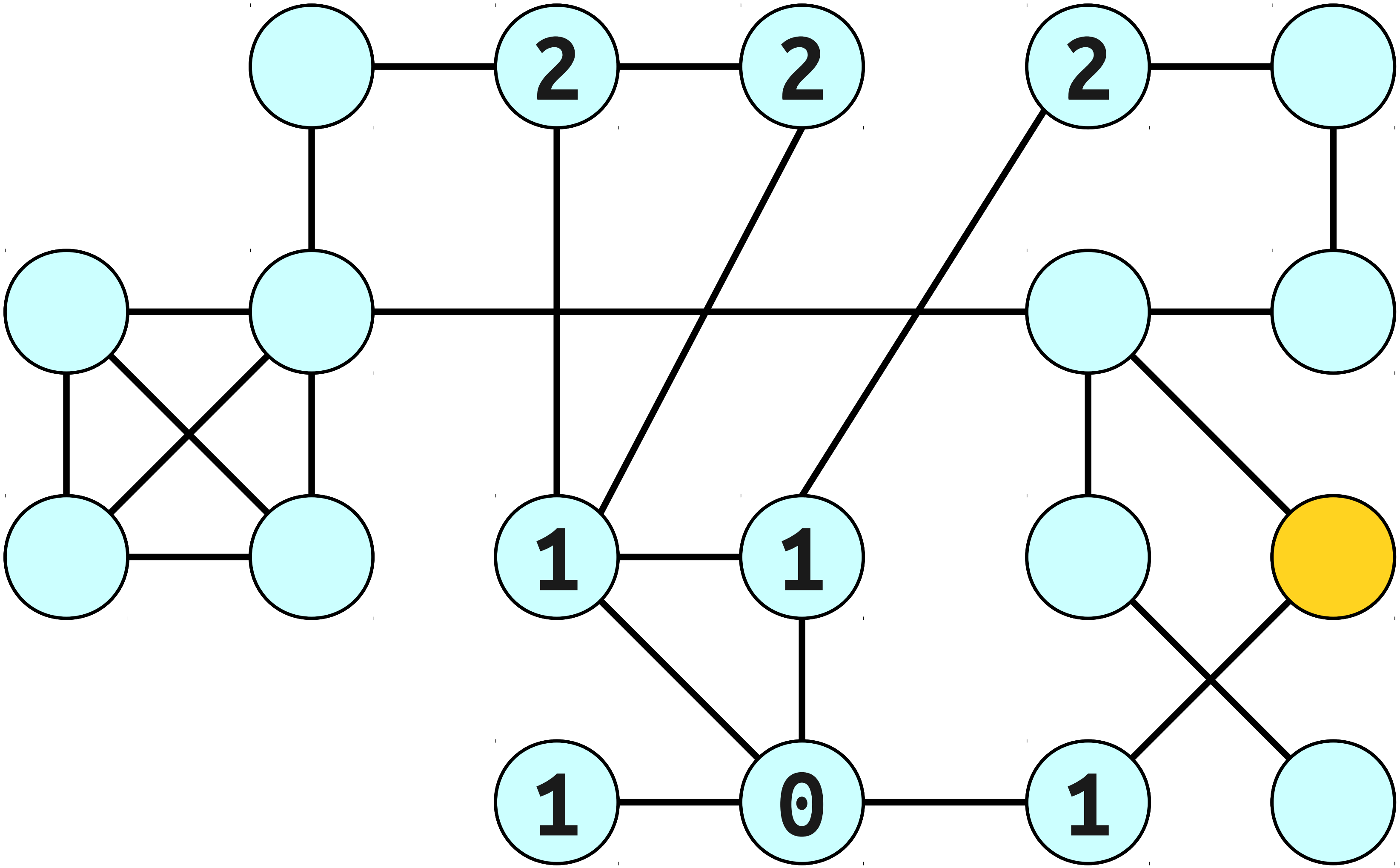

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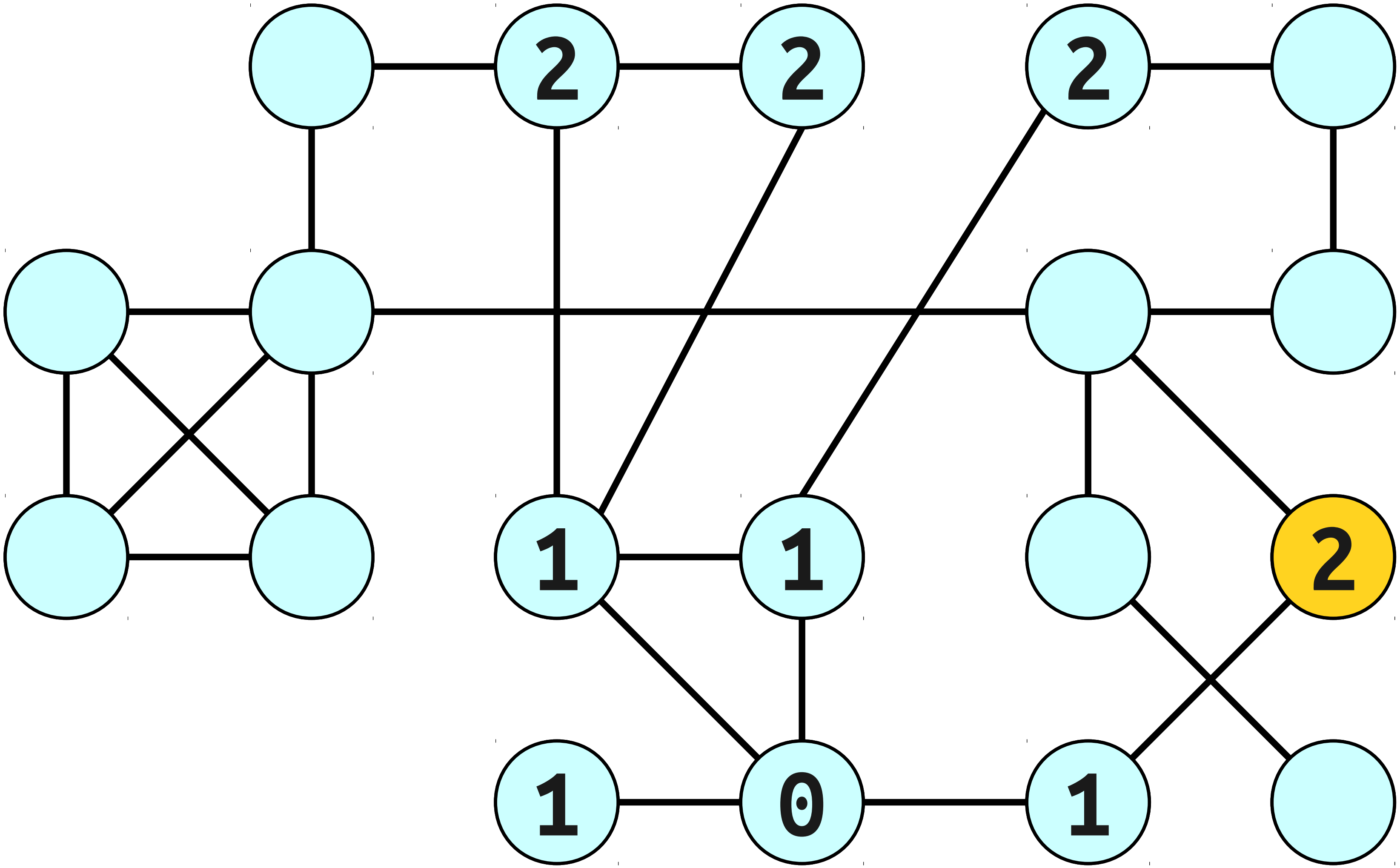

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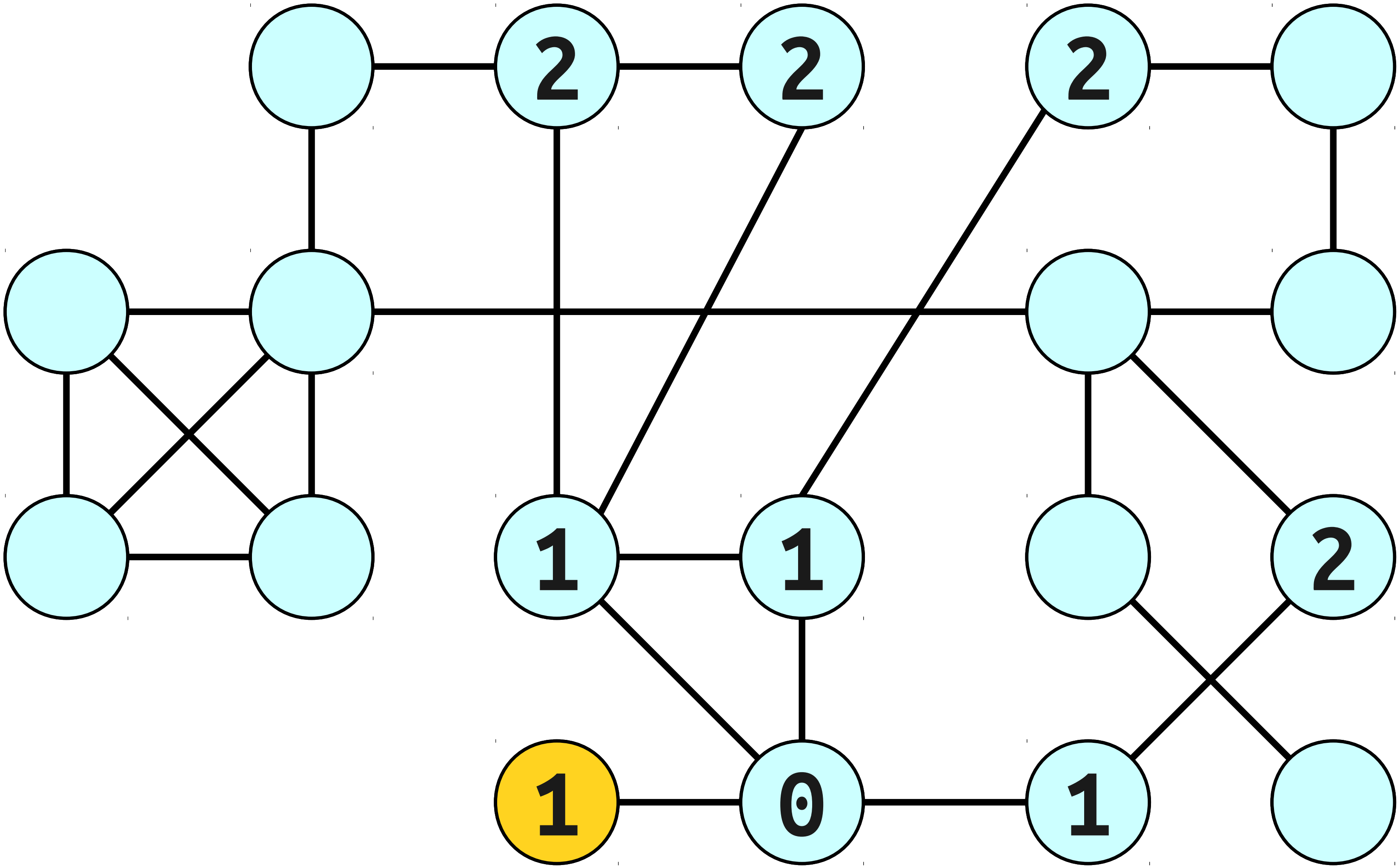

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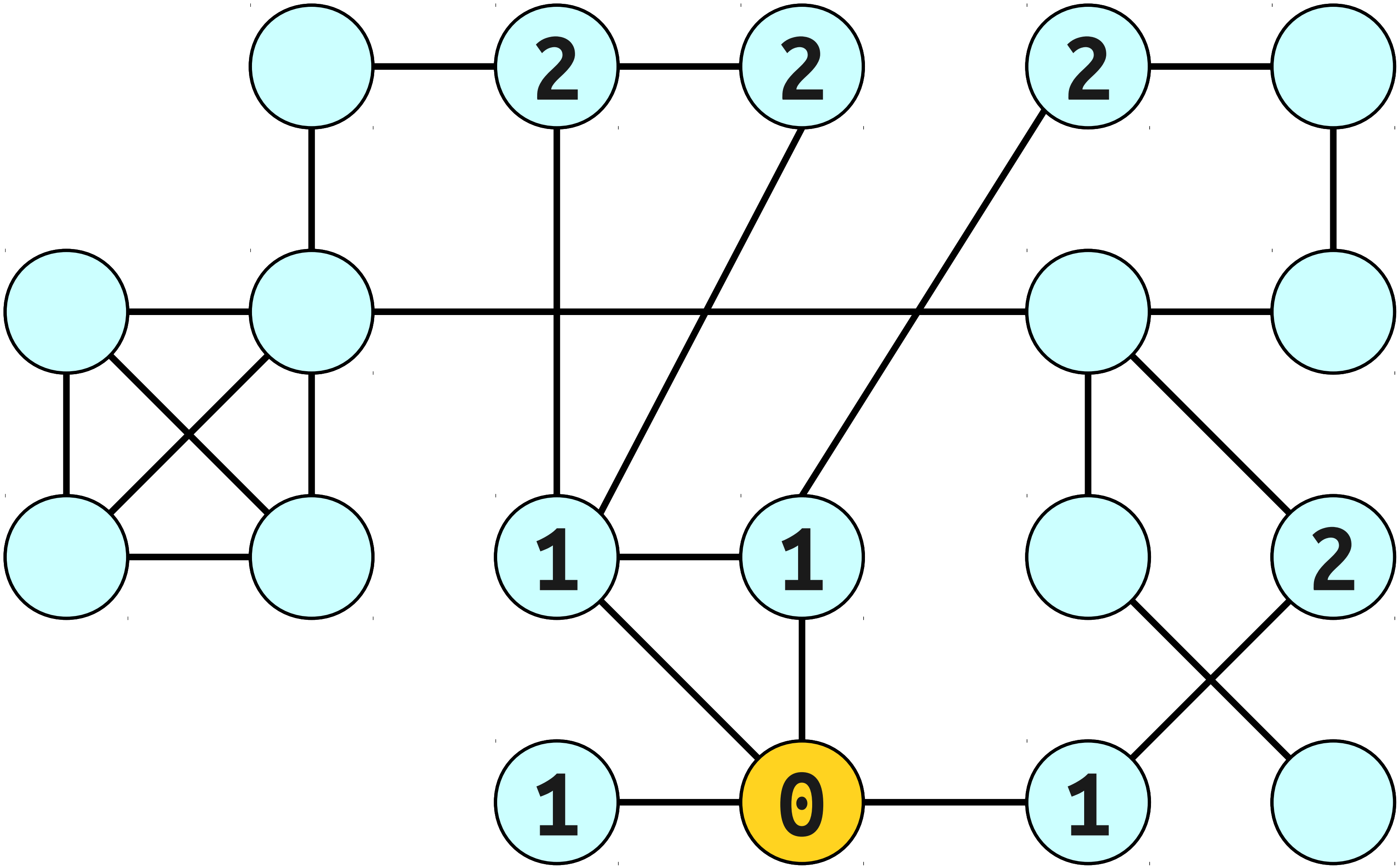

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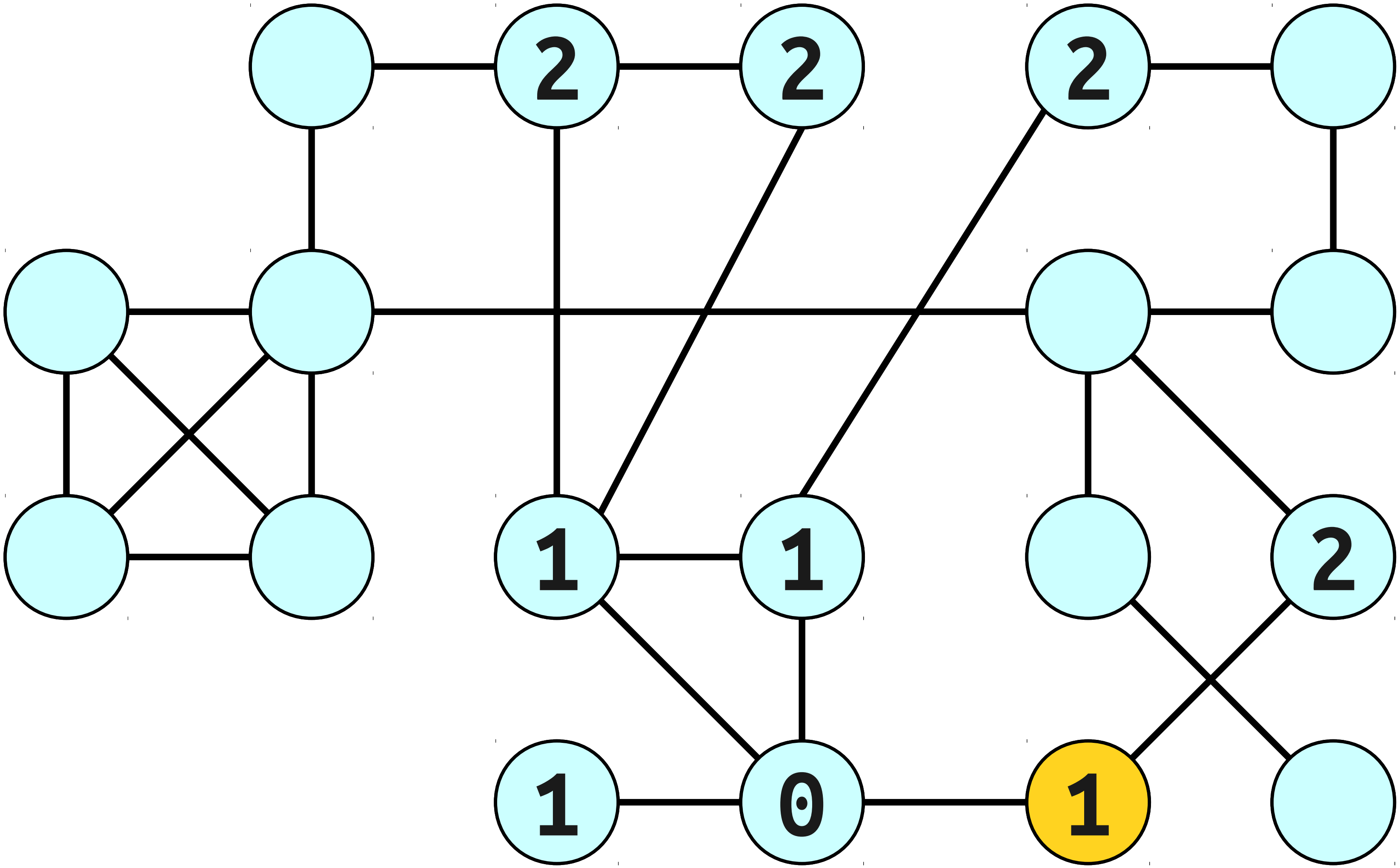

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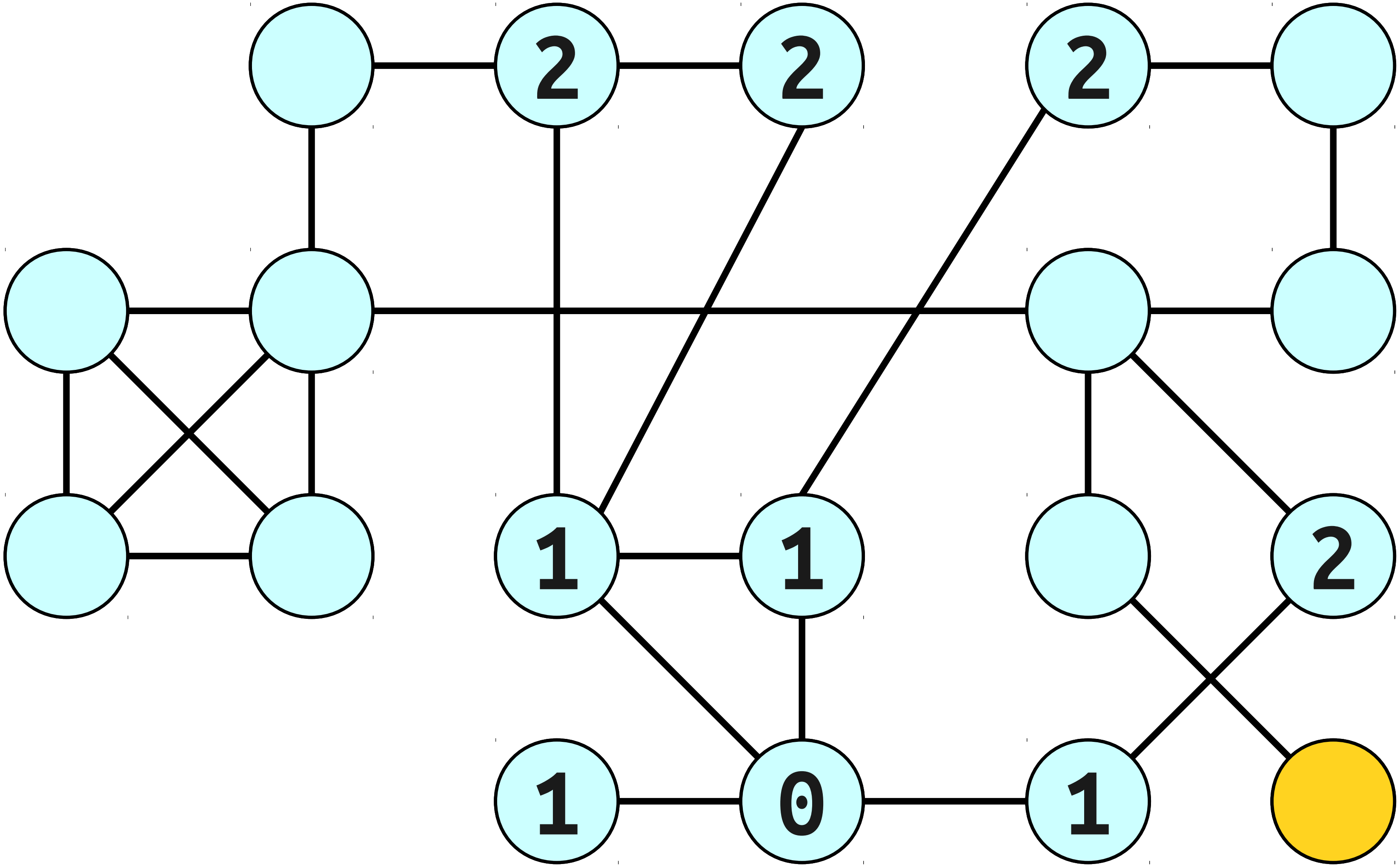

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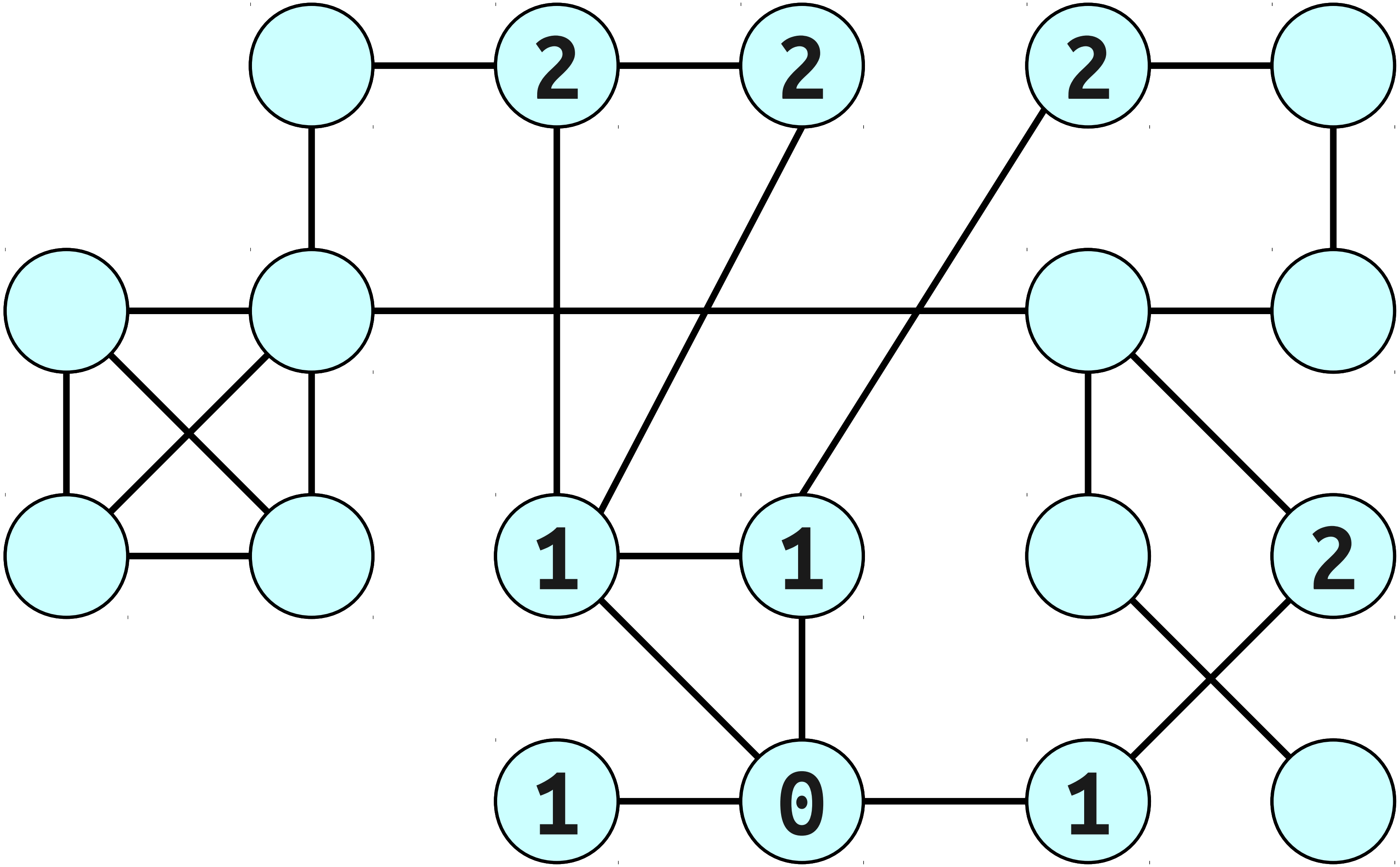

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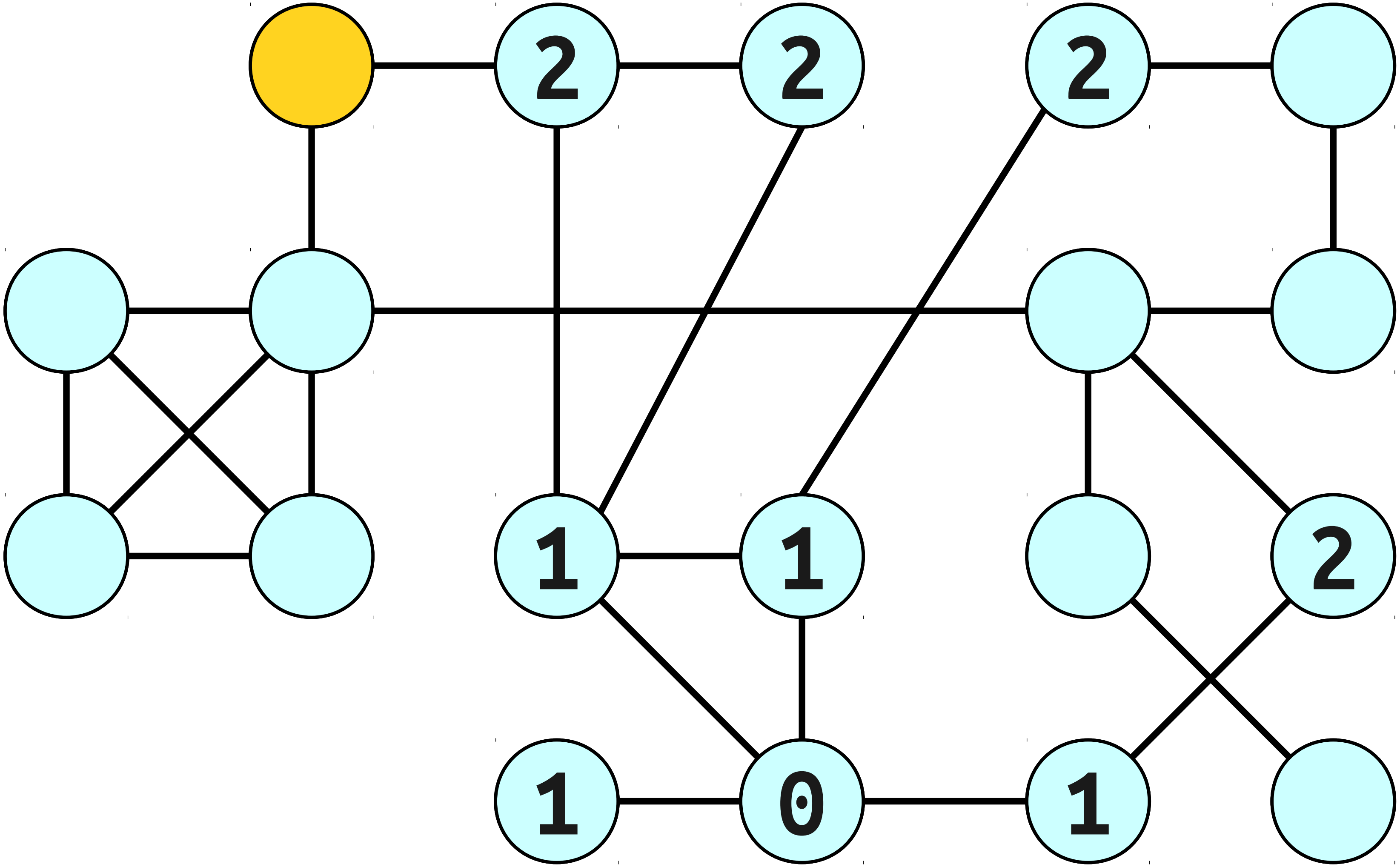

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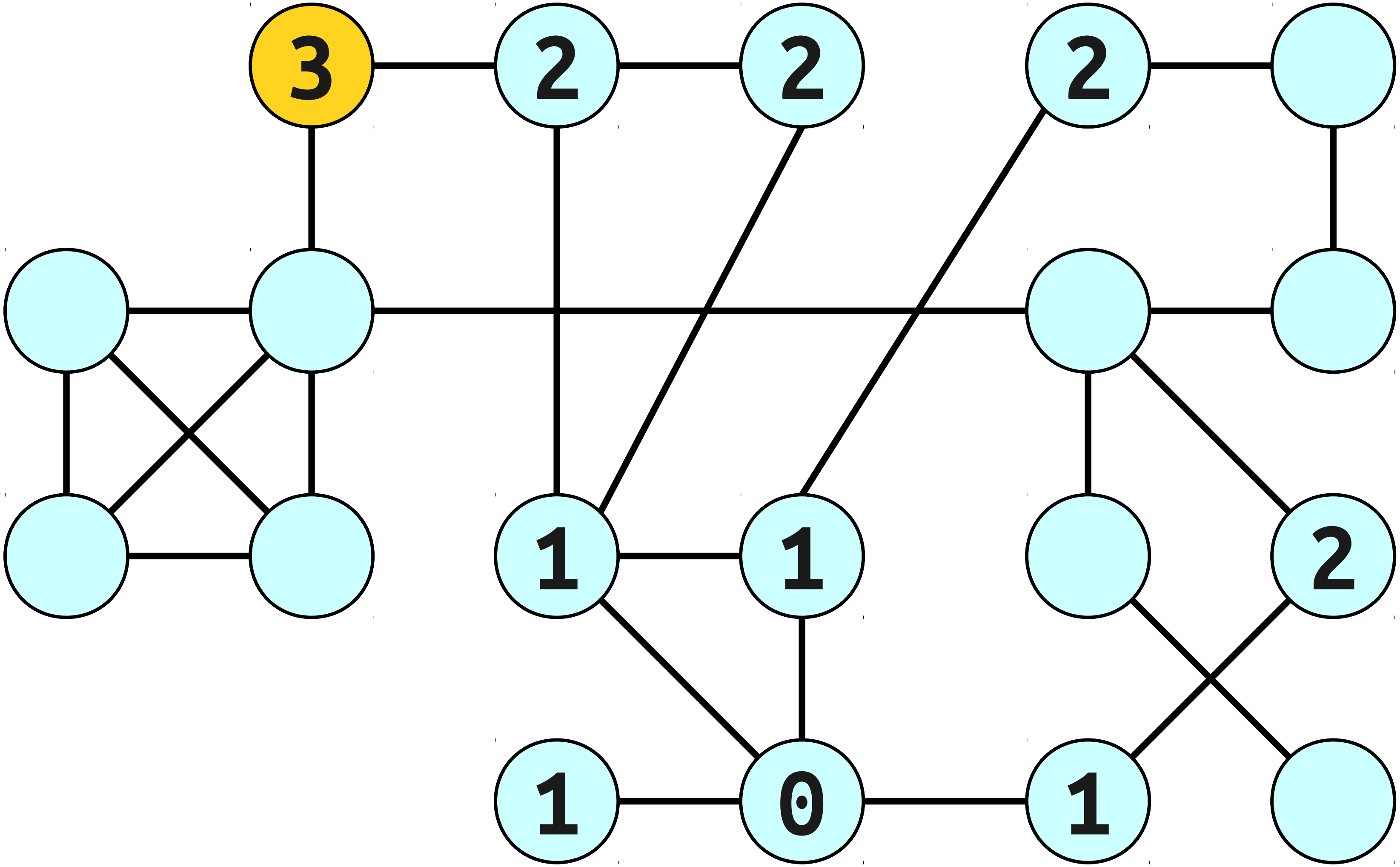

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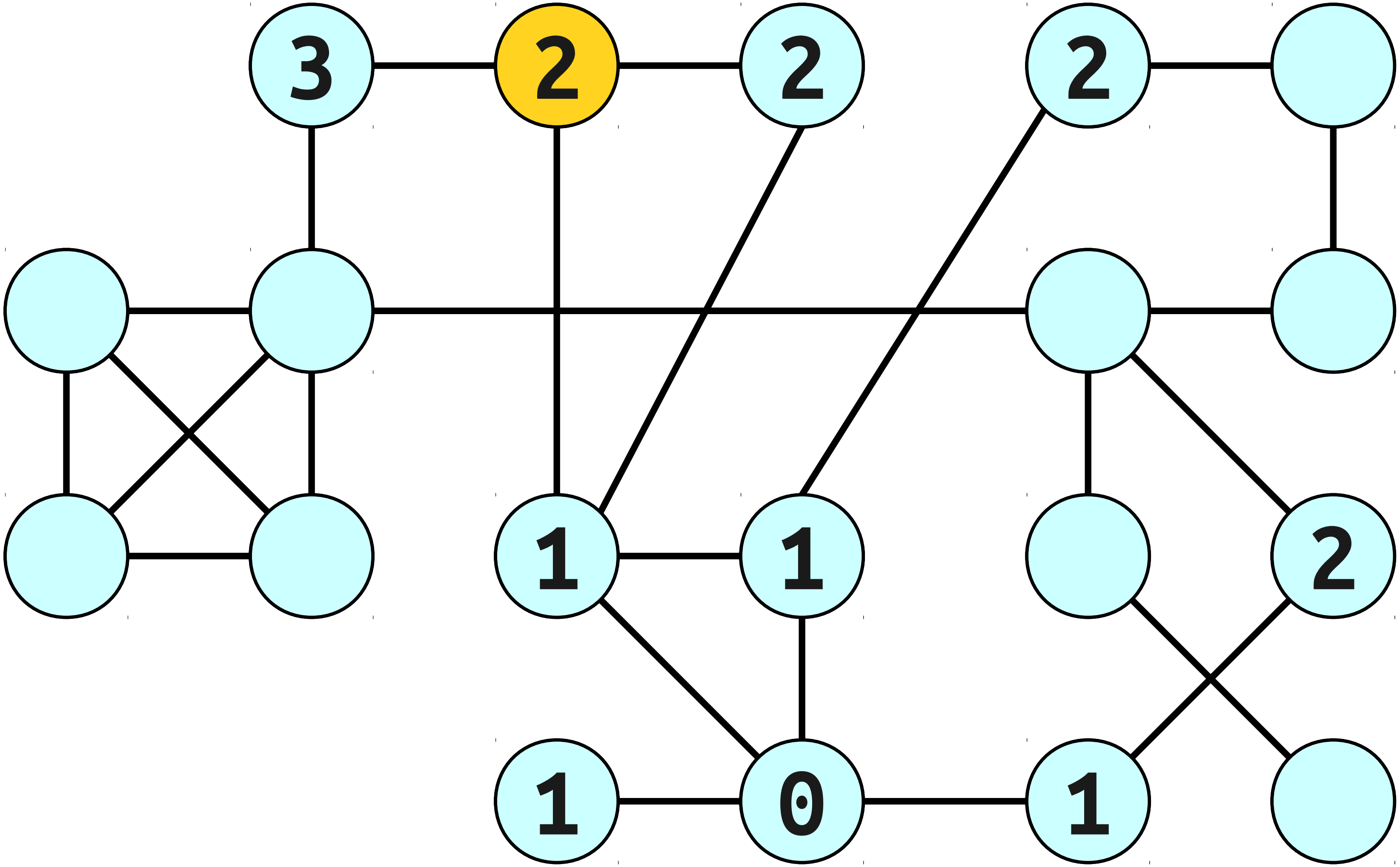

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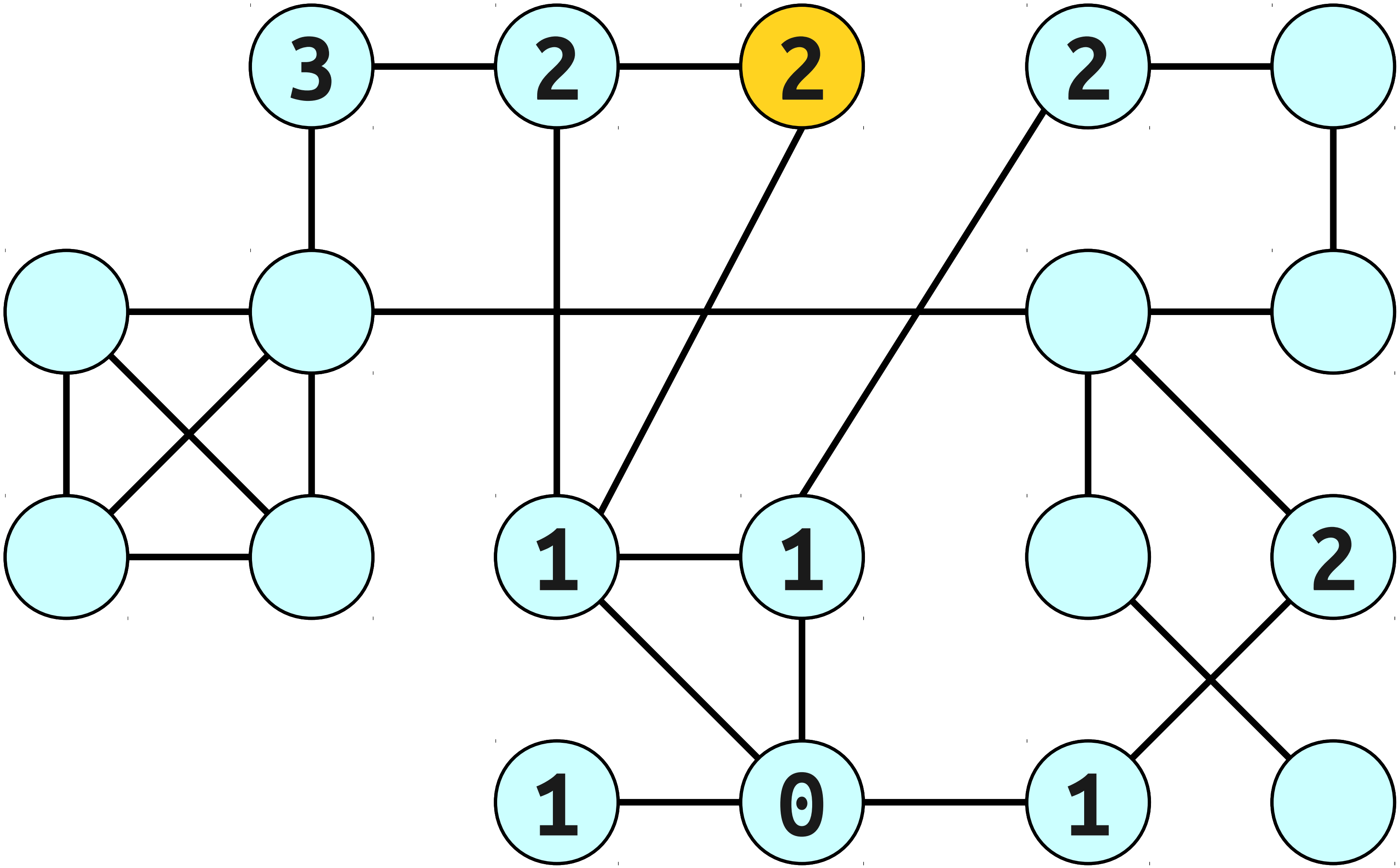

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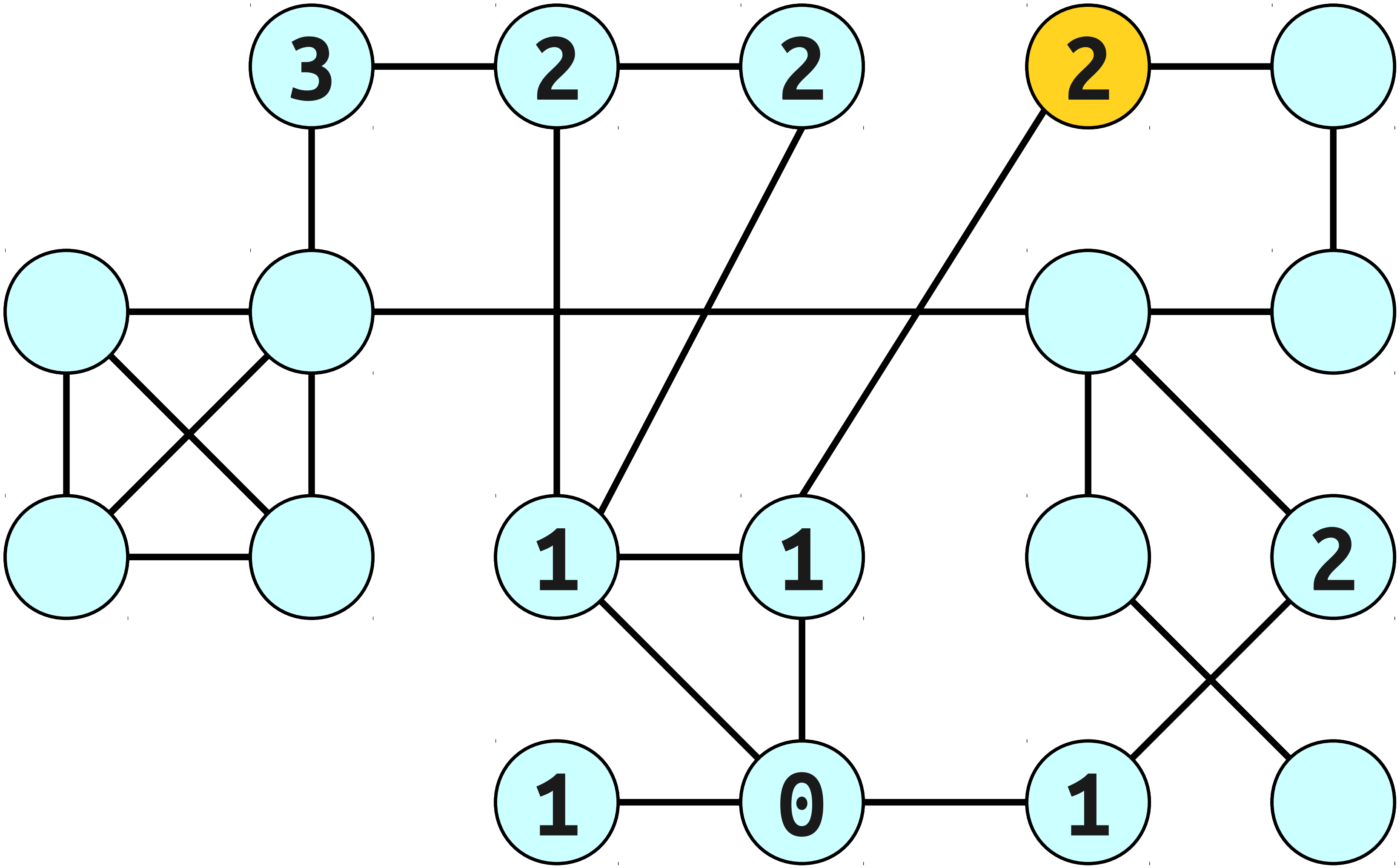

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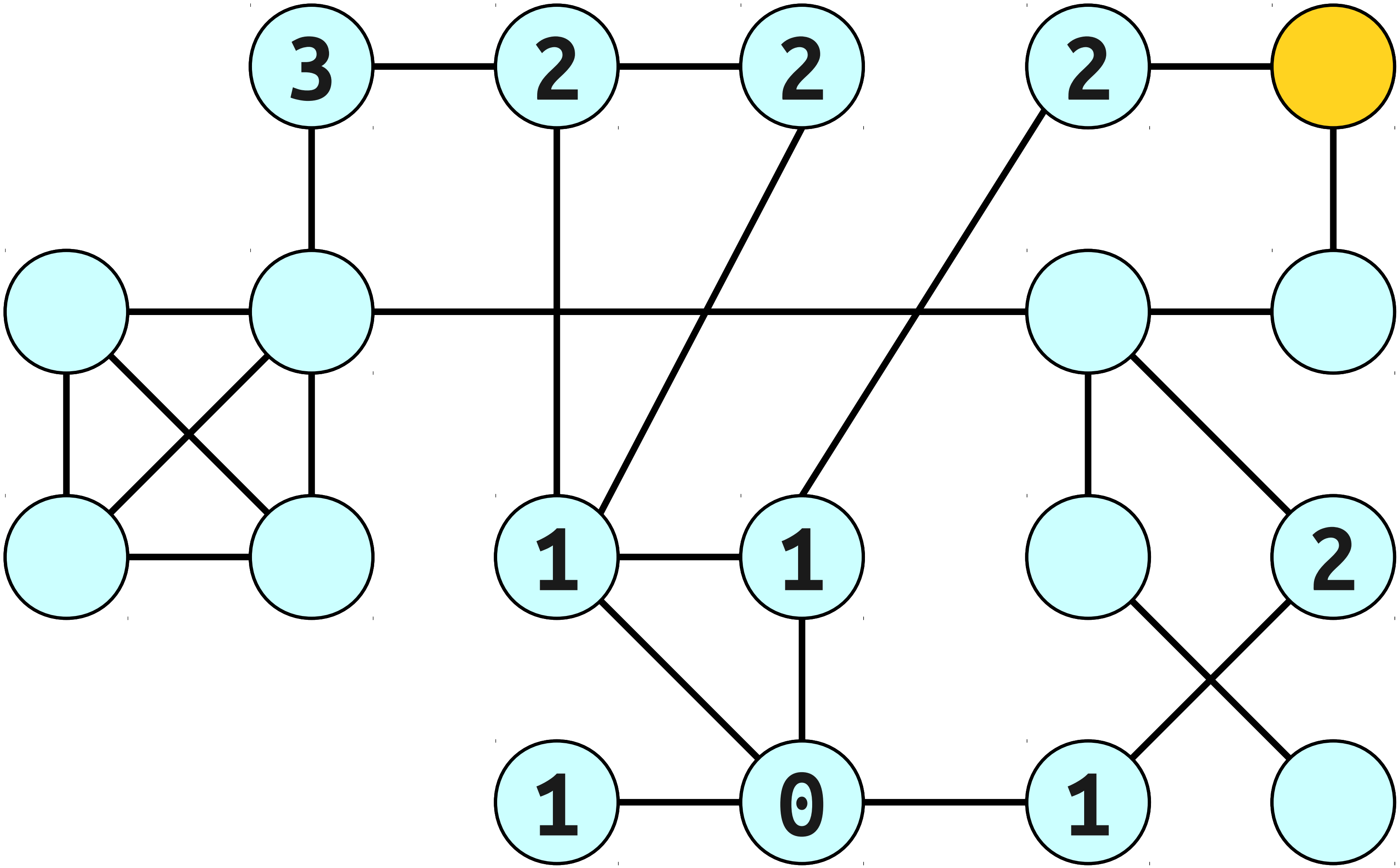

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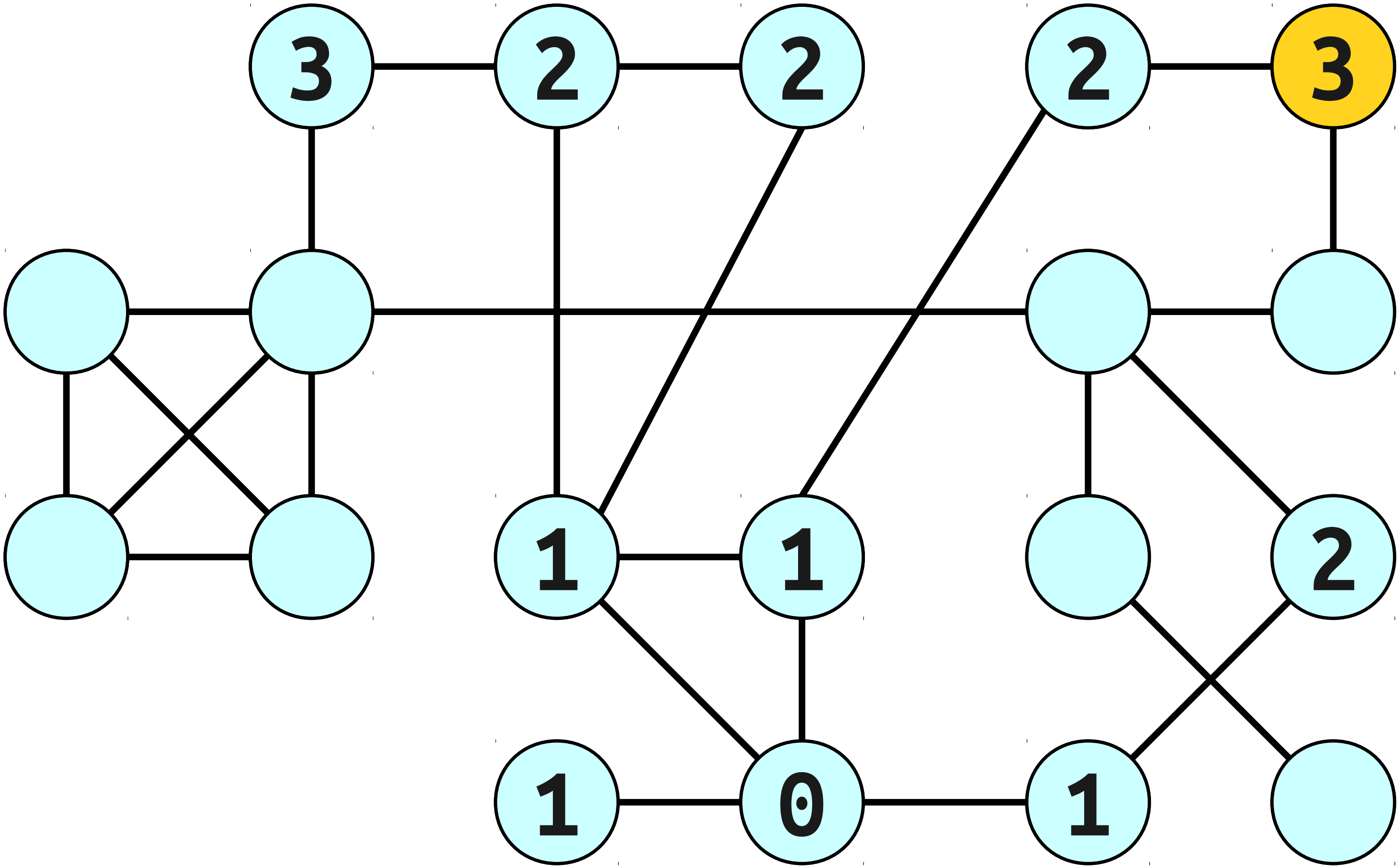

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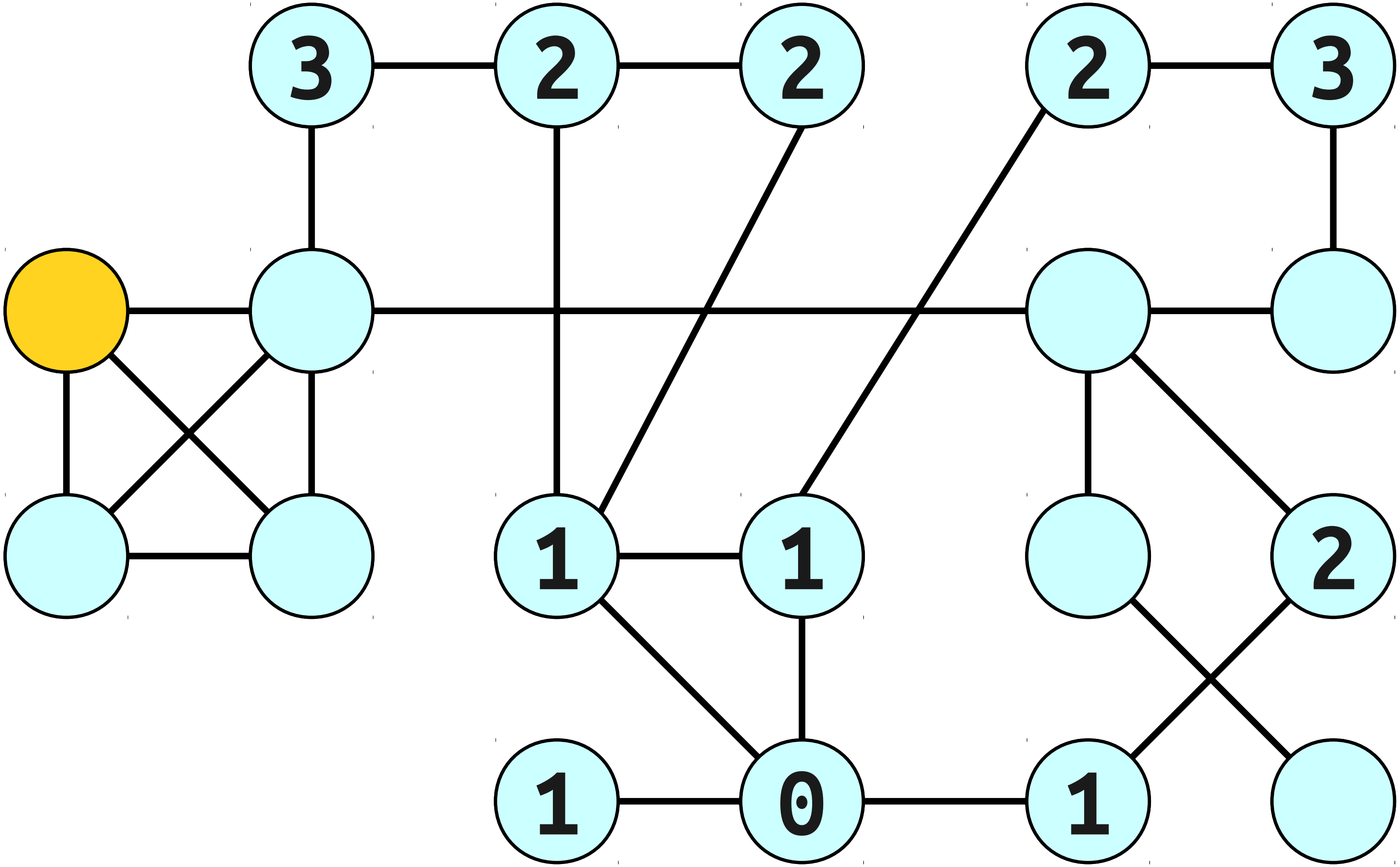

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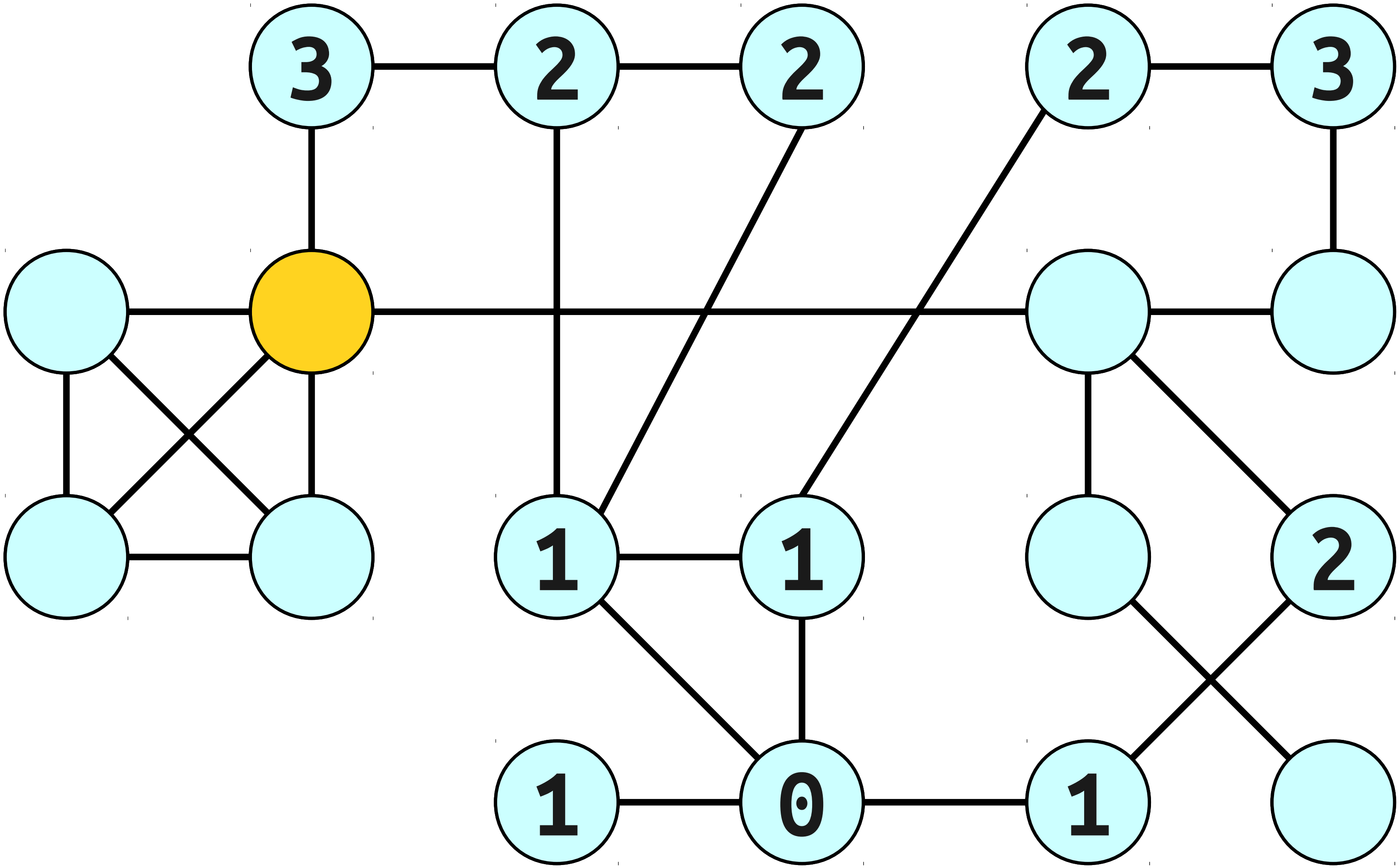

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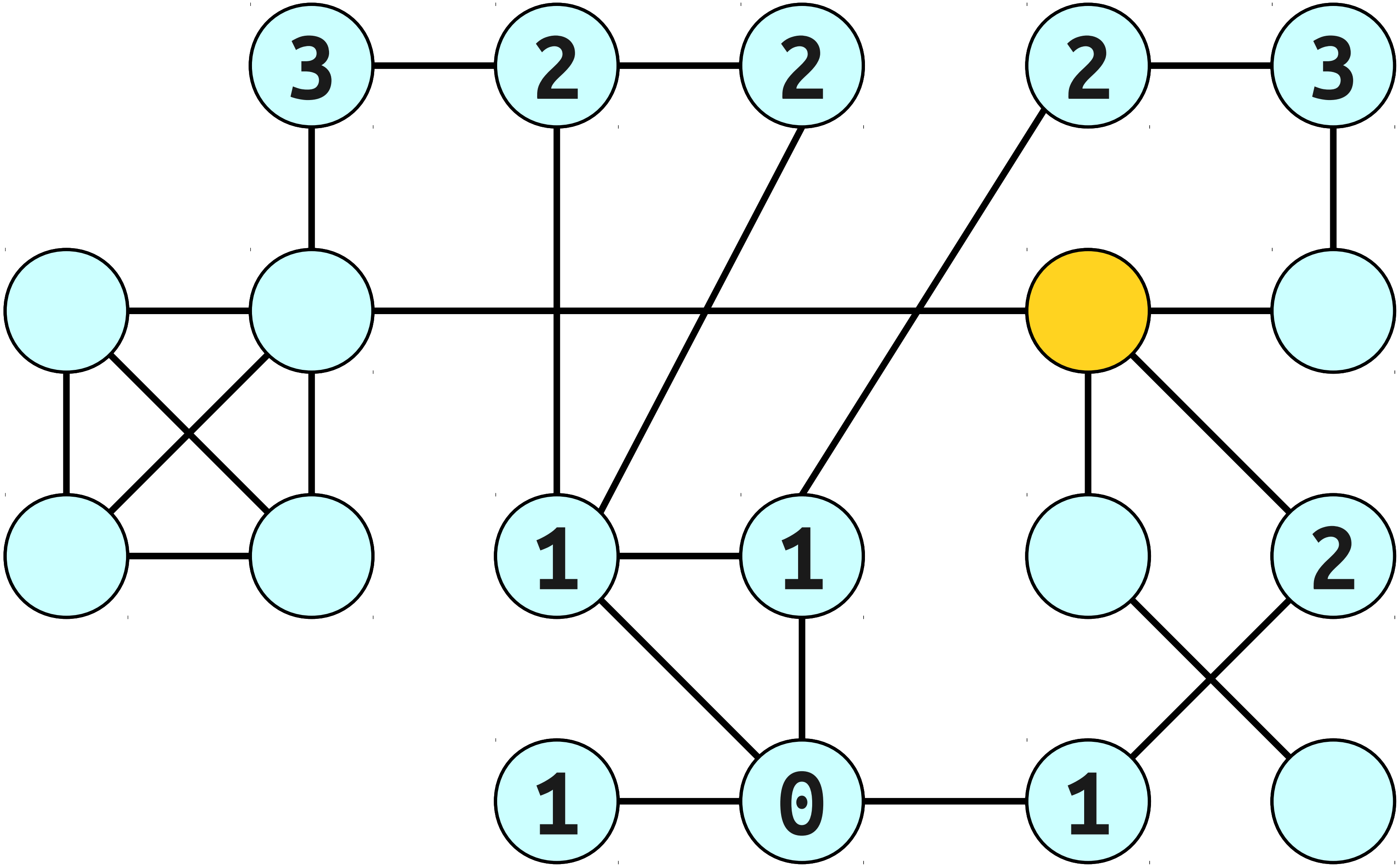

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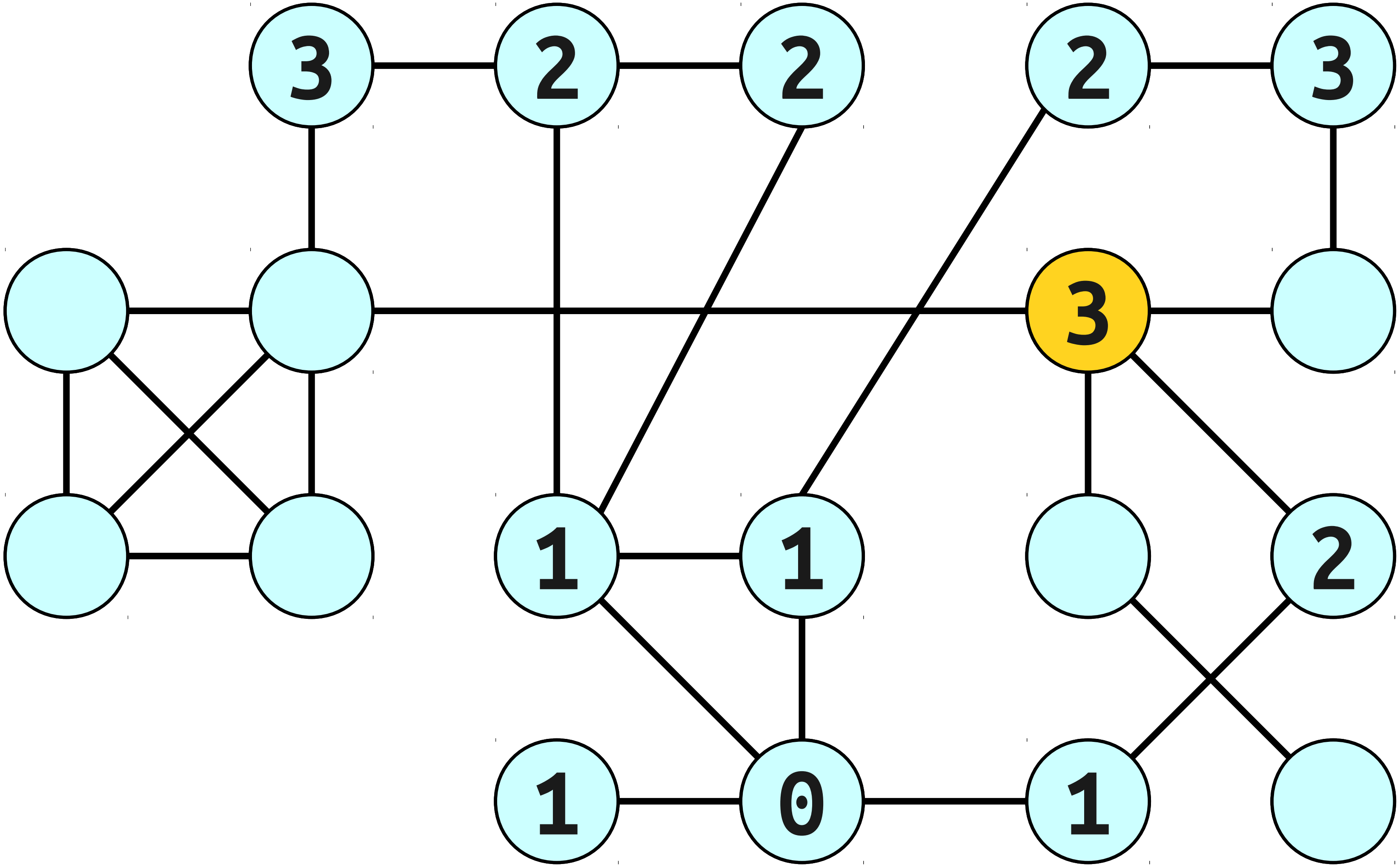

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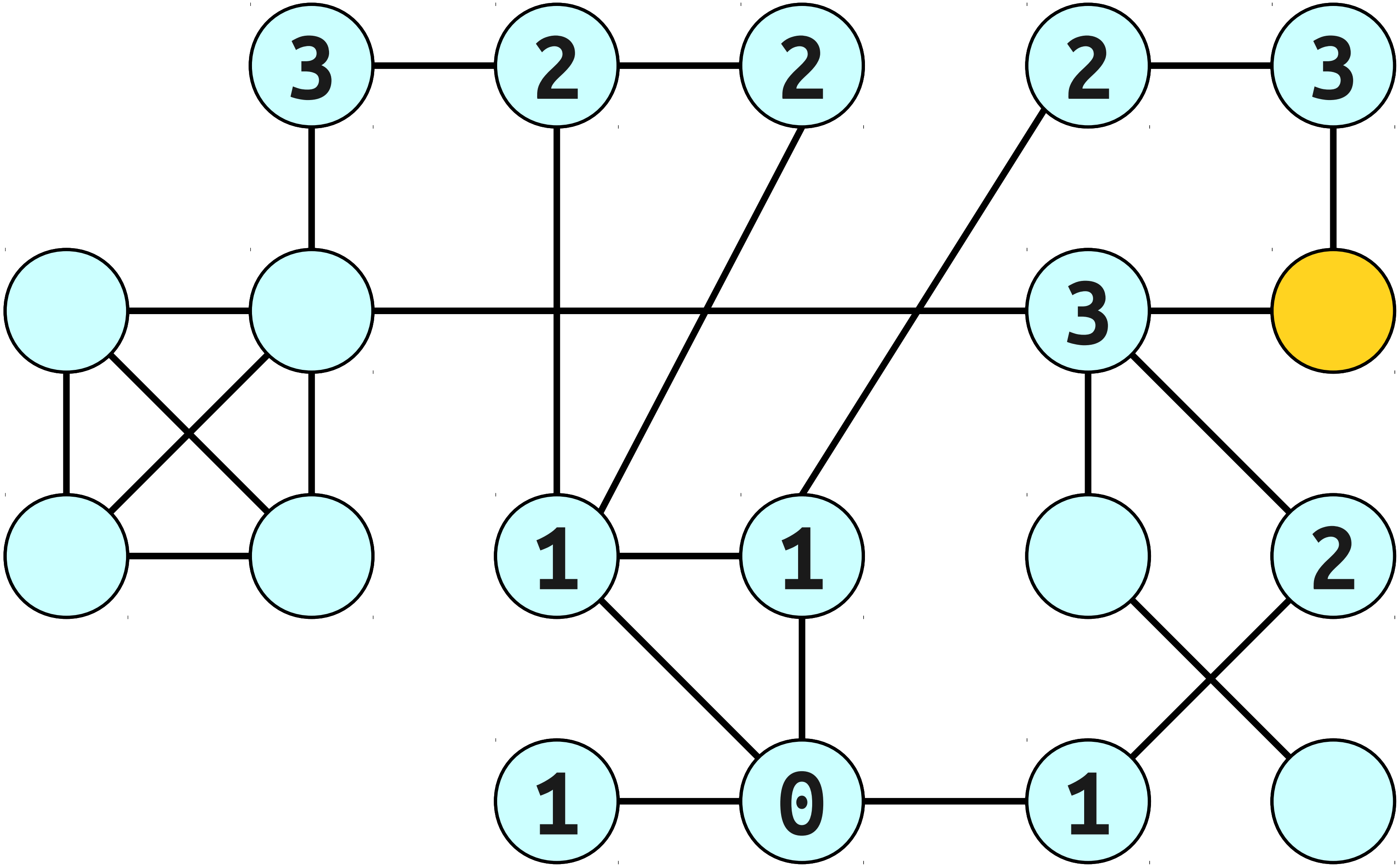

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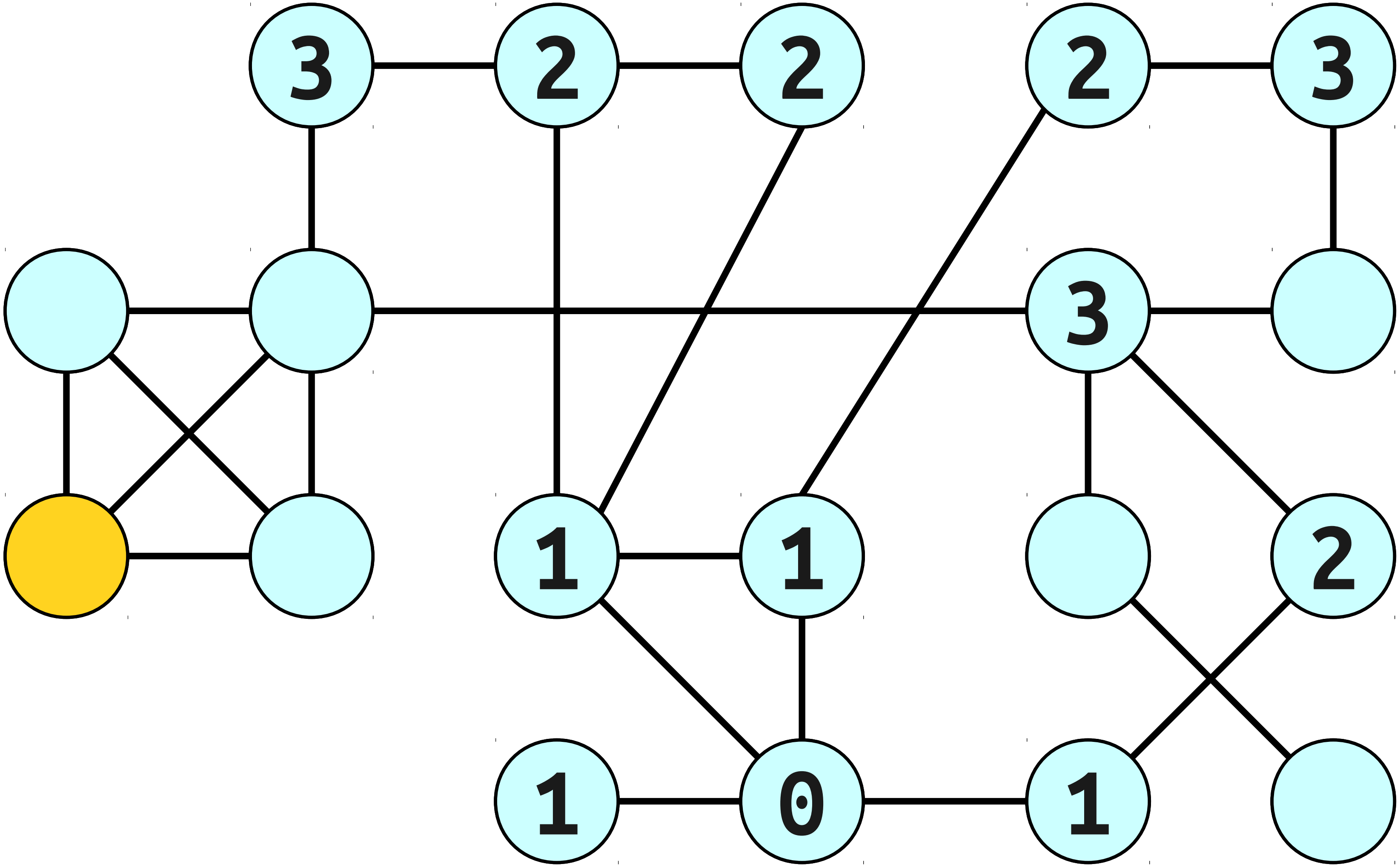

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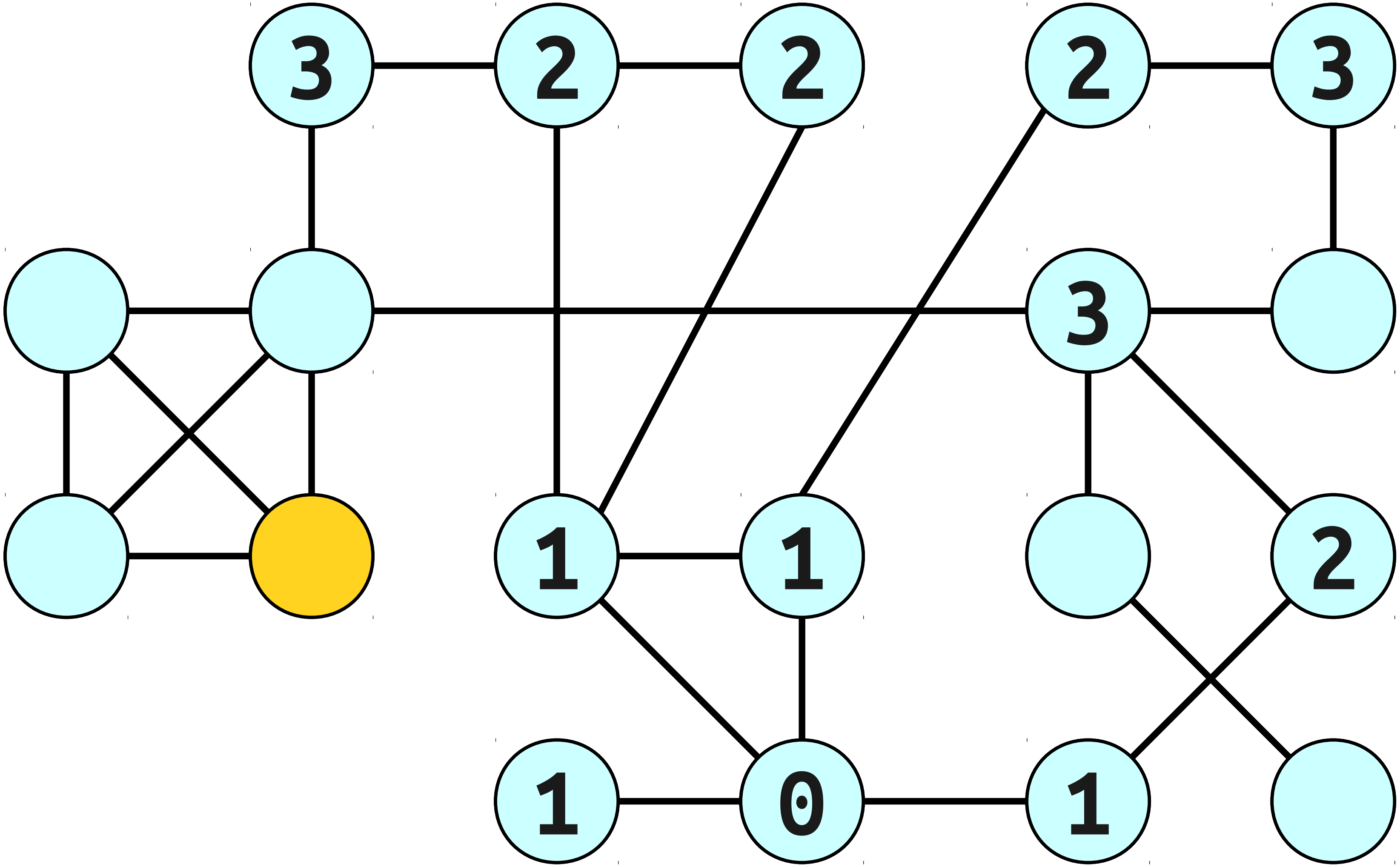

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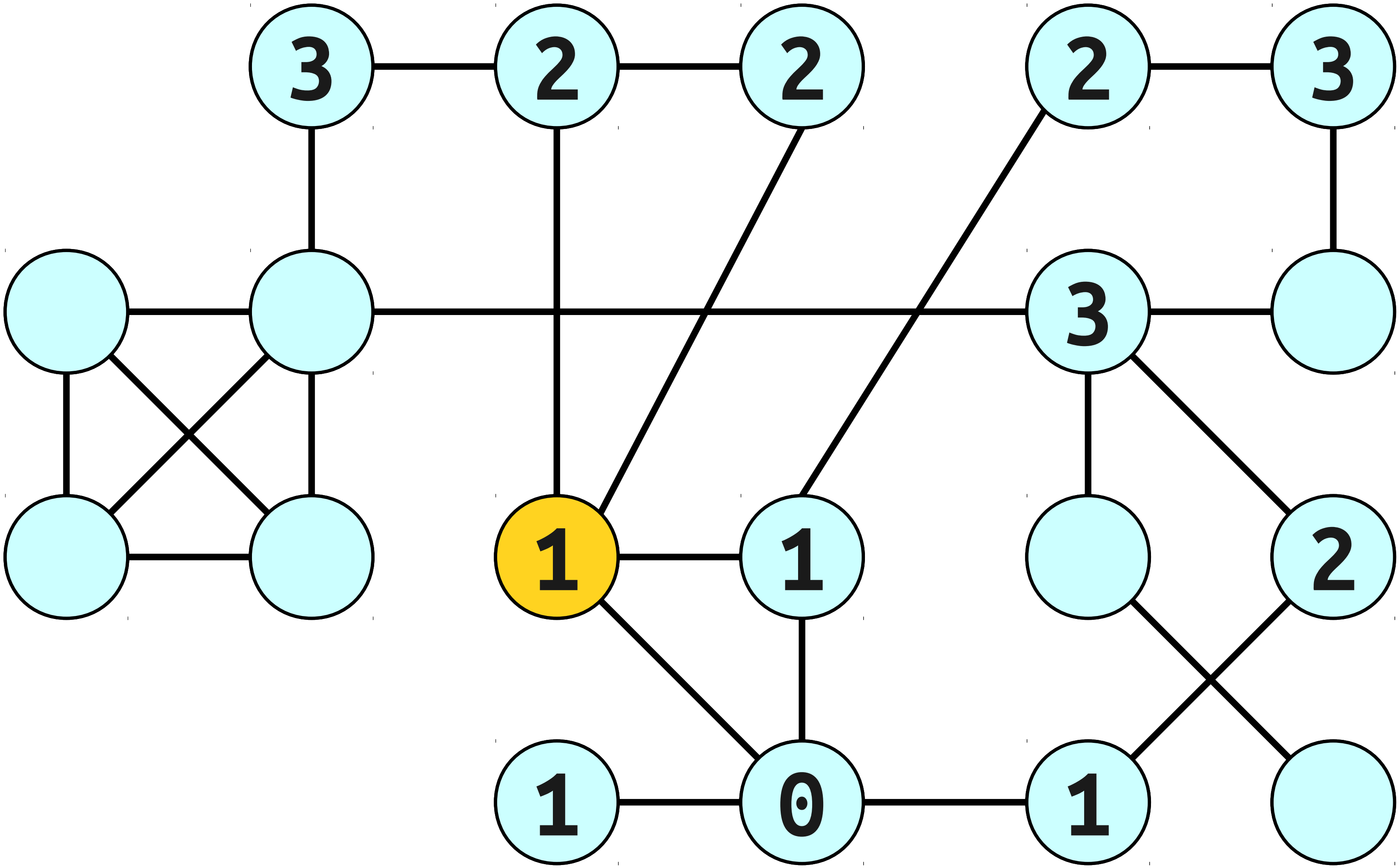

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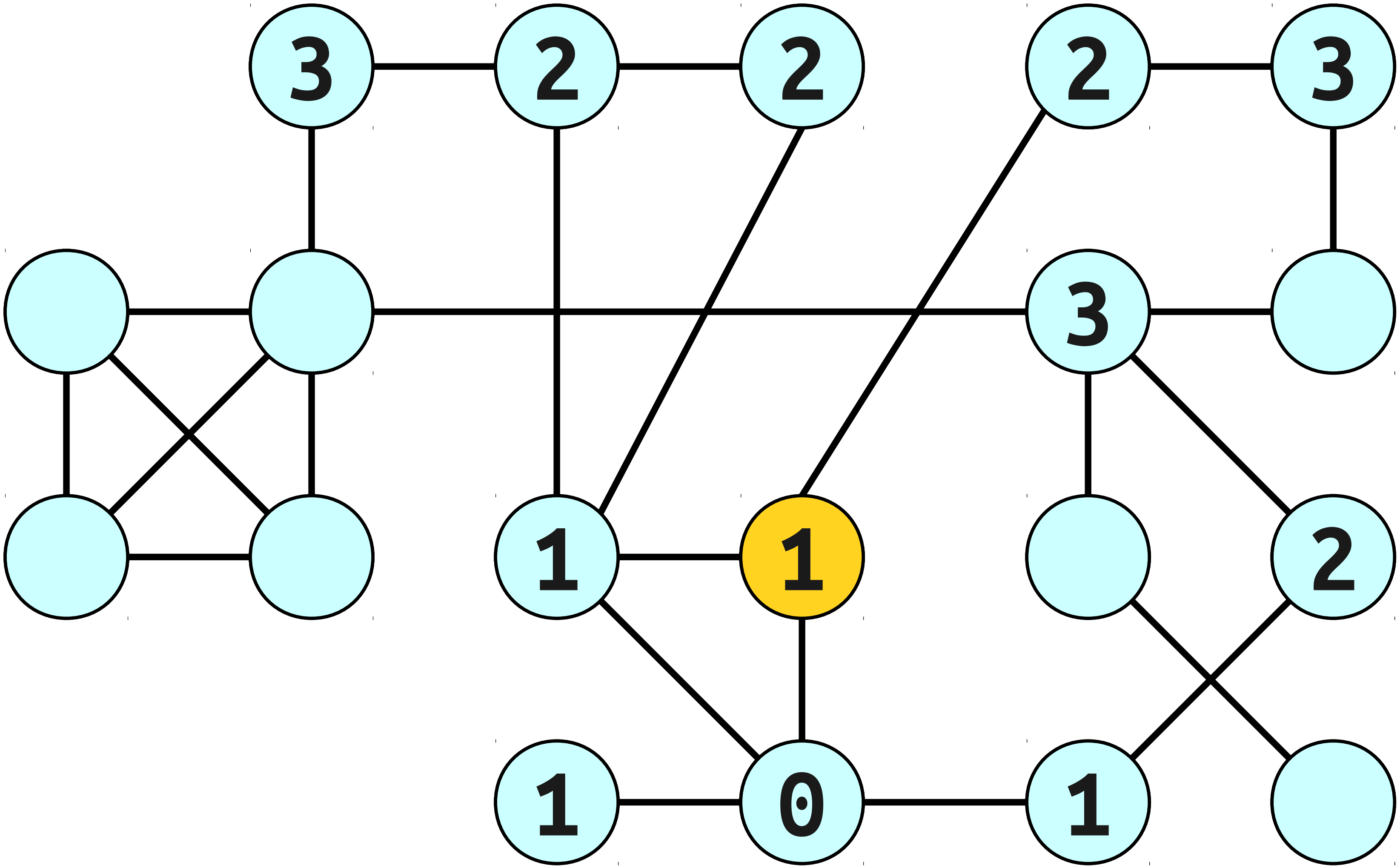

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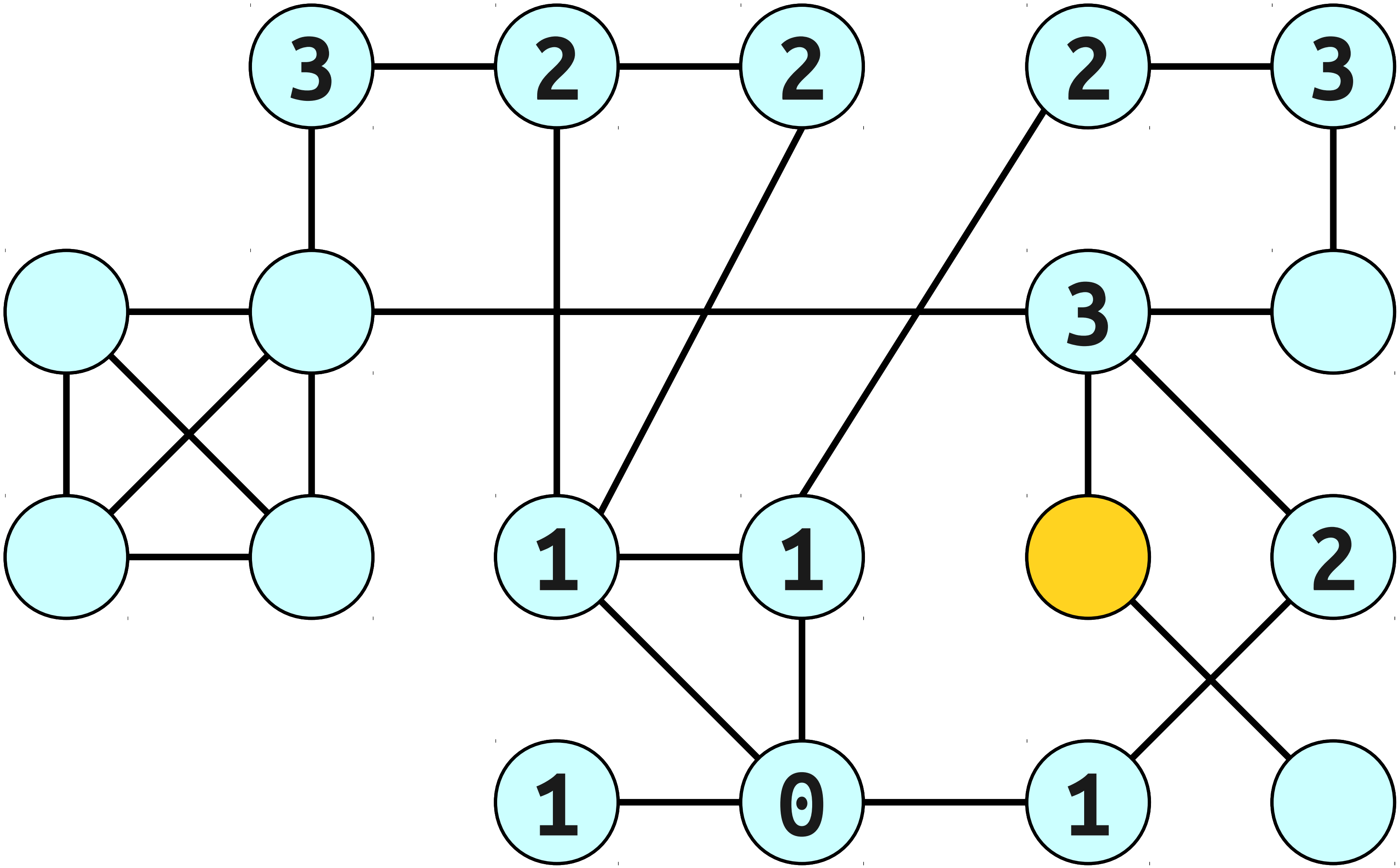

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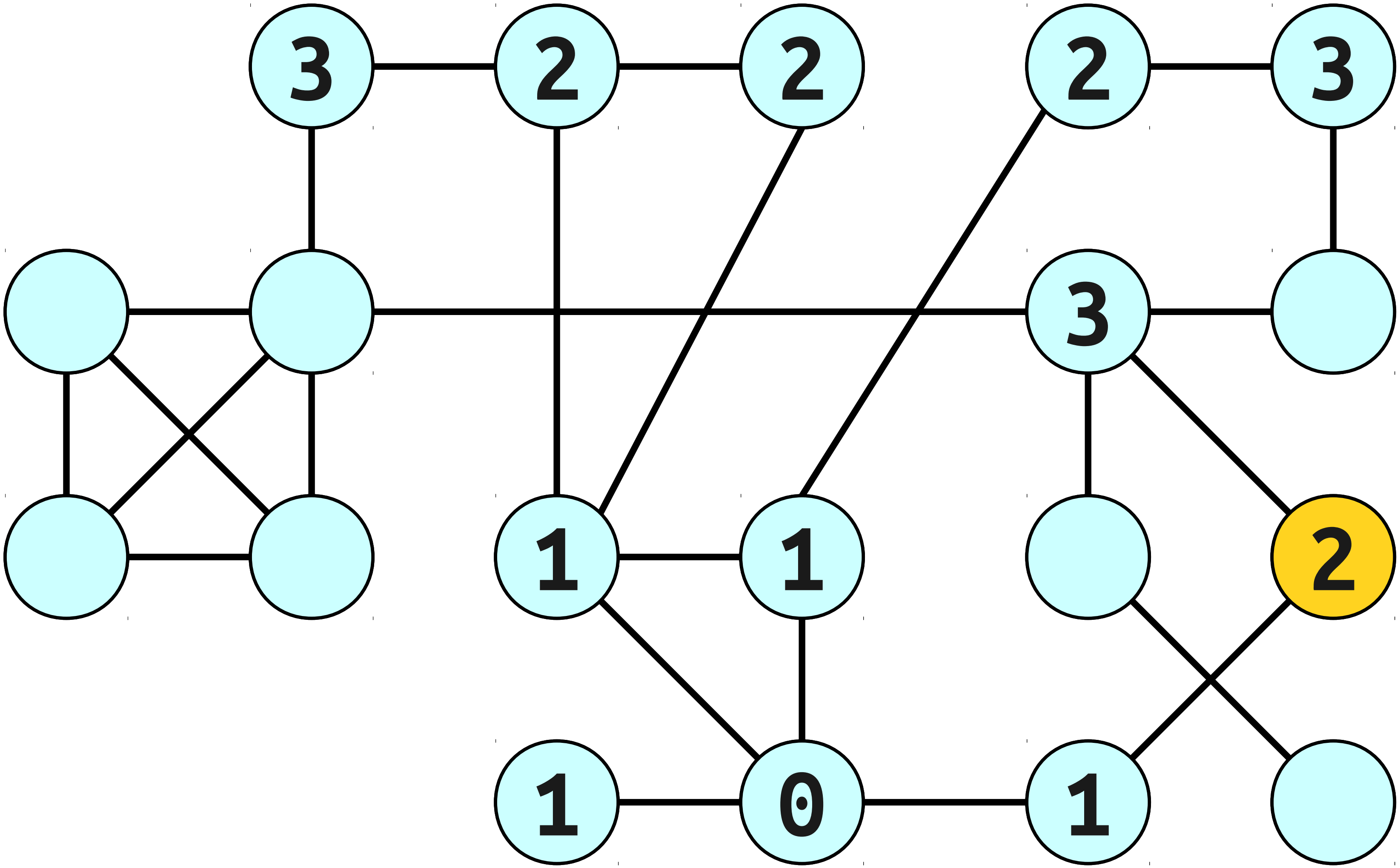

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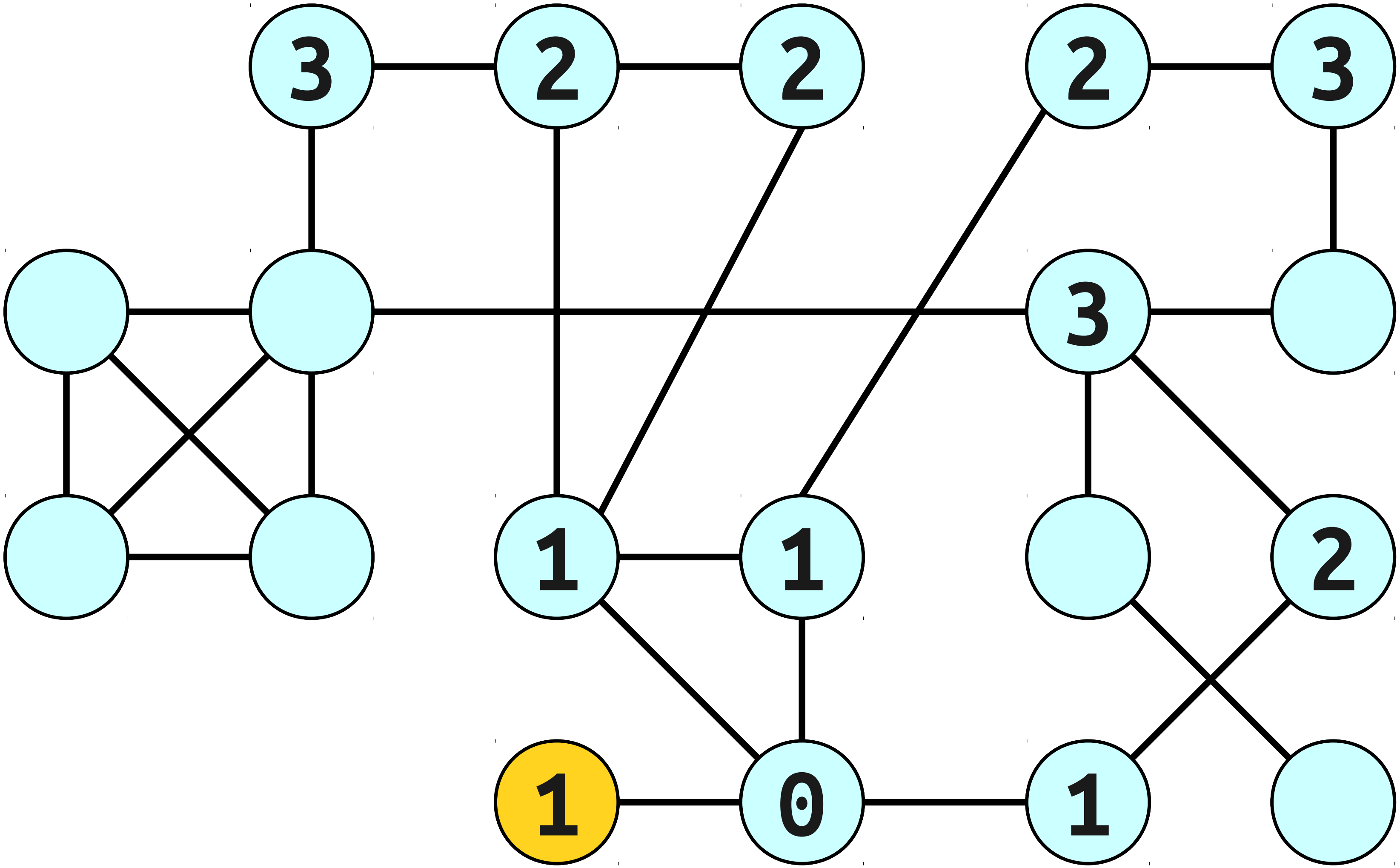

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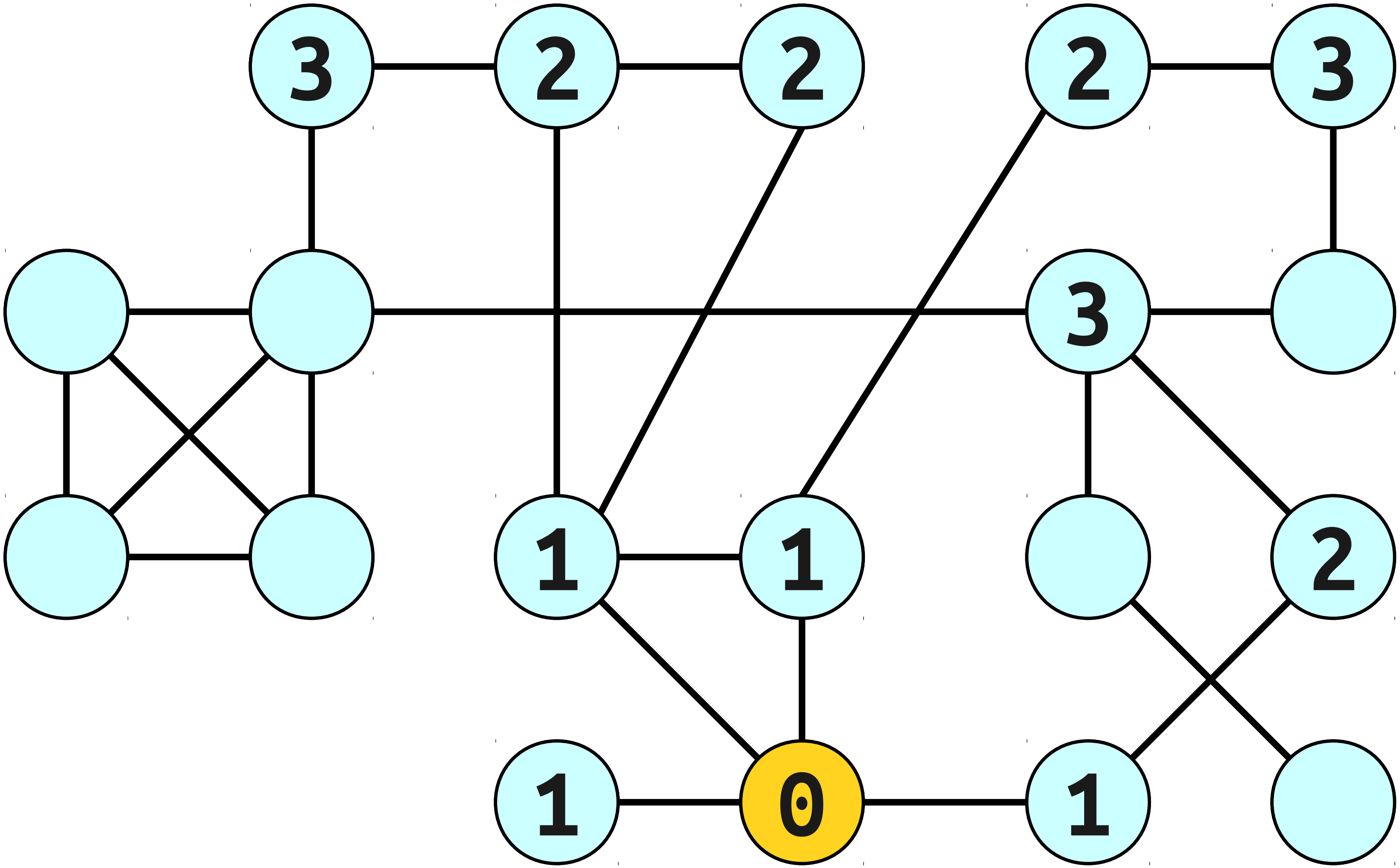

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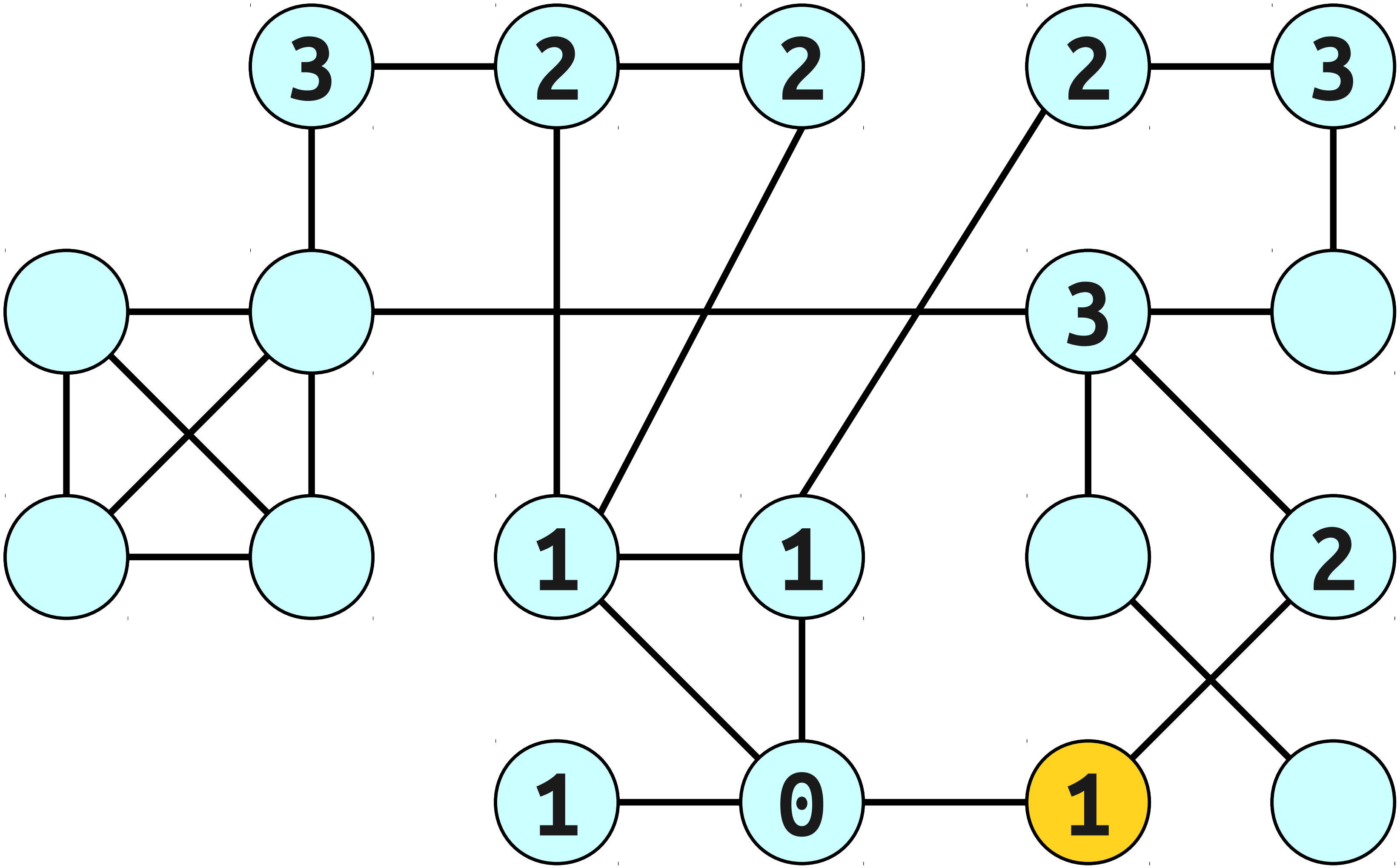

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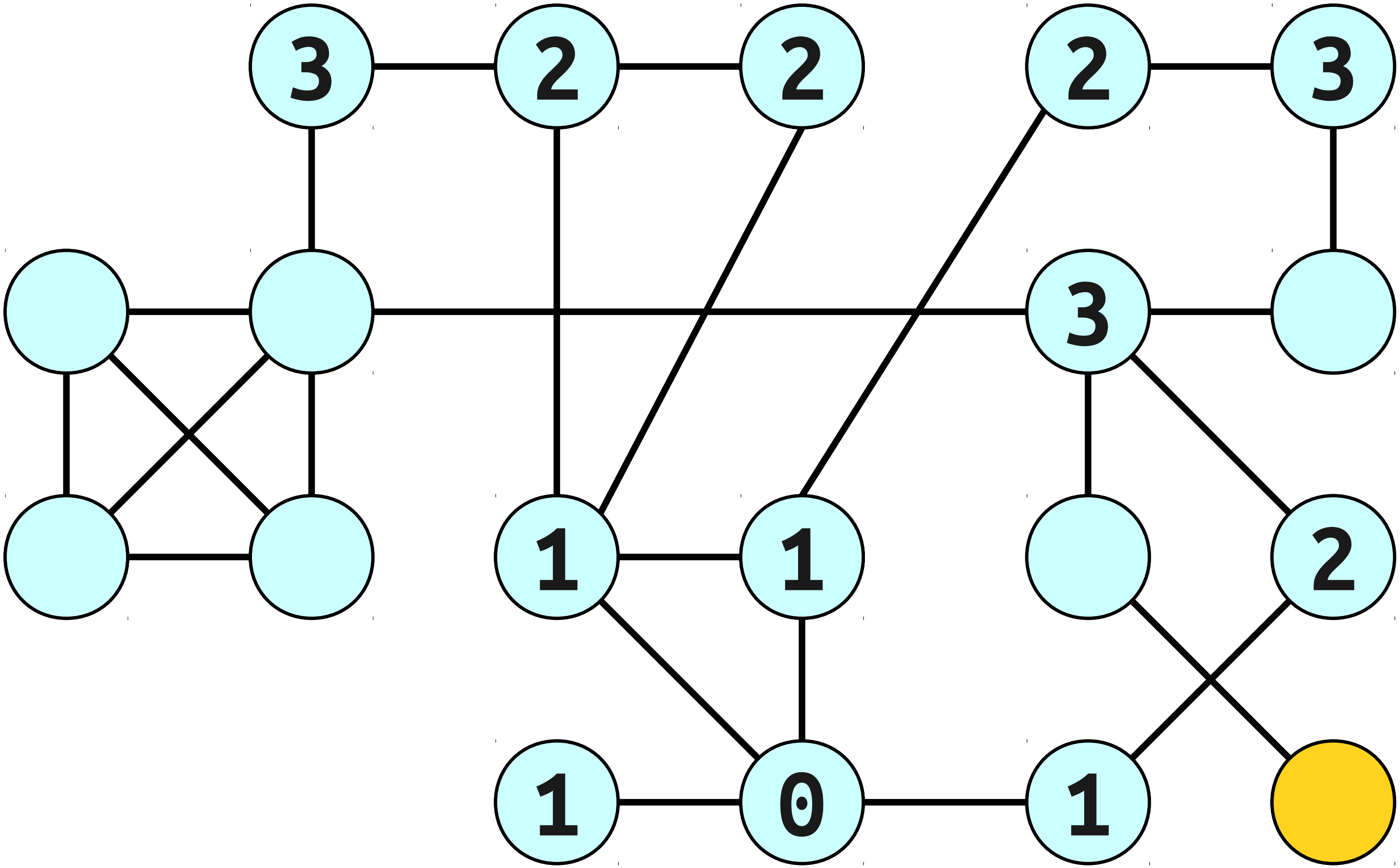

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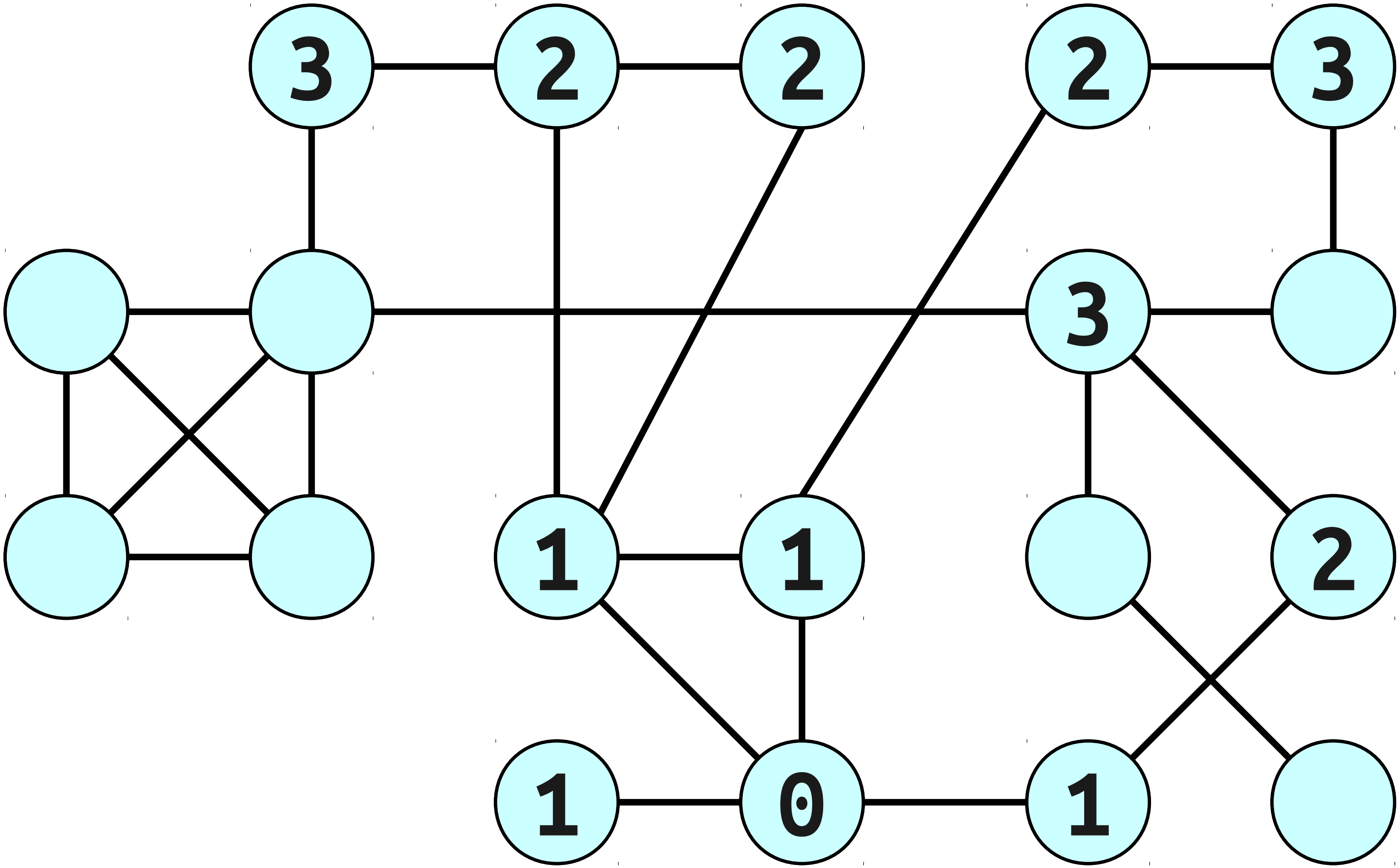
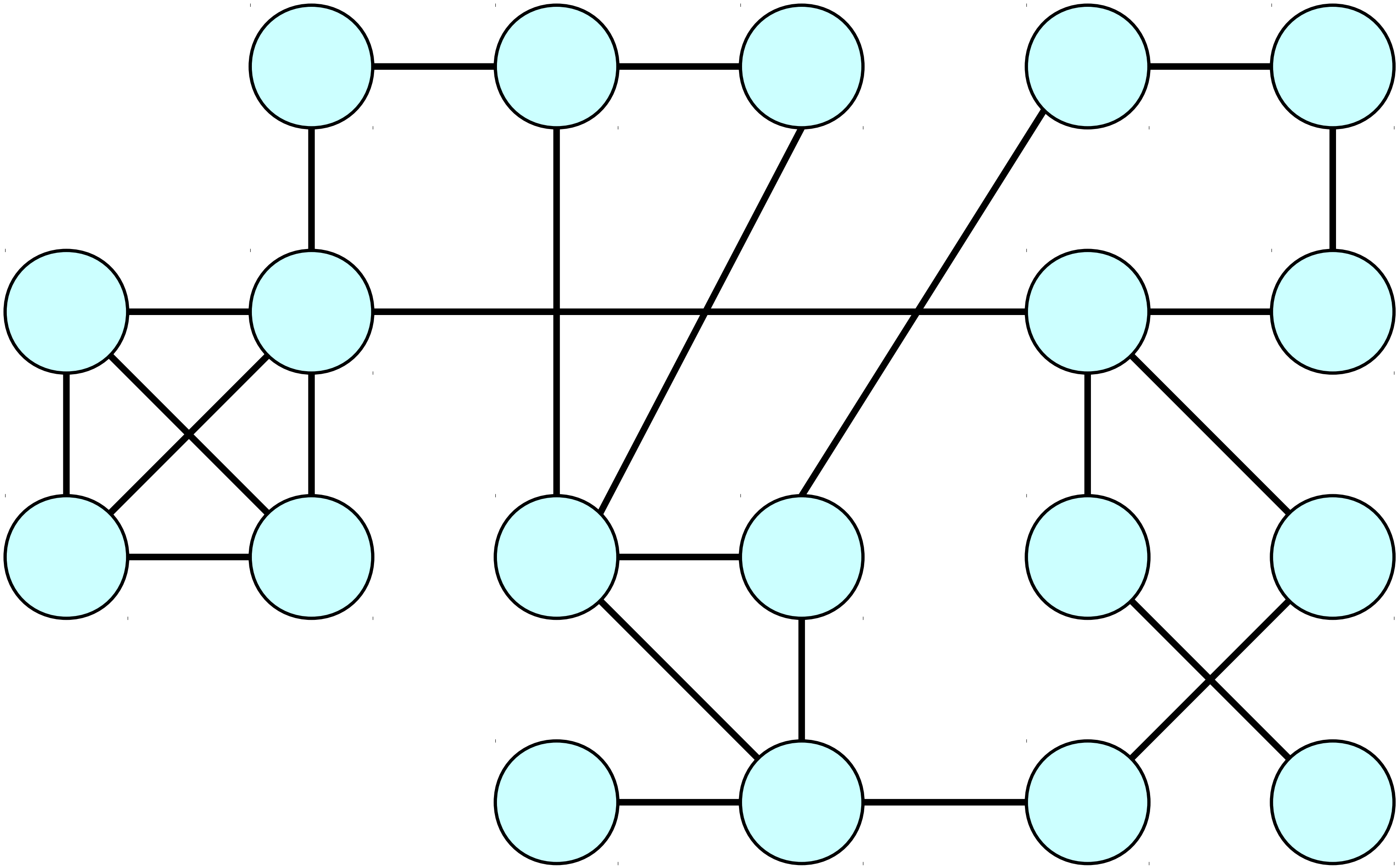
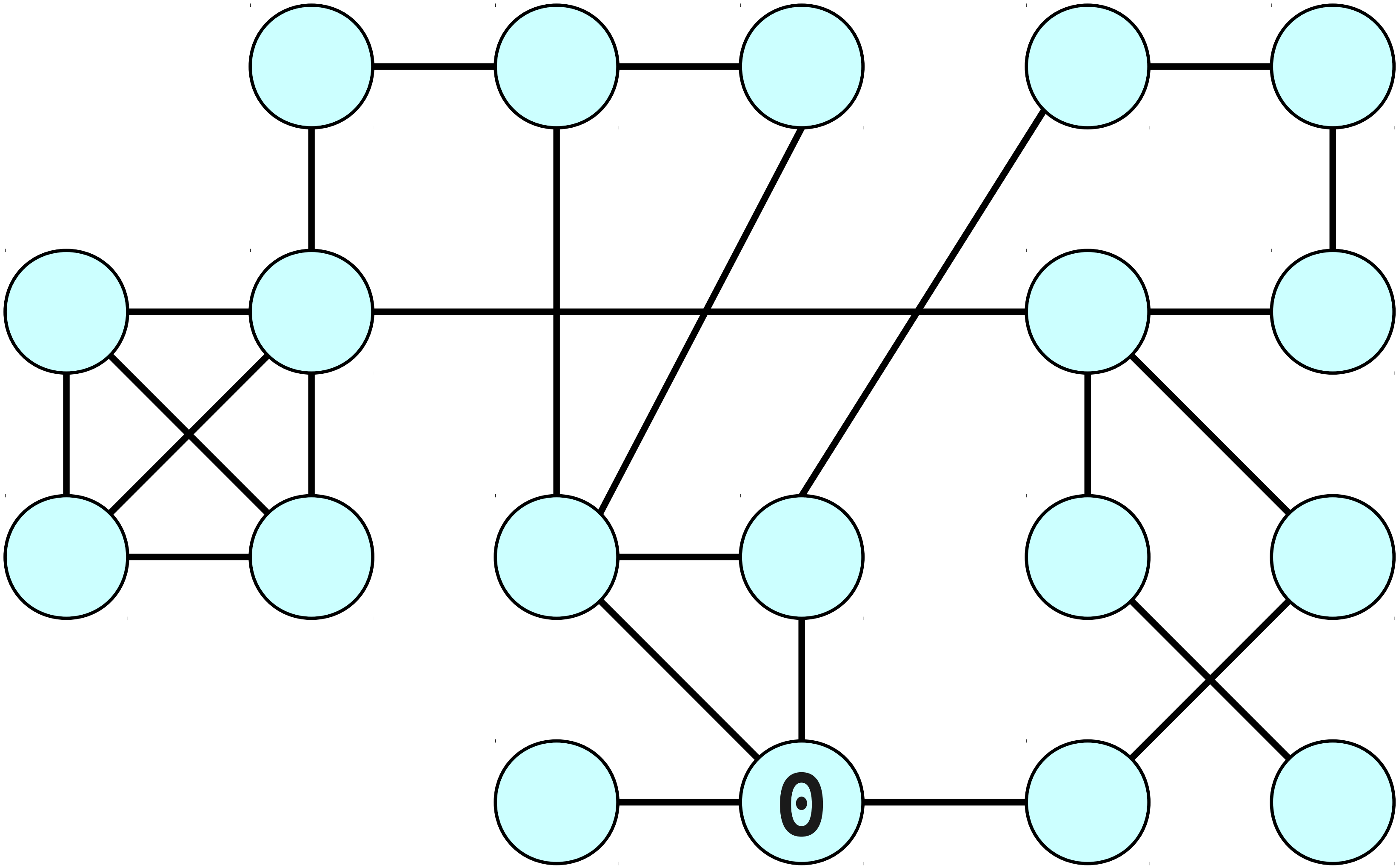


A Better Approach

Radiating Outward



Radiating Outward



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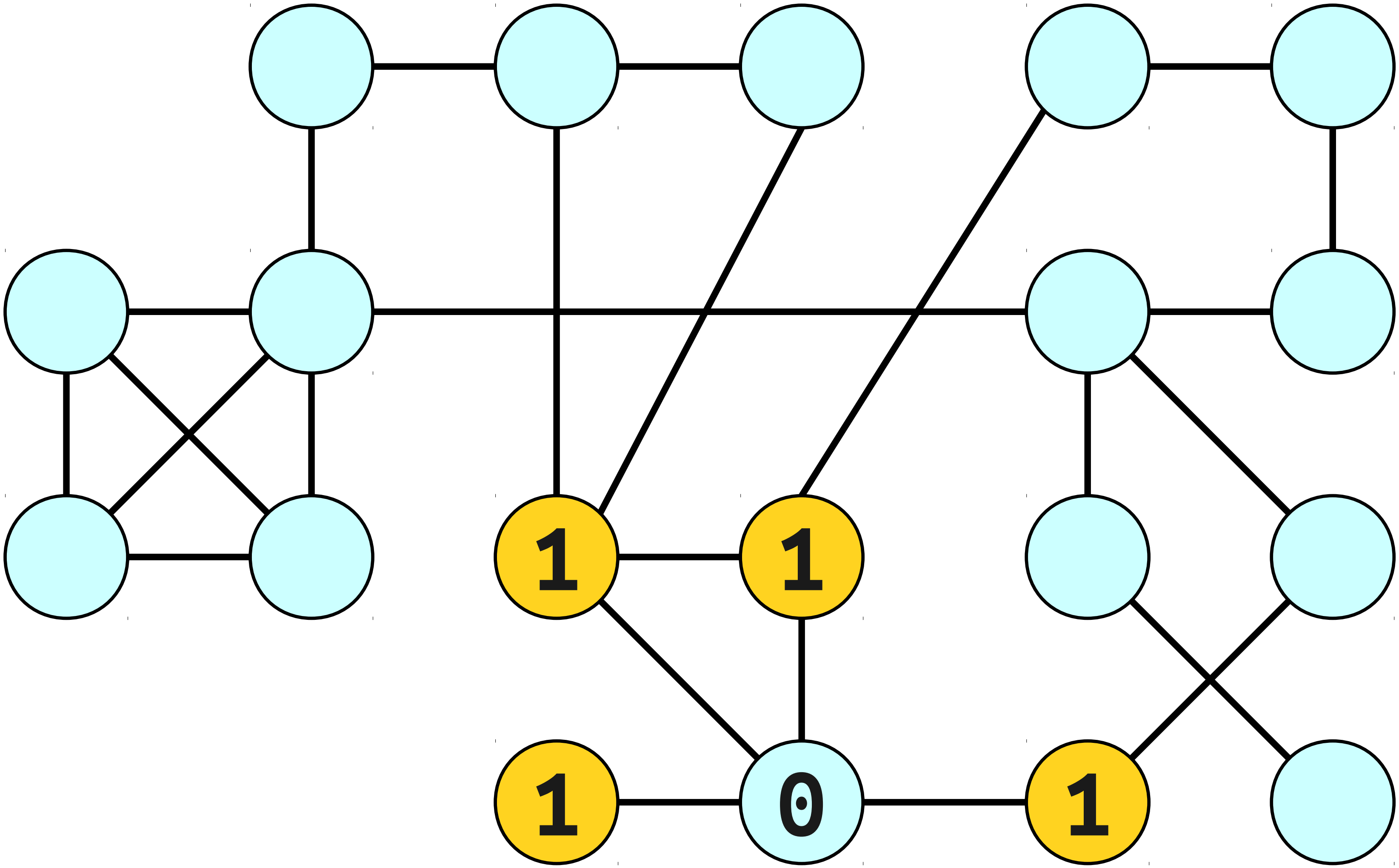

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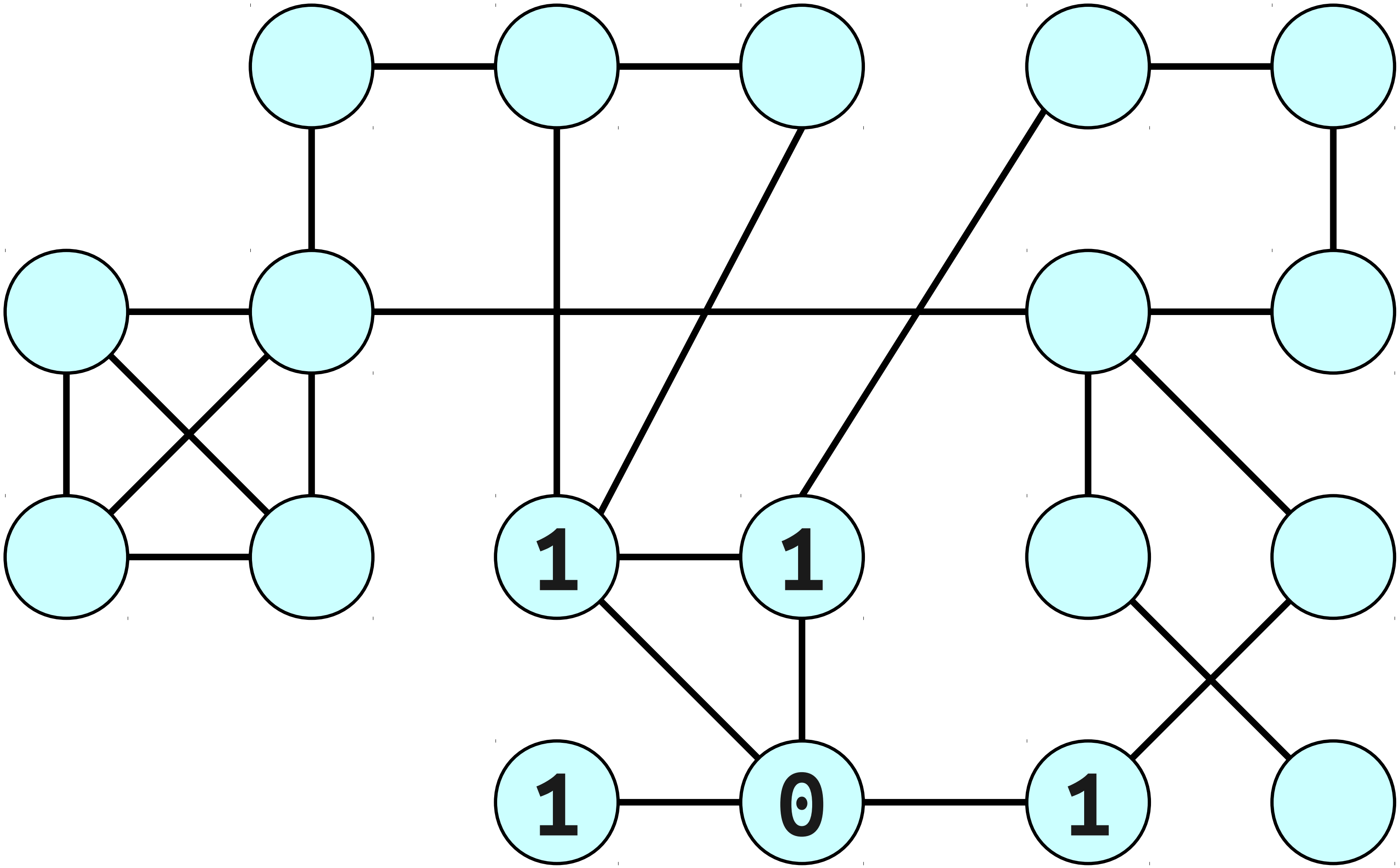

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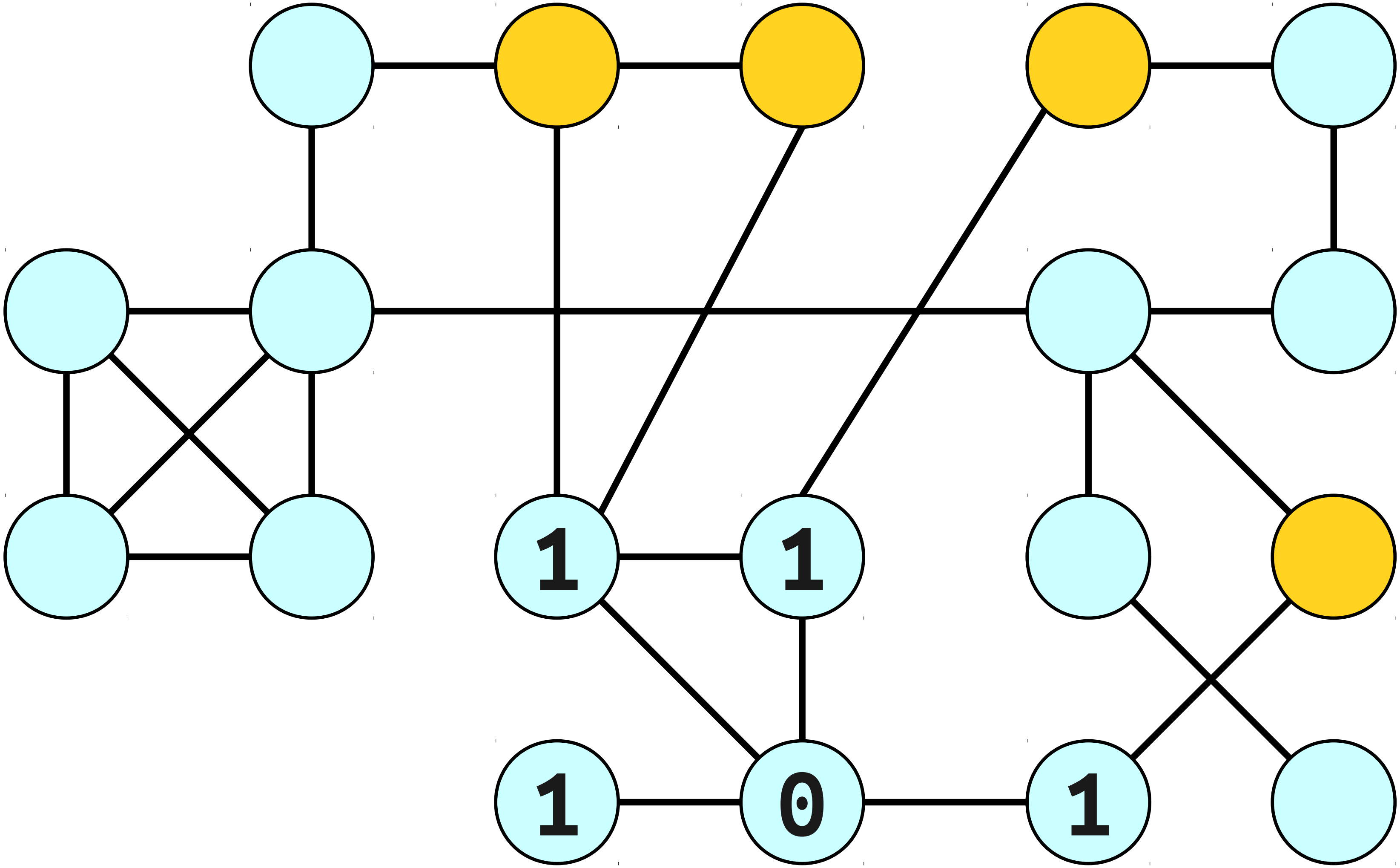

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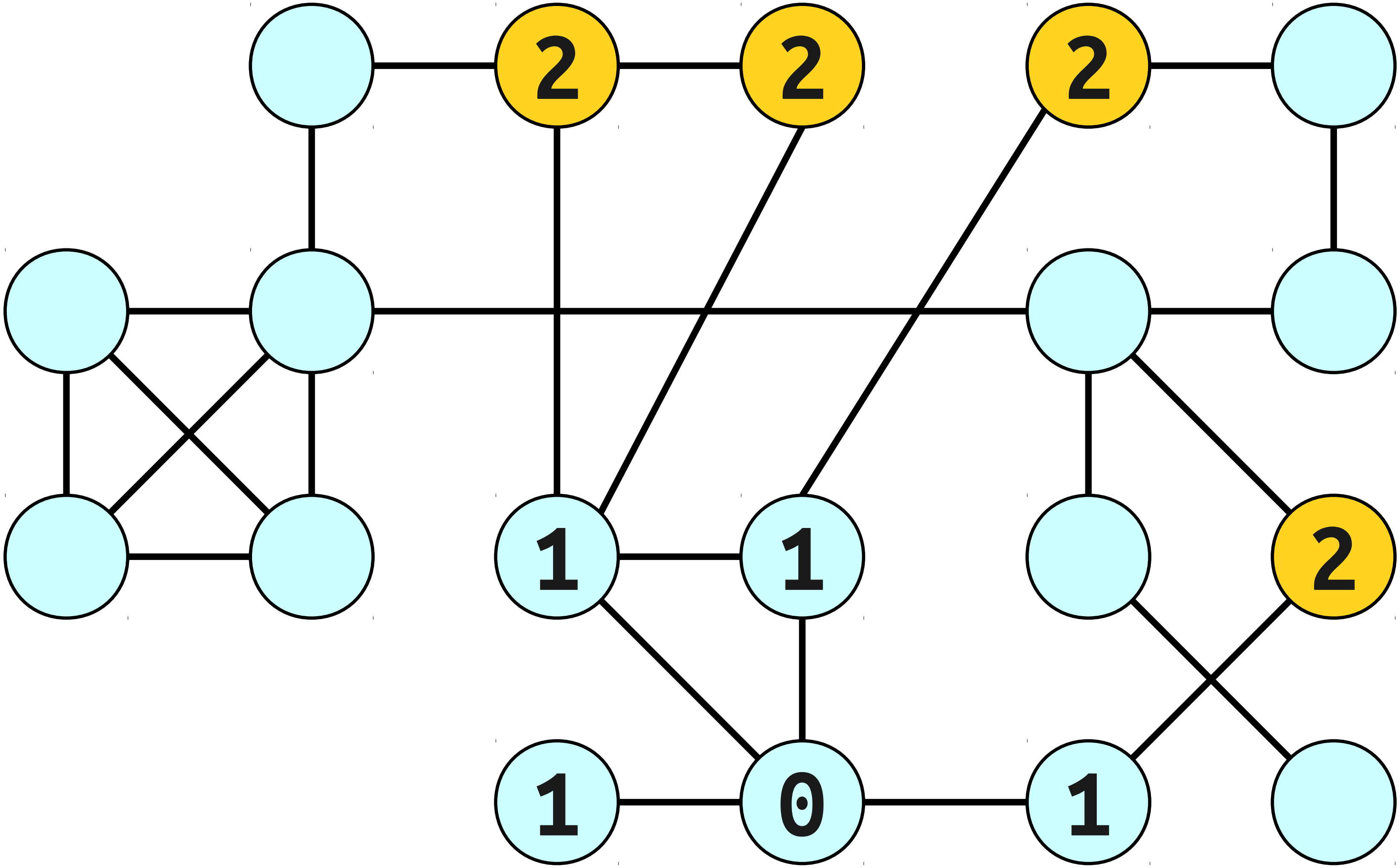

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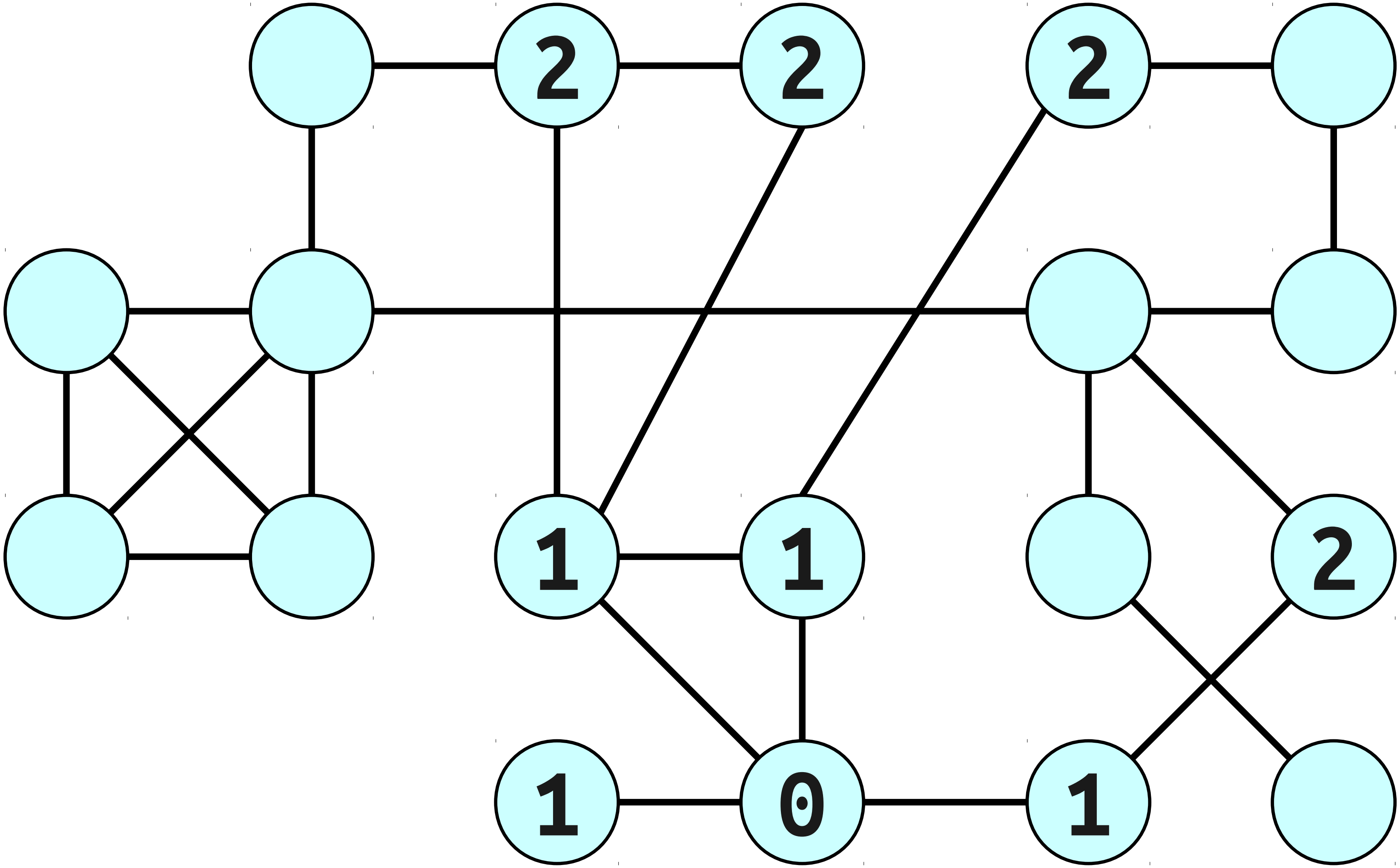

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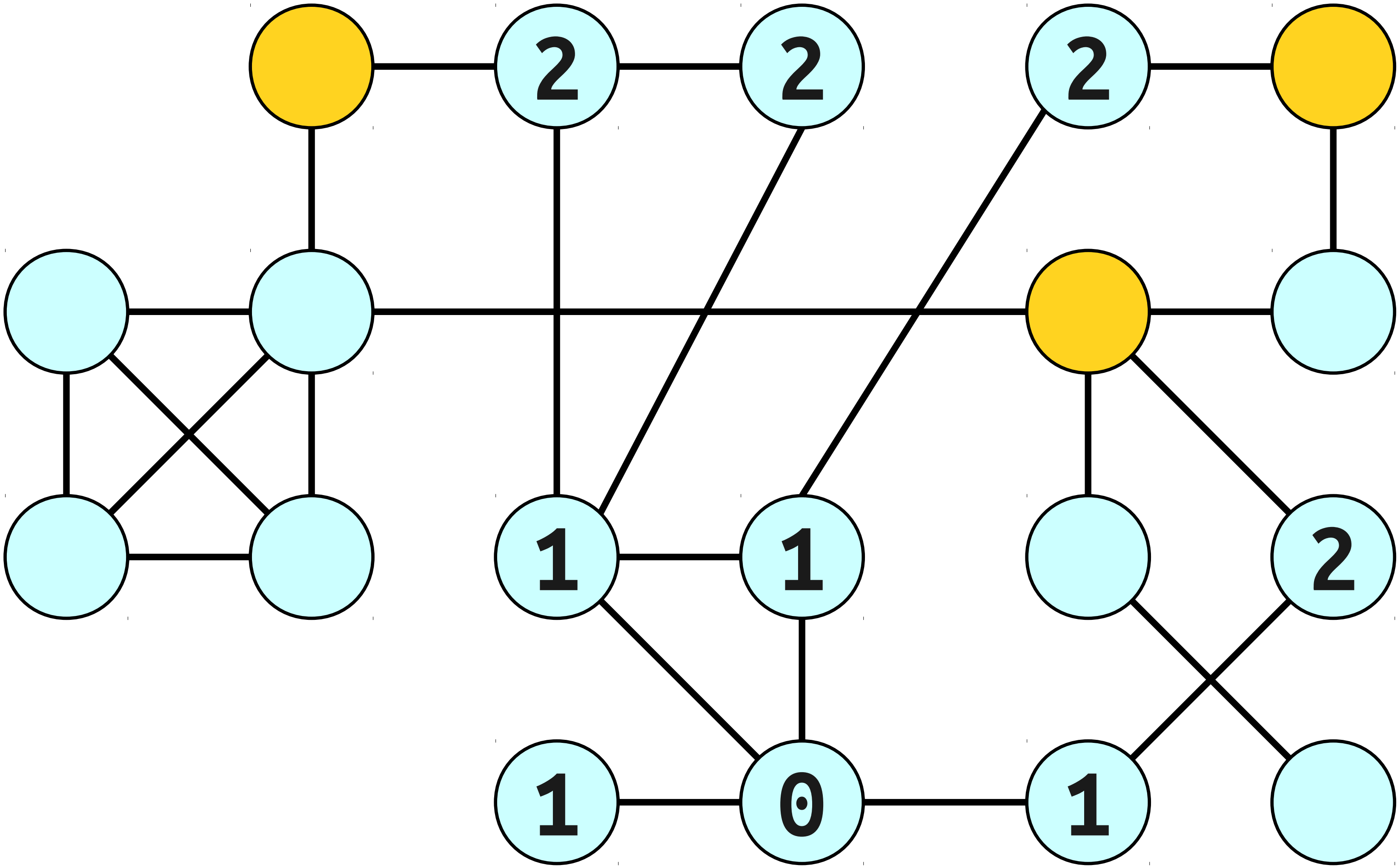

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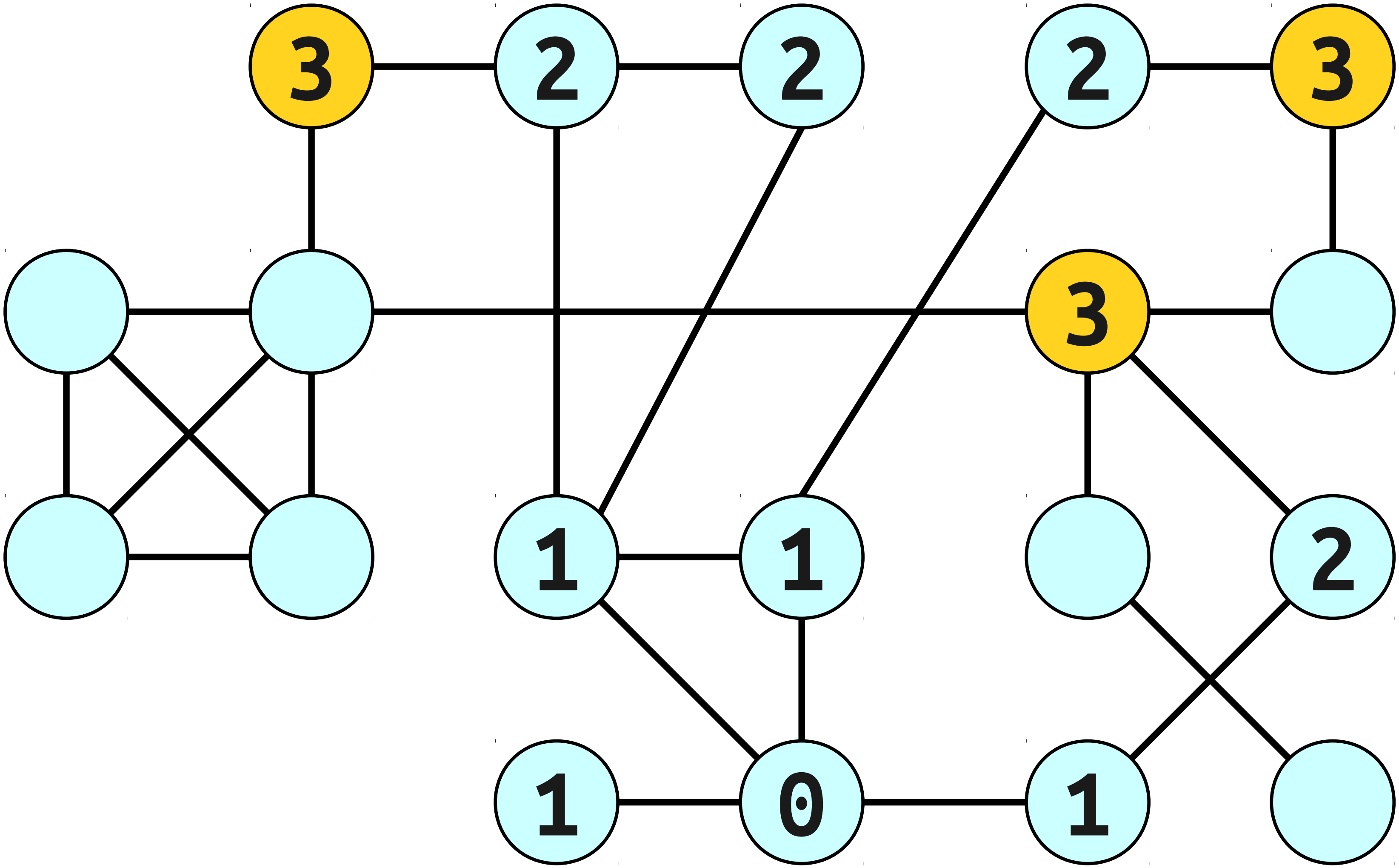

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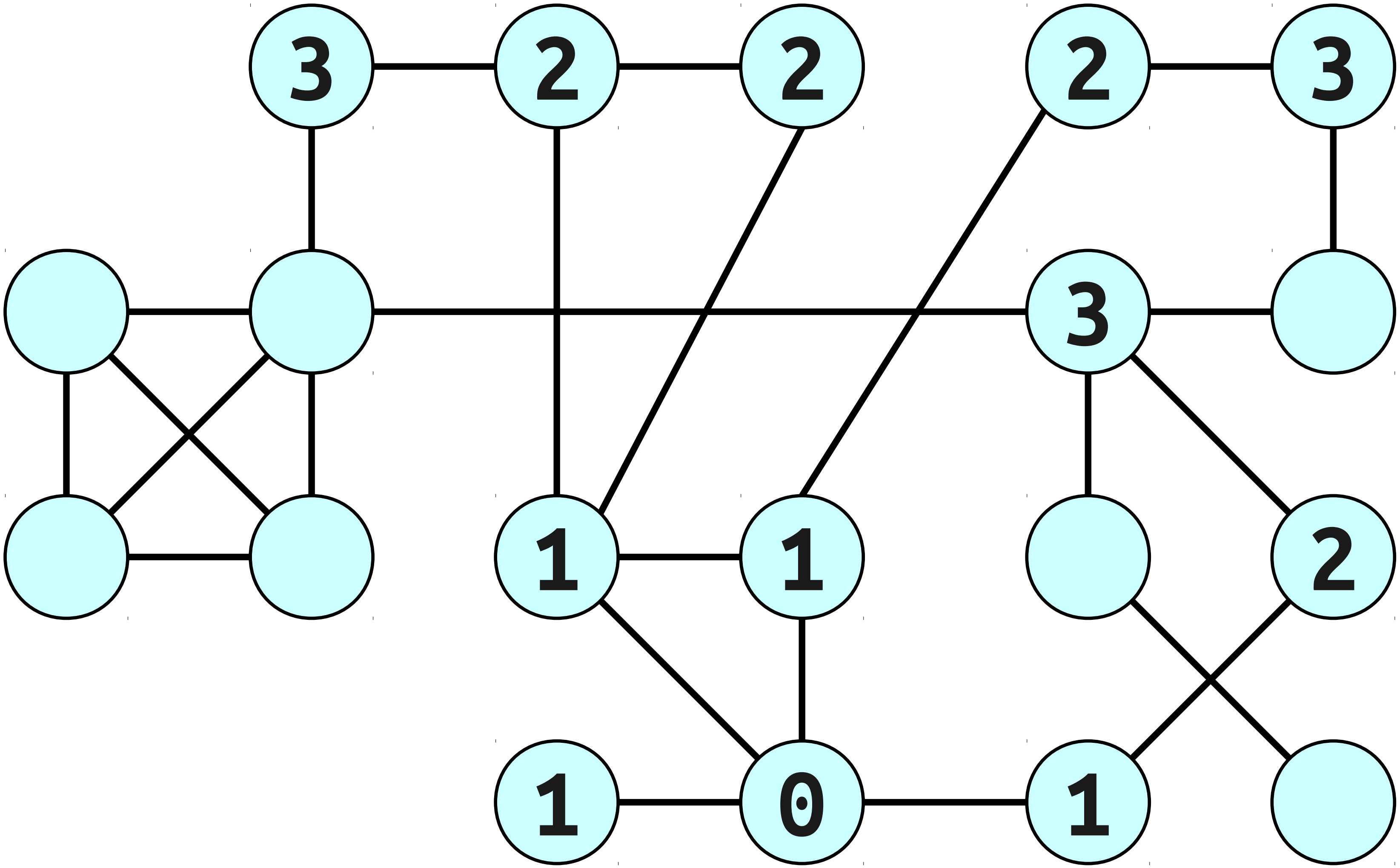

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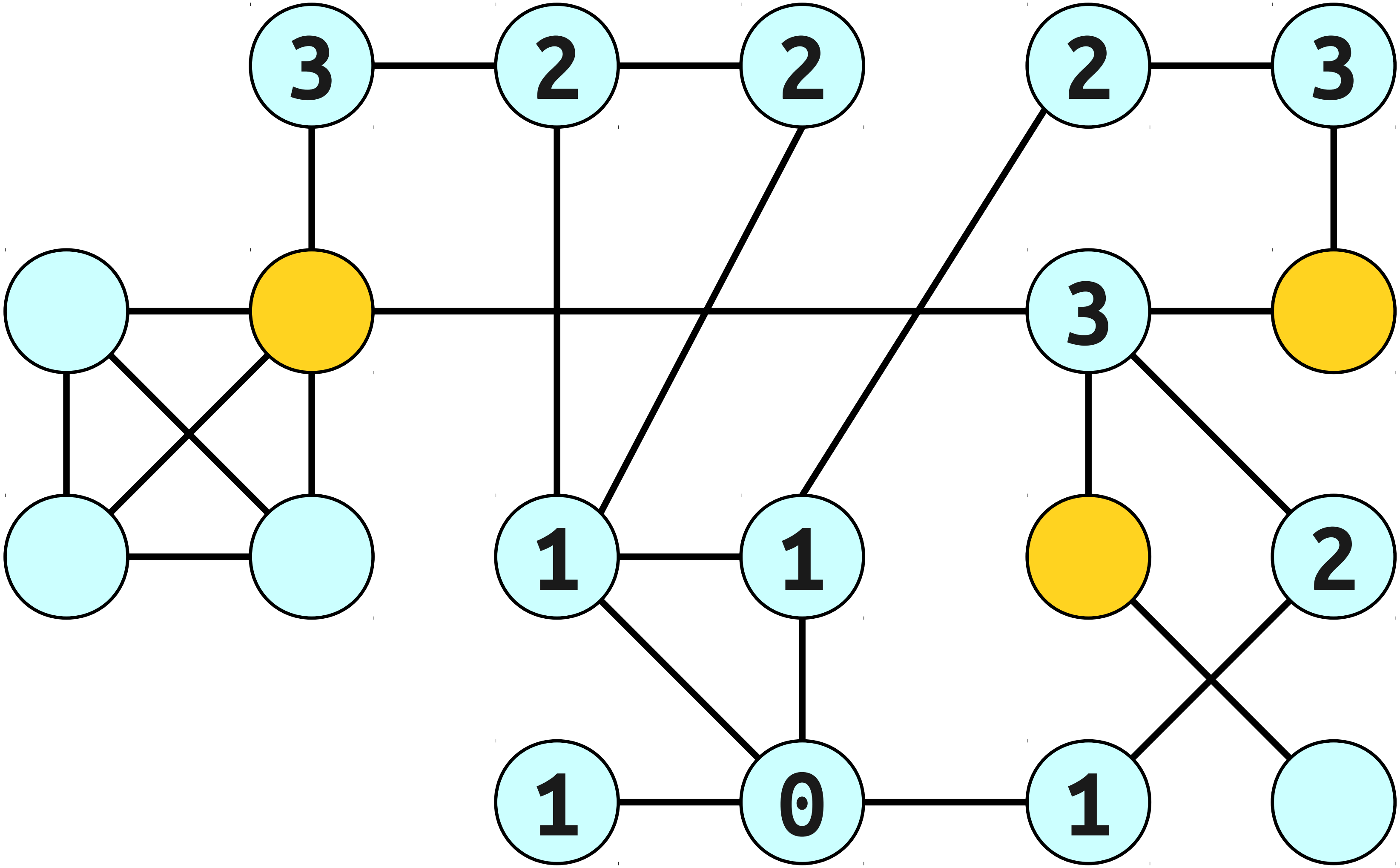

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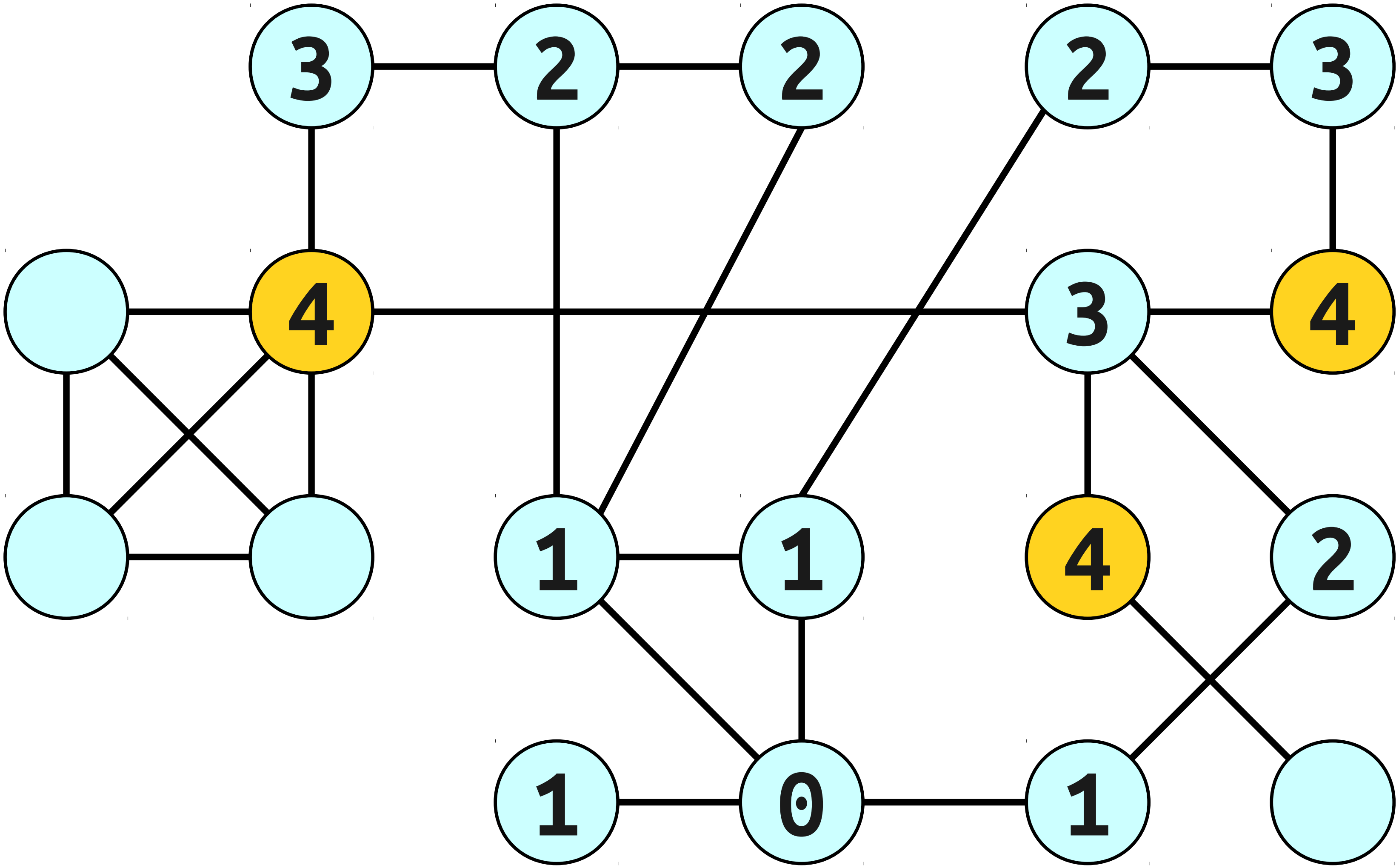

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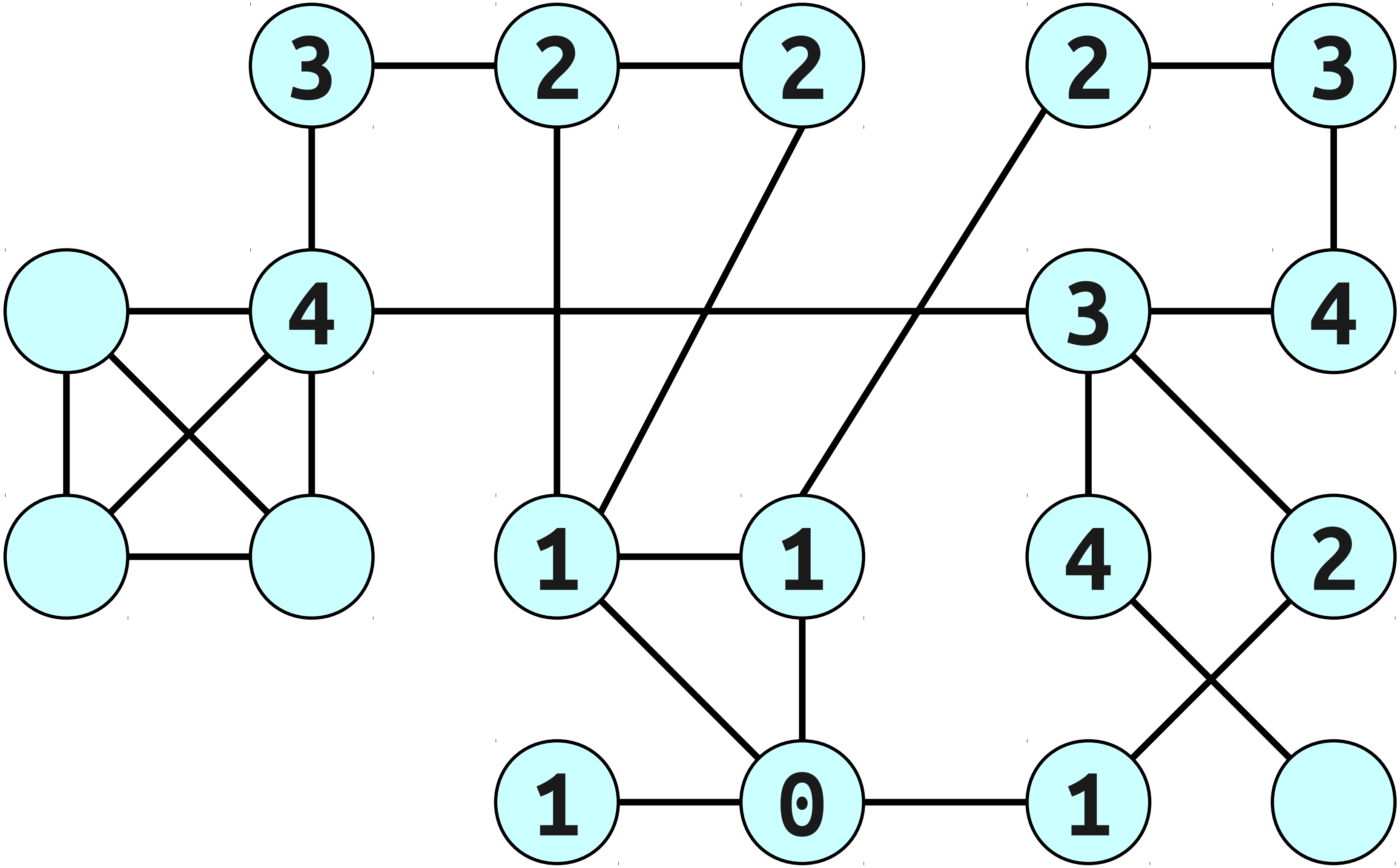

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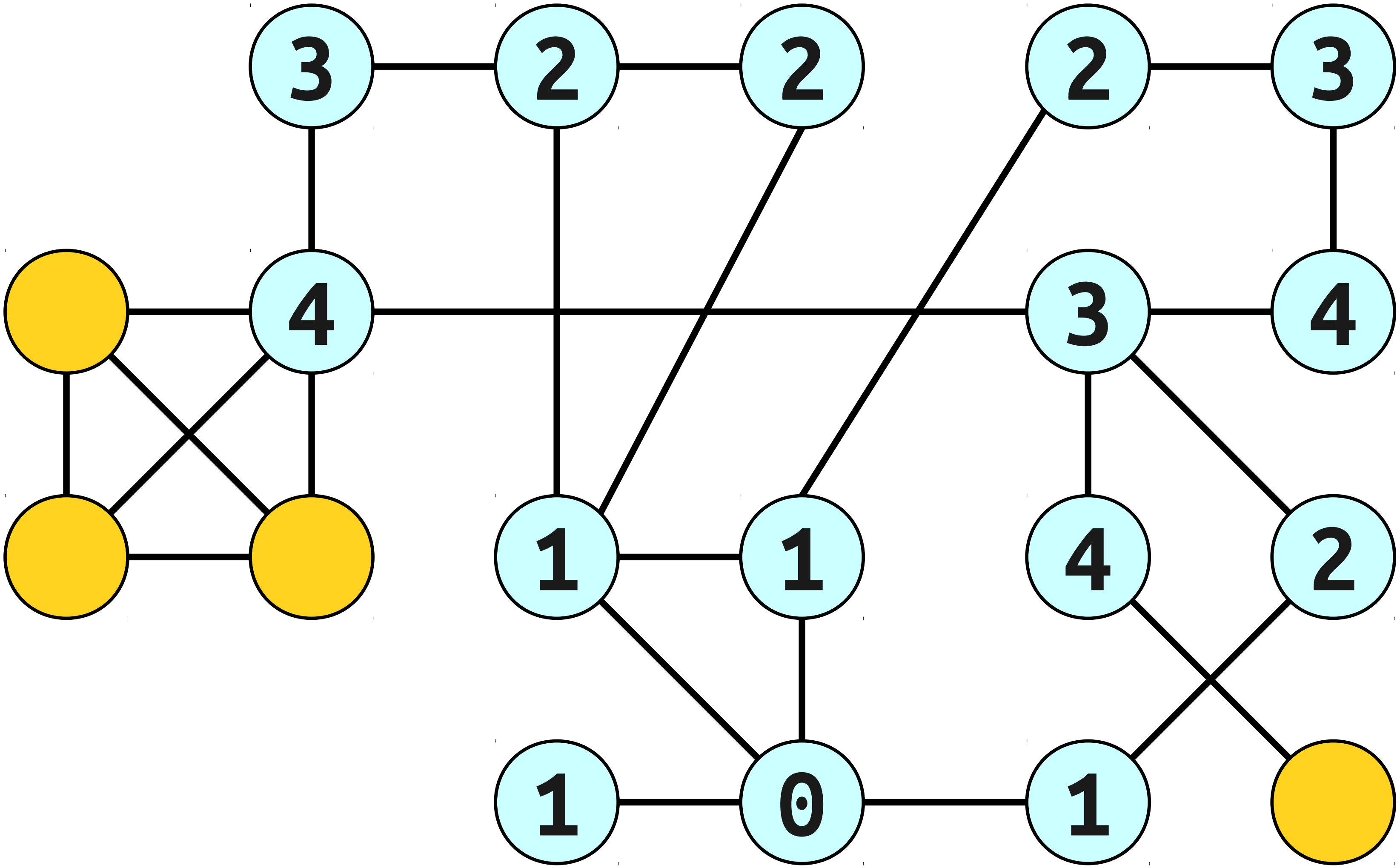

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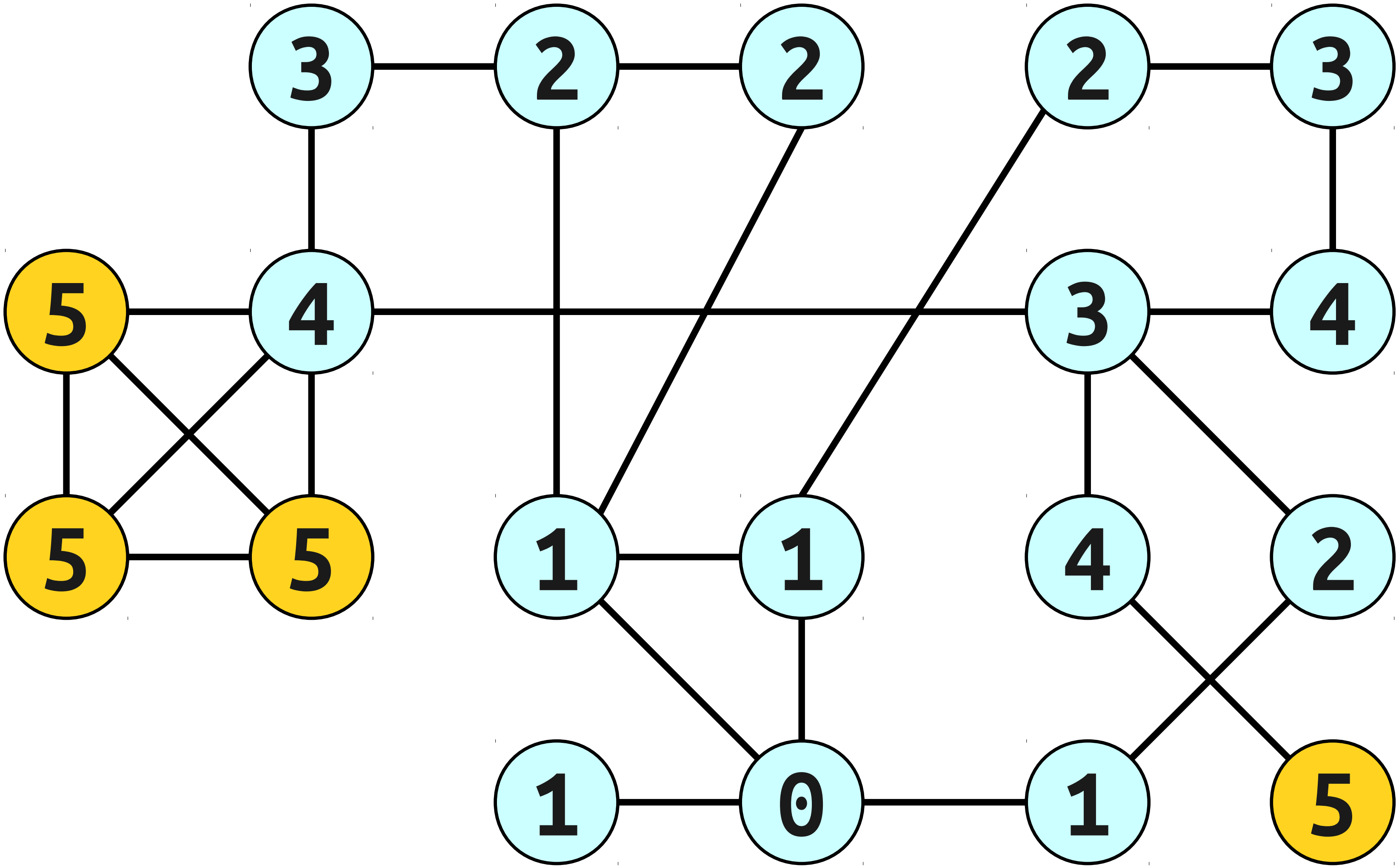

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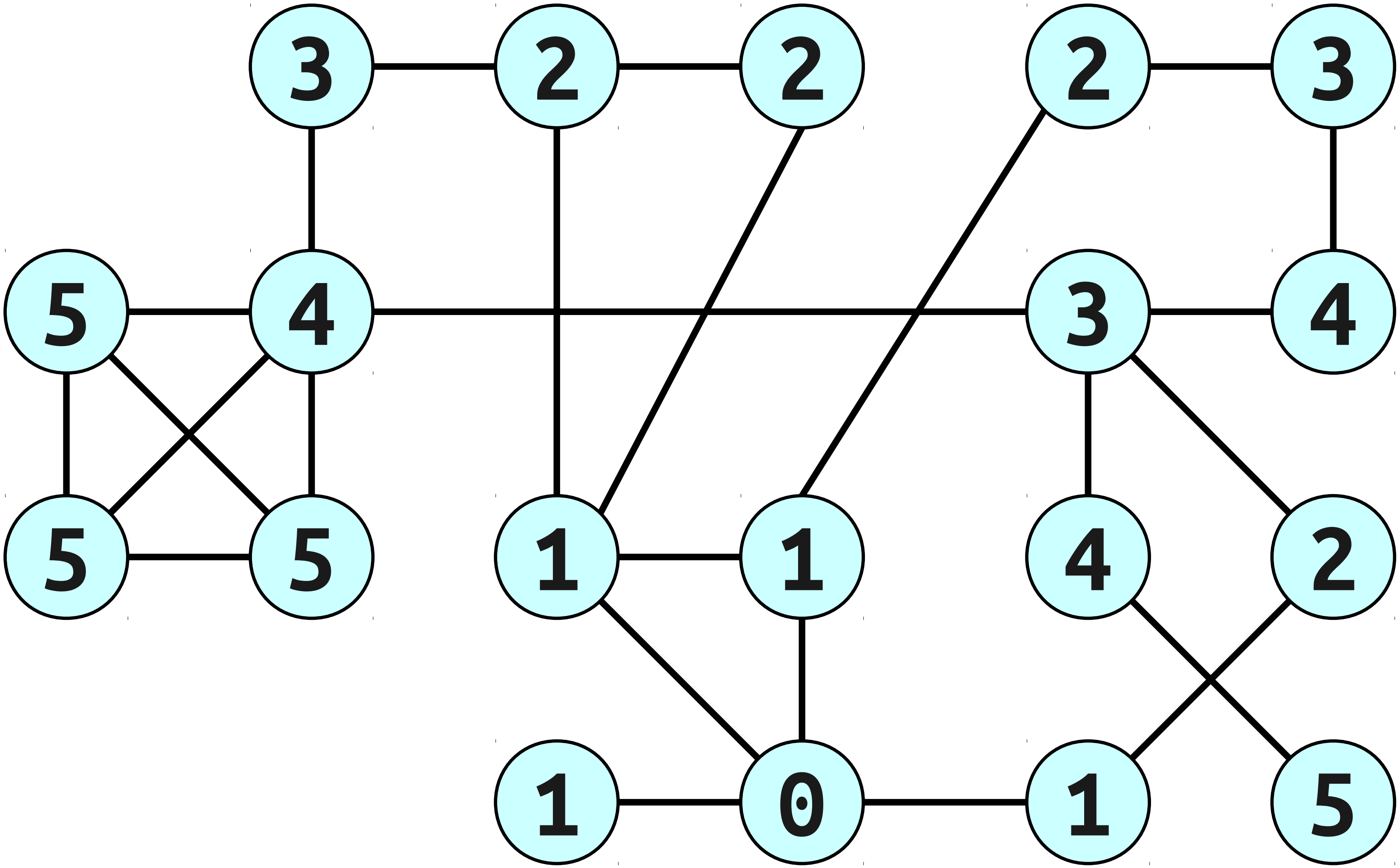
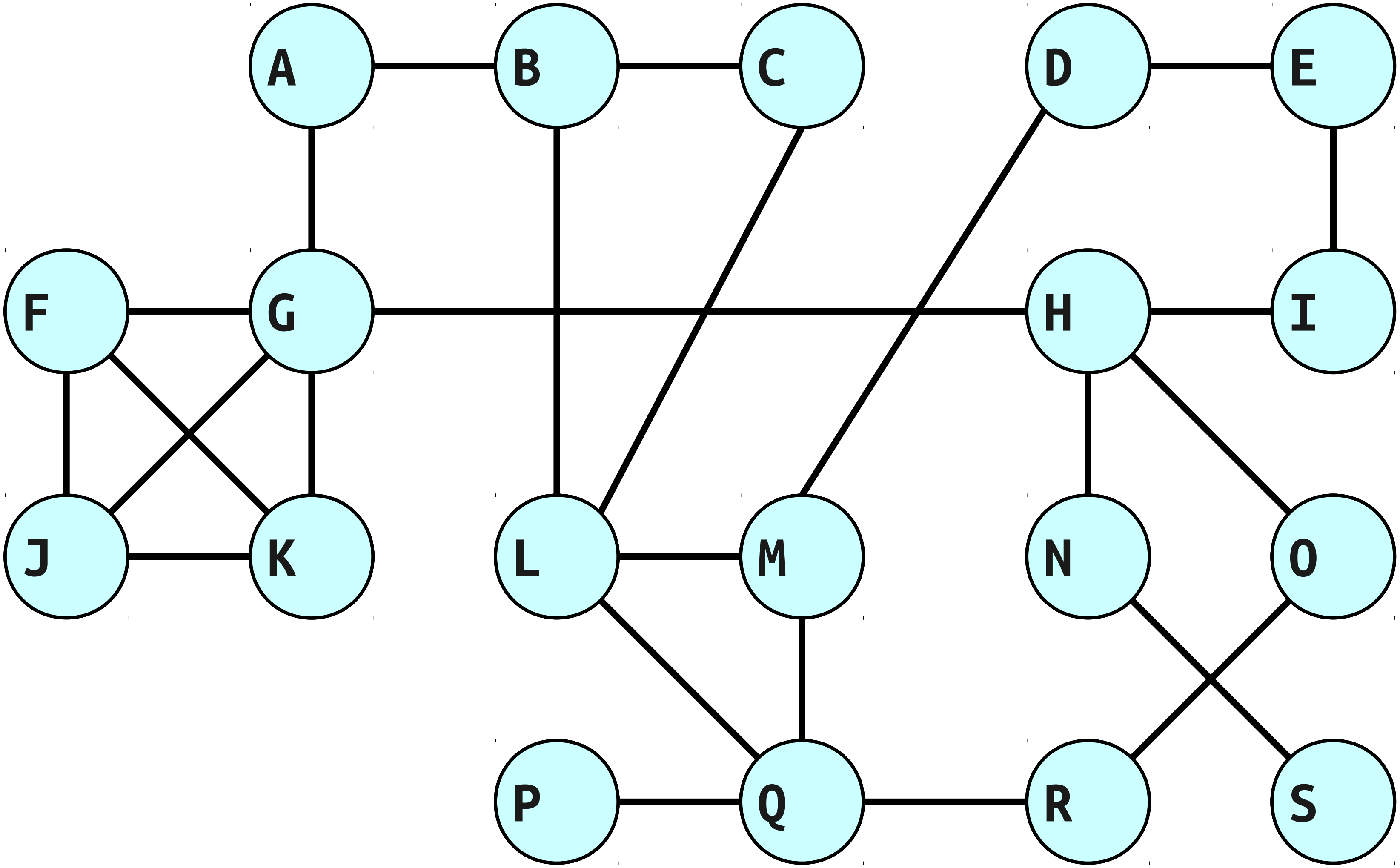


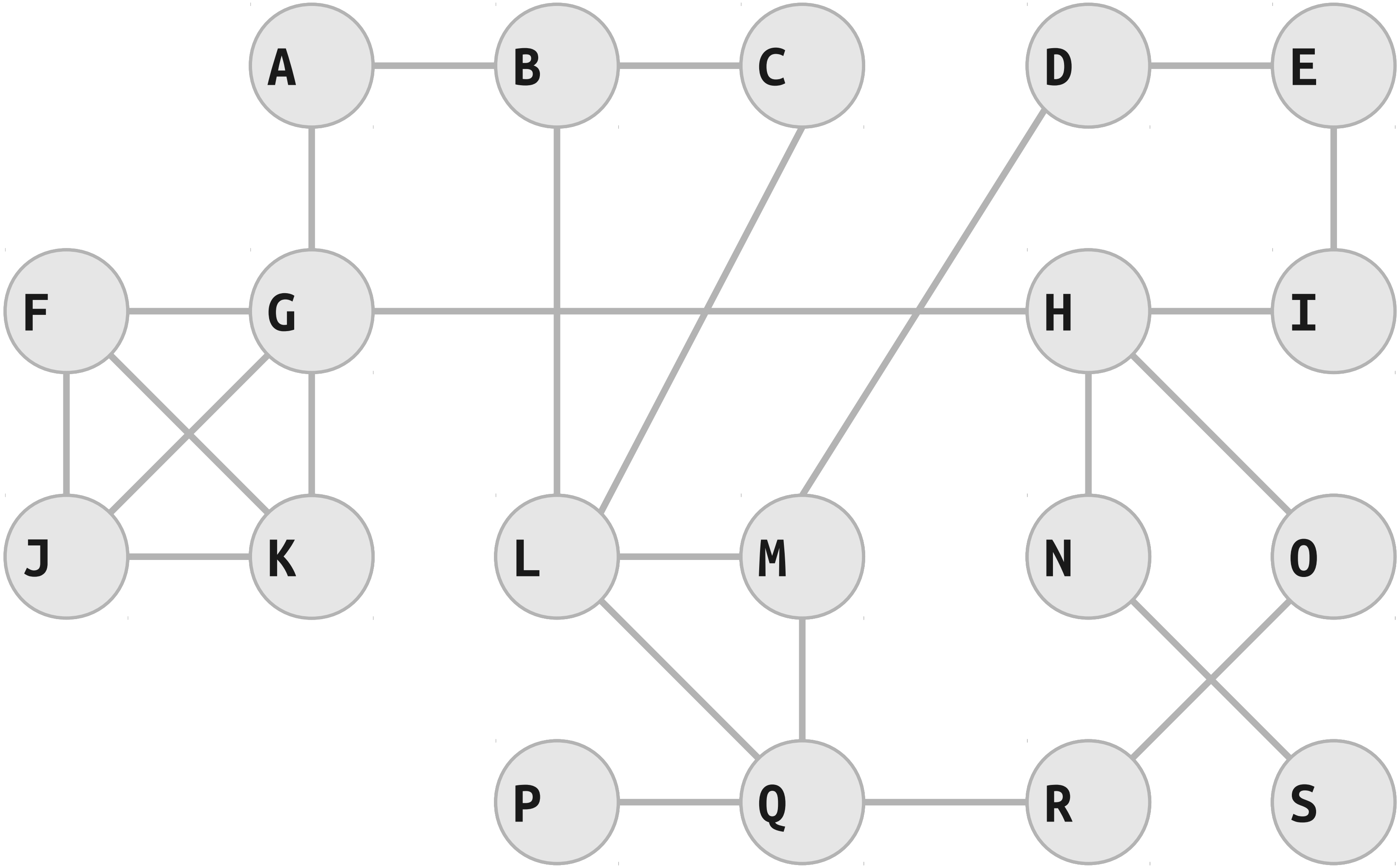
A Secondary Idea

- Proceed outward from the source node *s* in “layers.”
  - The first layer is all nodes of distance 0.
  - The second layer is all nodes of distance 1.
  - The third layer is all nodes of distance 2.
  - etc.
- This gives rise to **breadth-first search**.

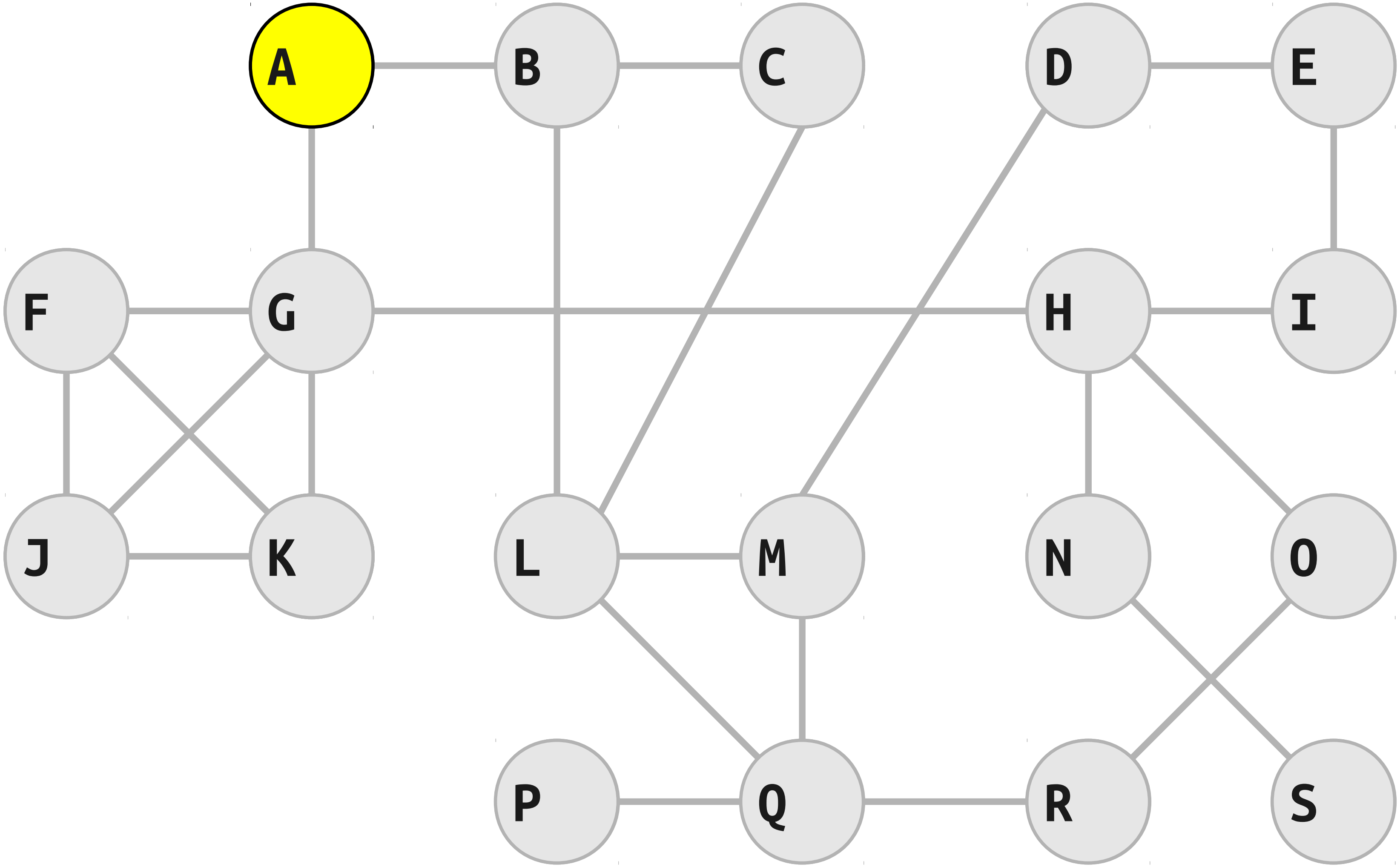
Breadth-First Search



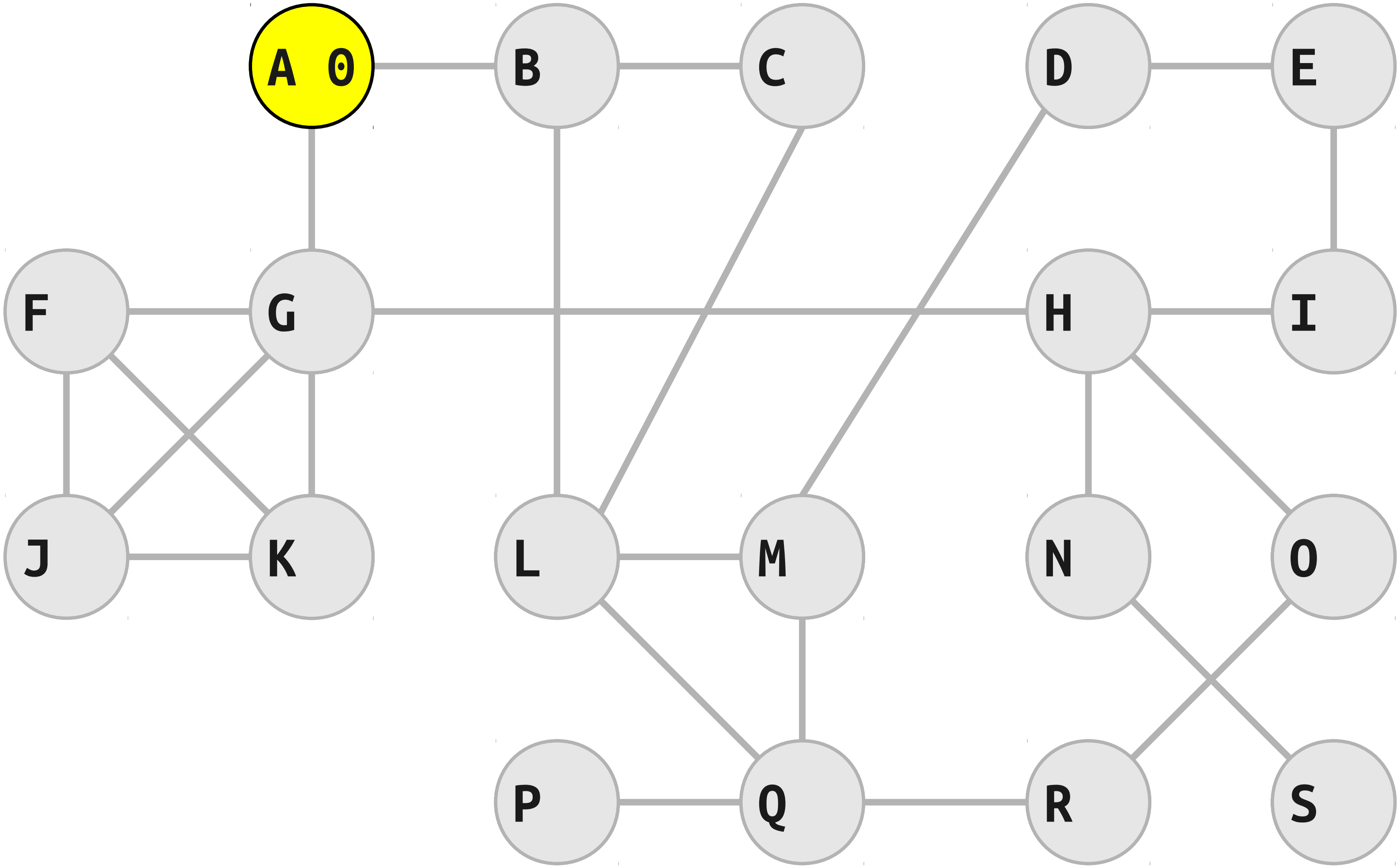
Breadth-First Search



Breadth-First Search



Breadth-First Search



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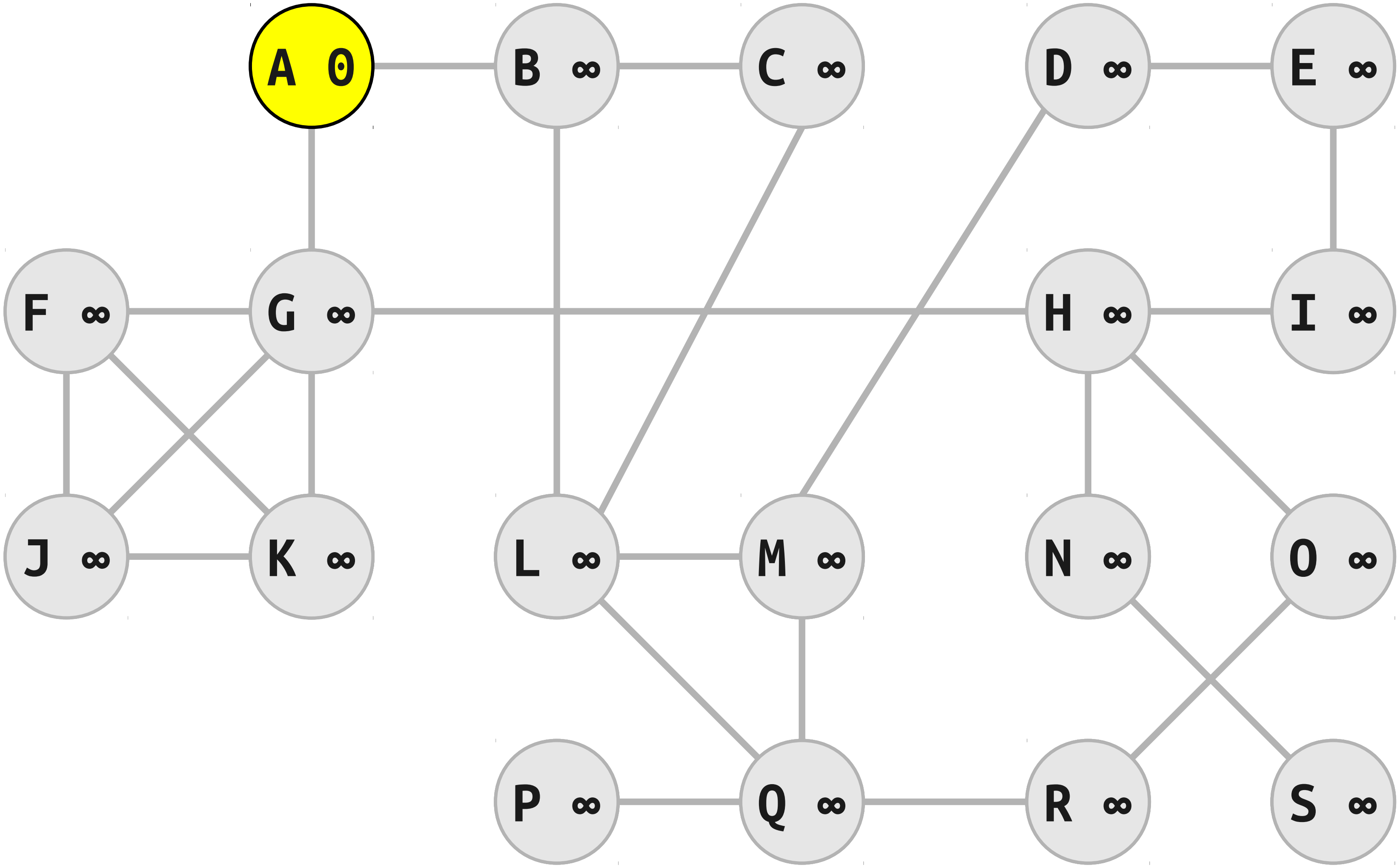

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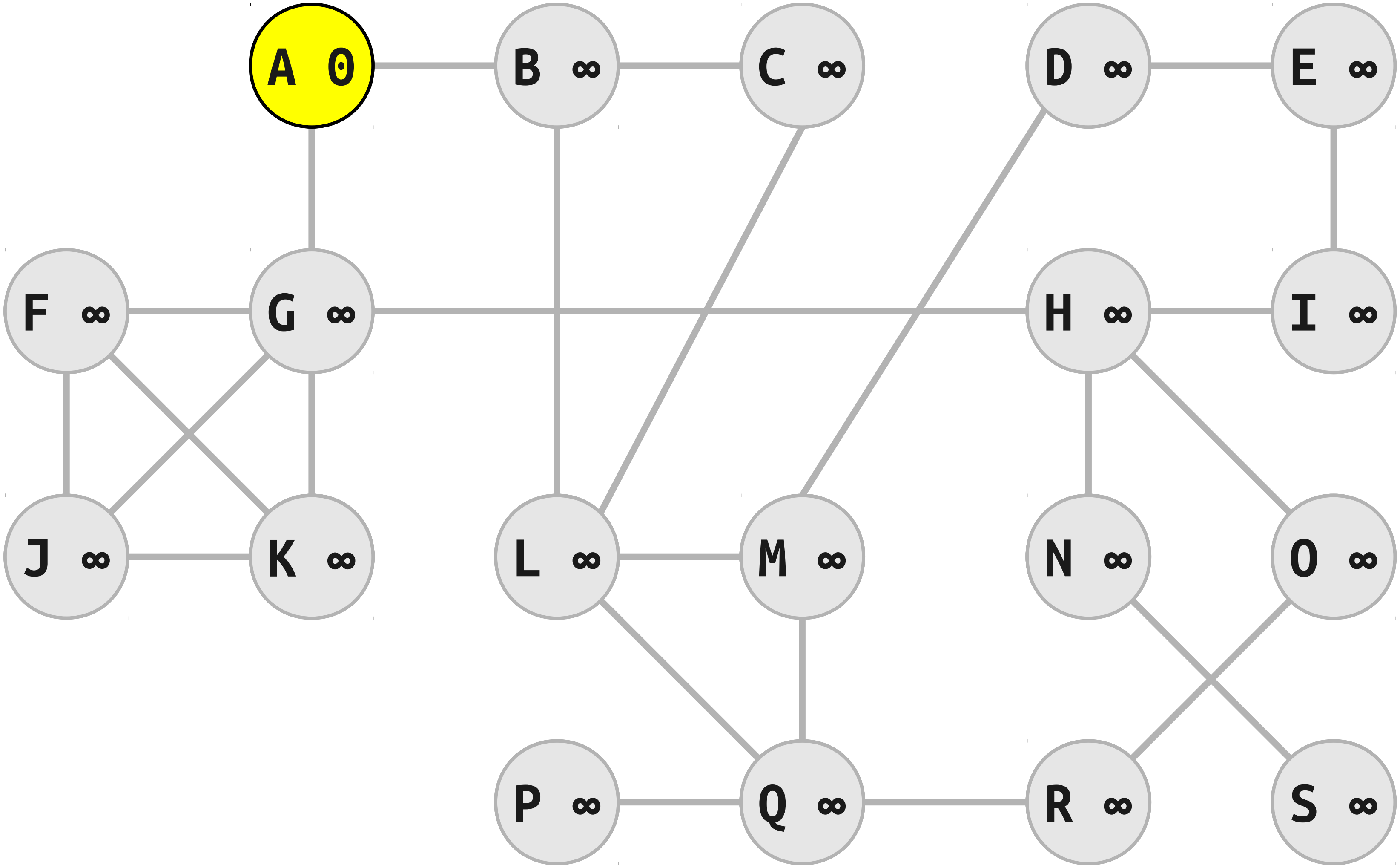

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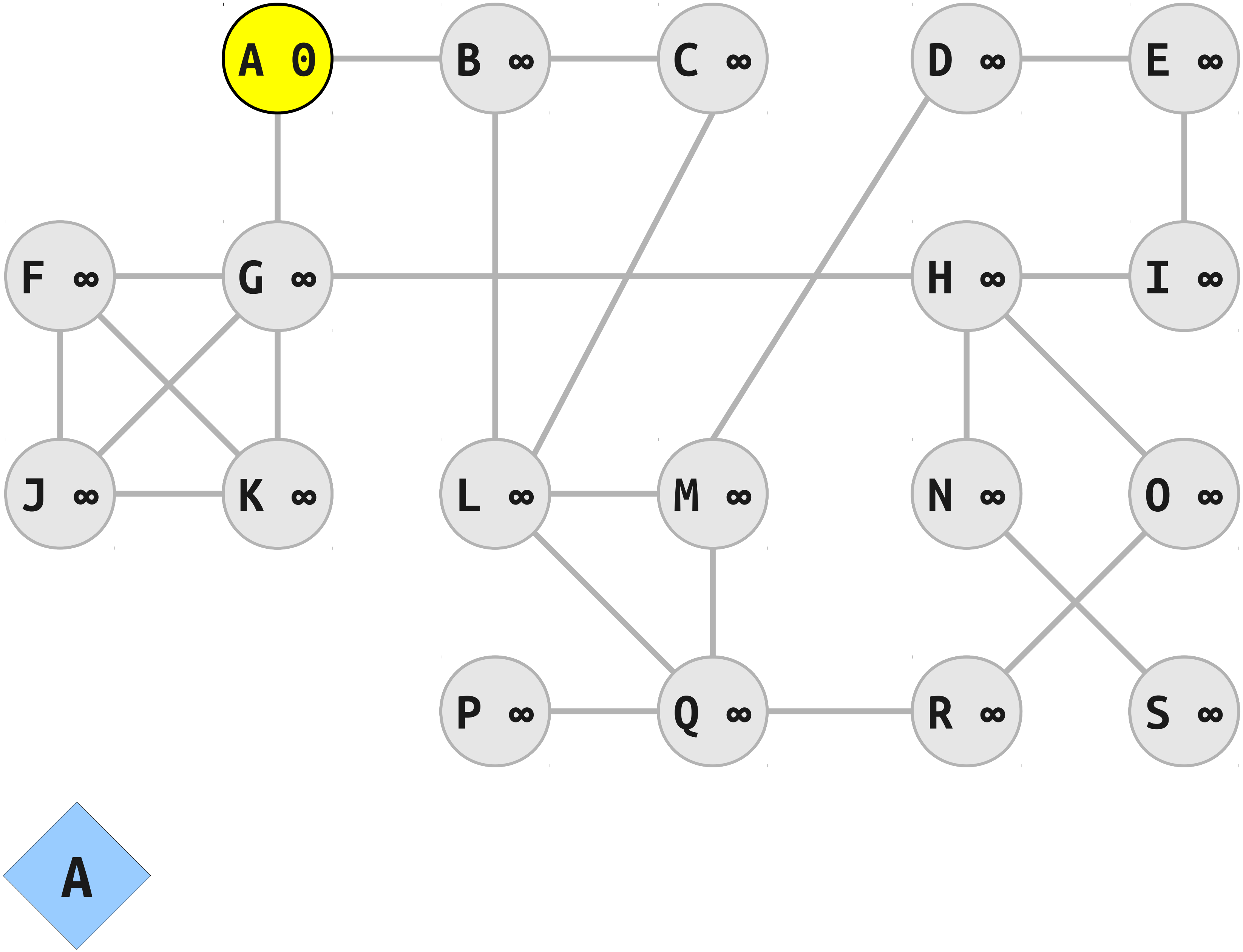

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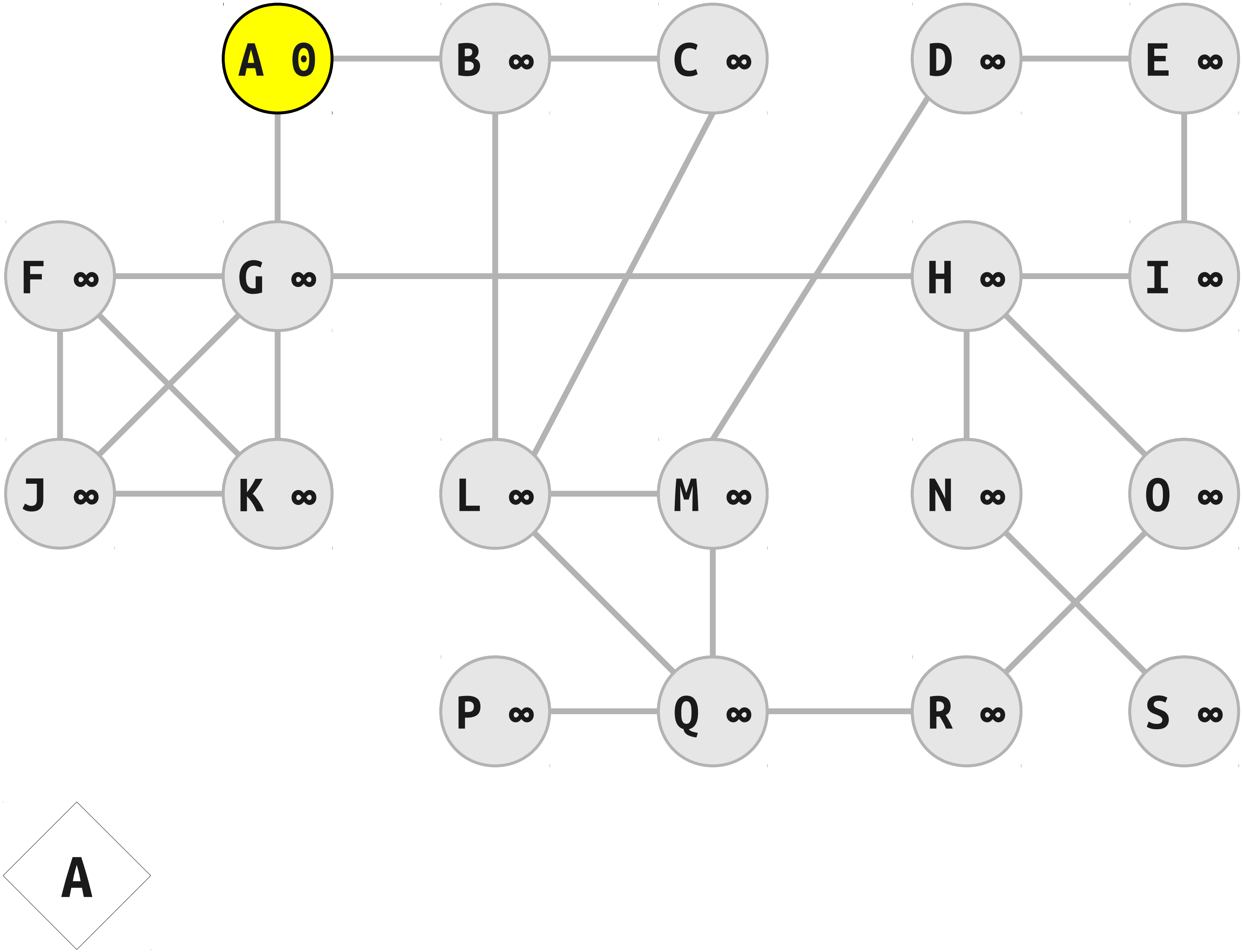

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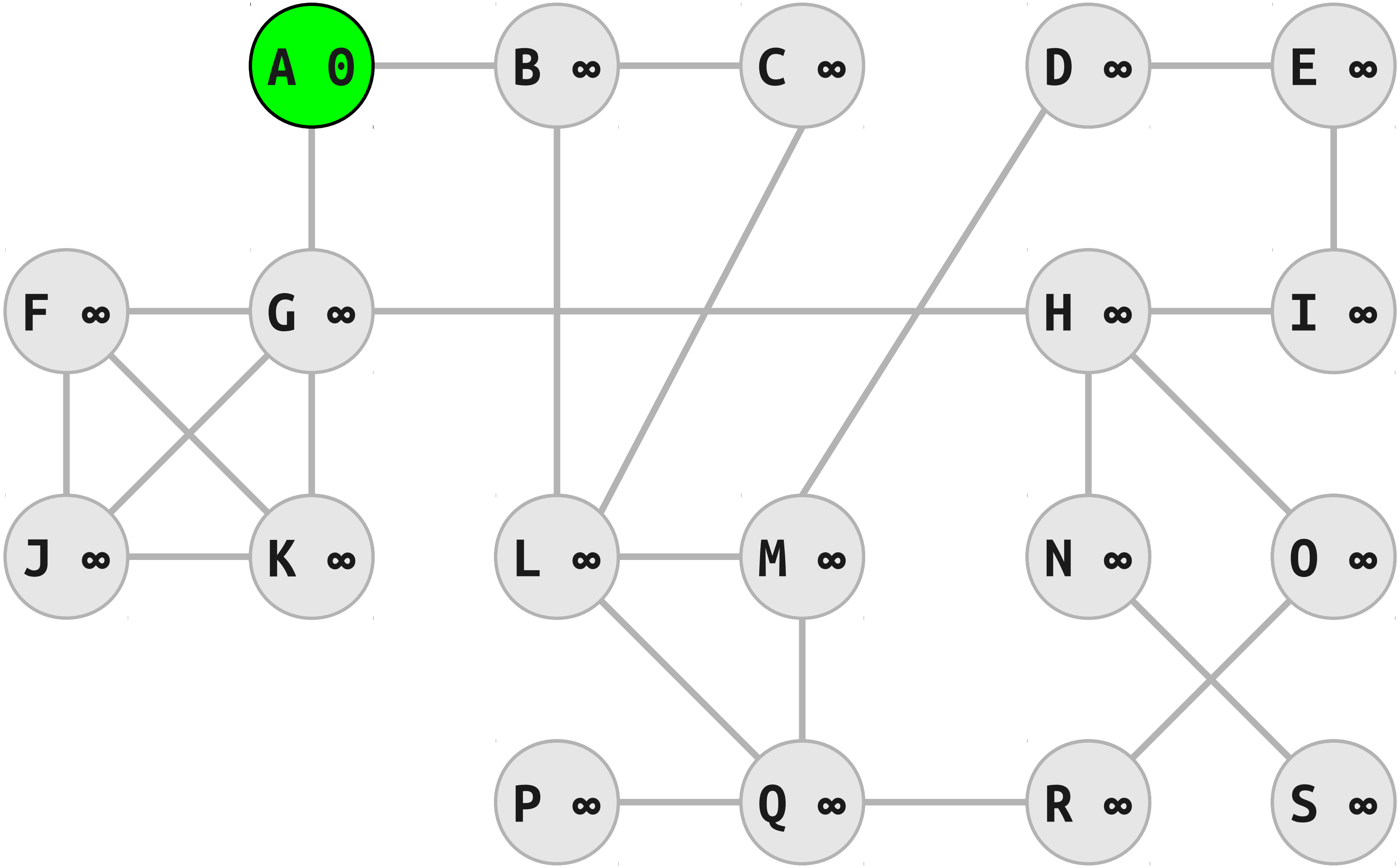

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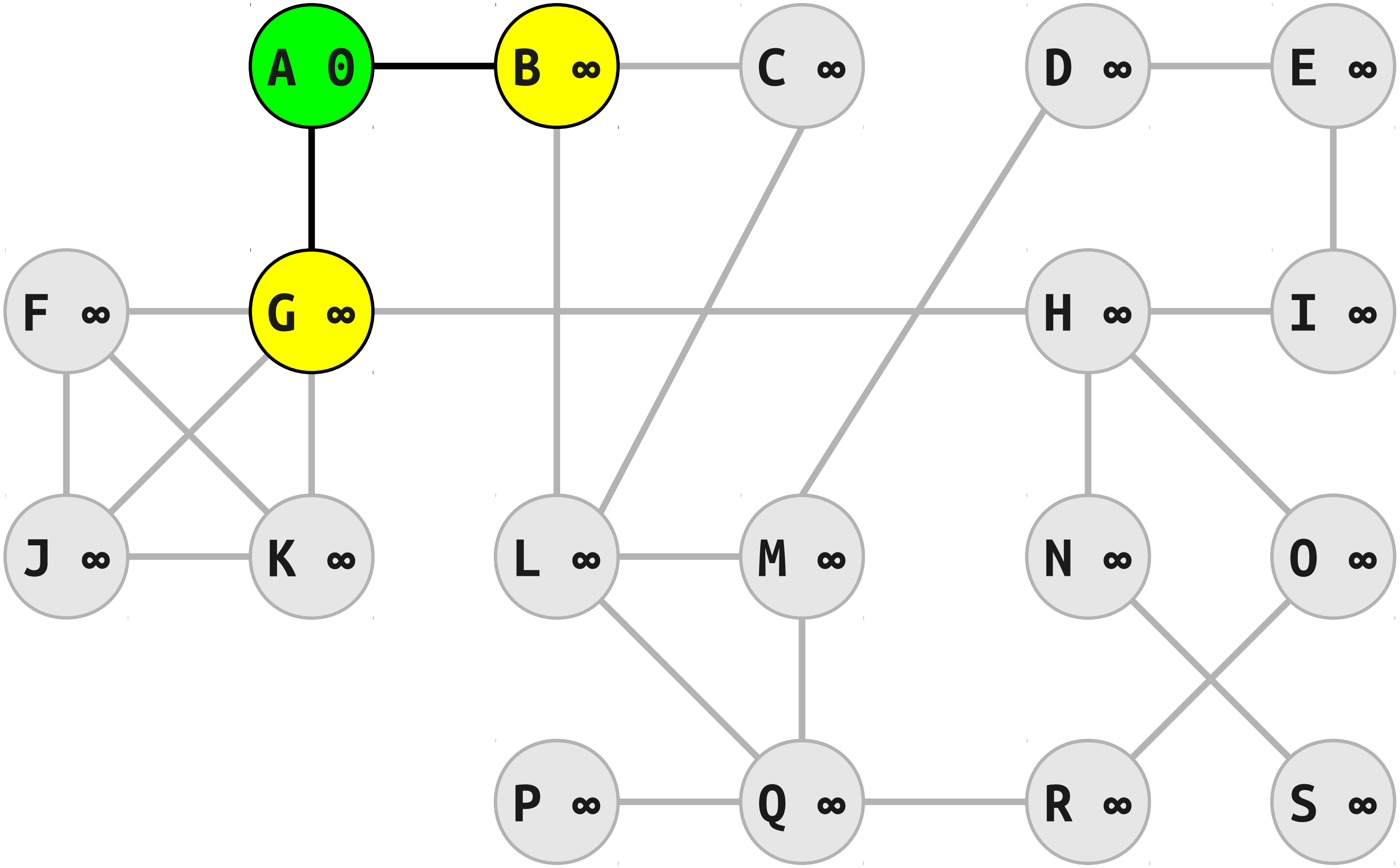

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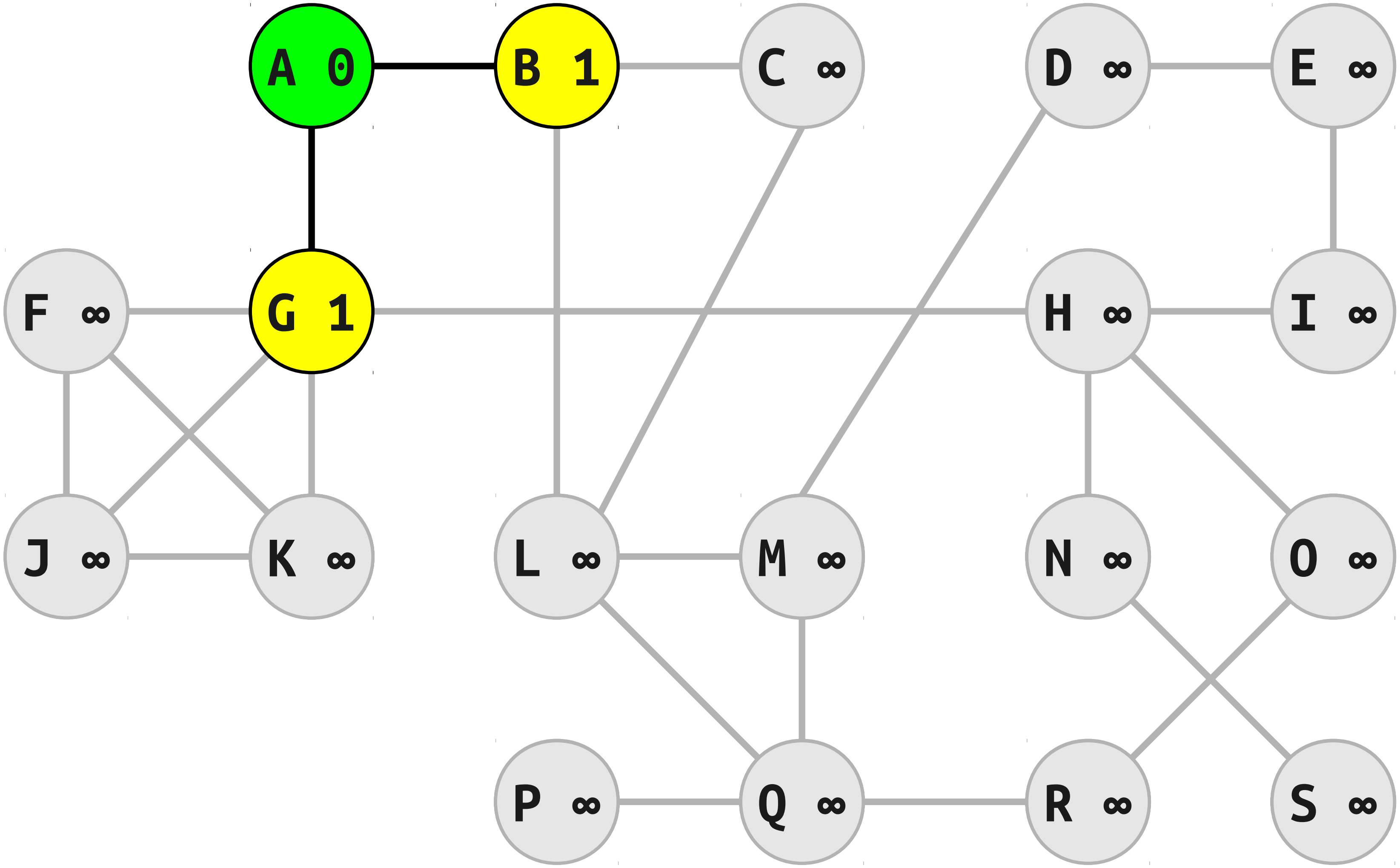

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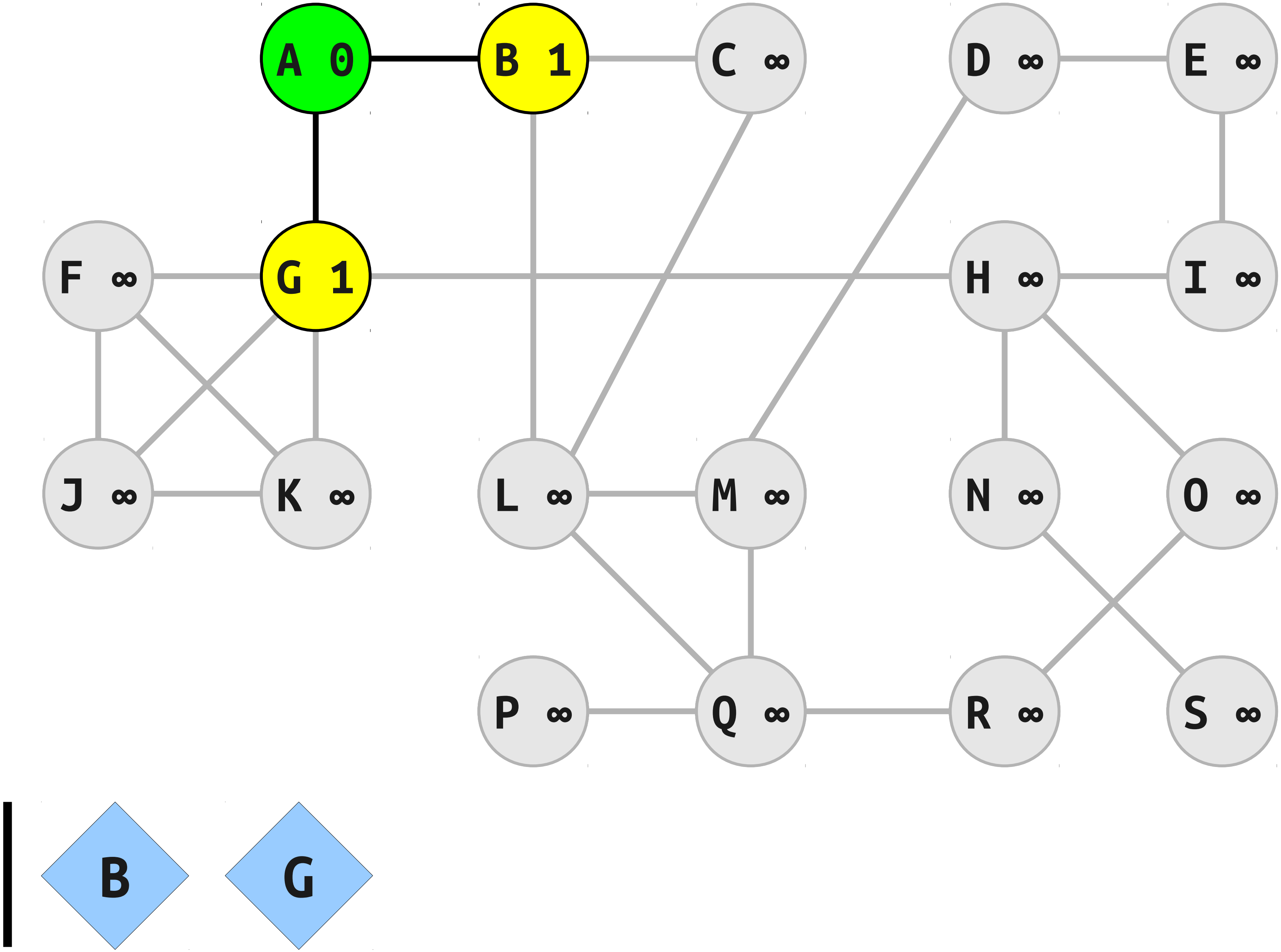

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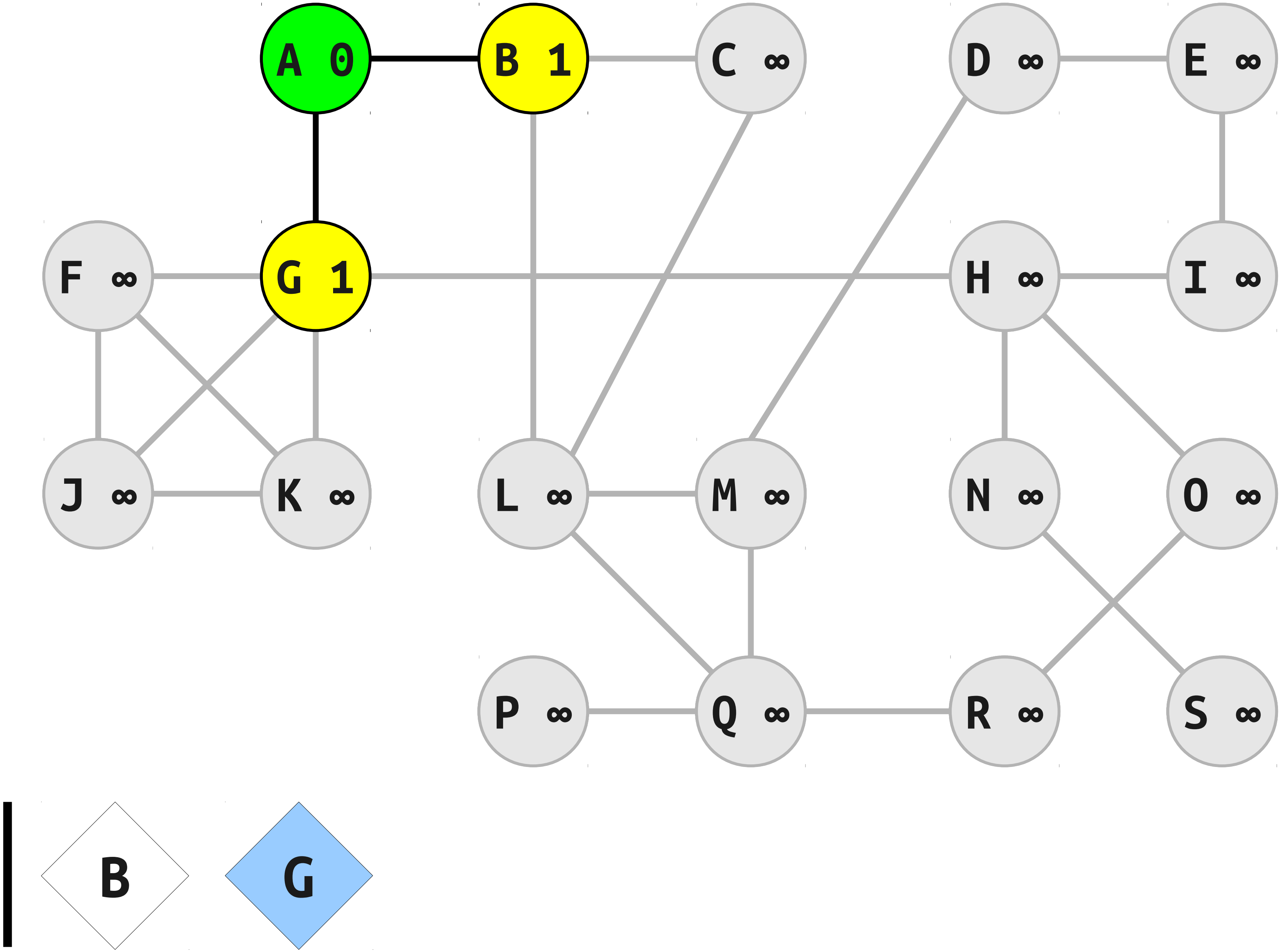

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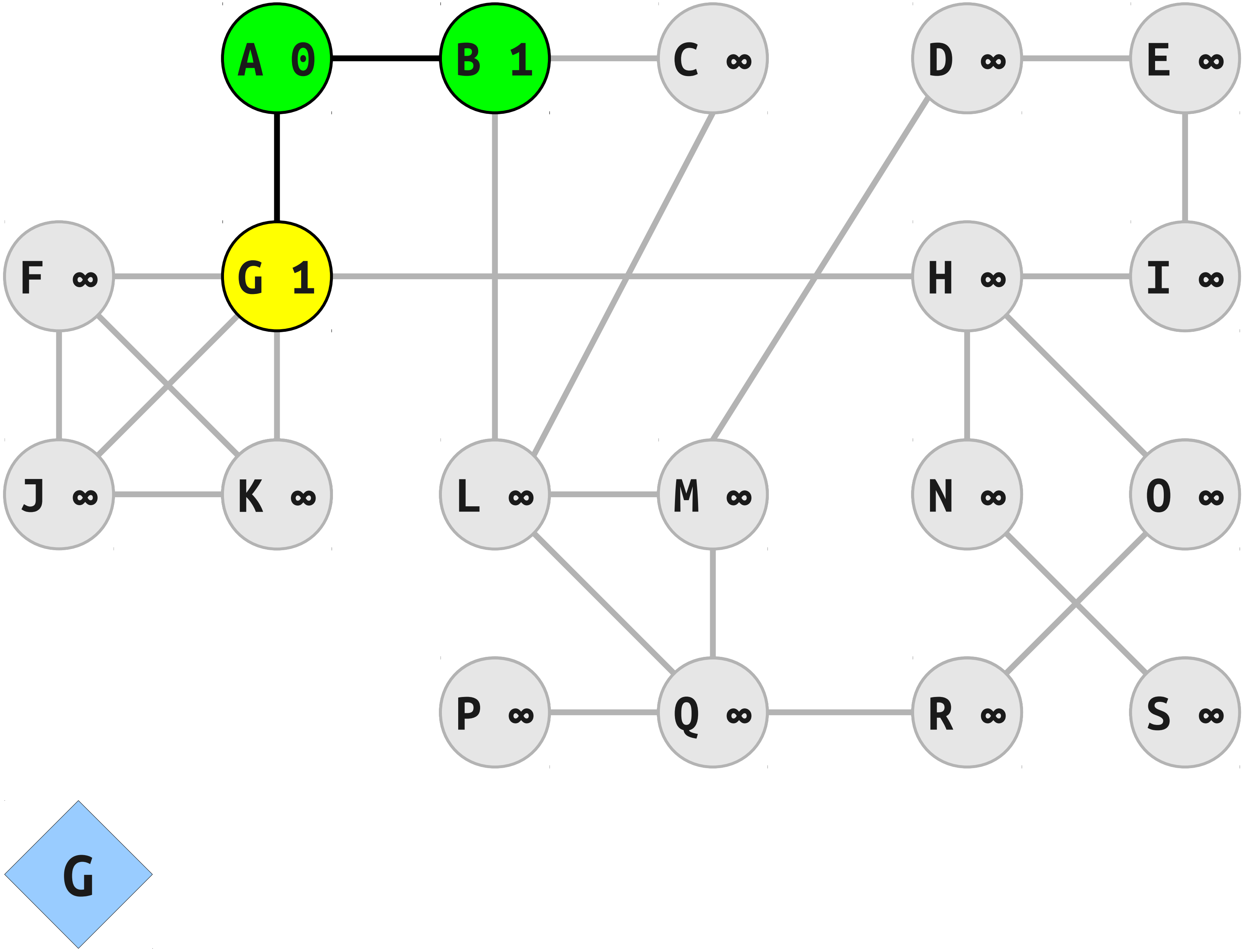

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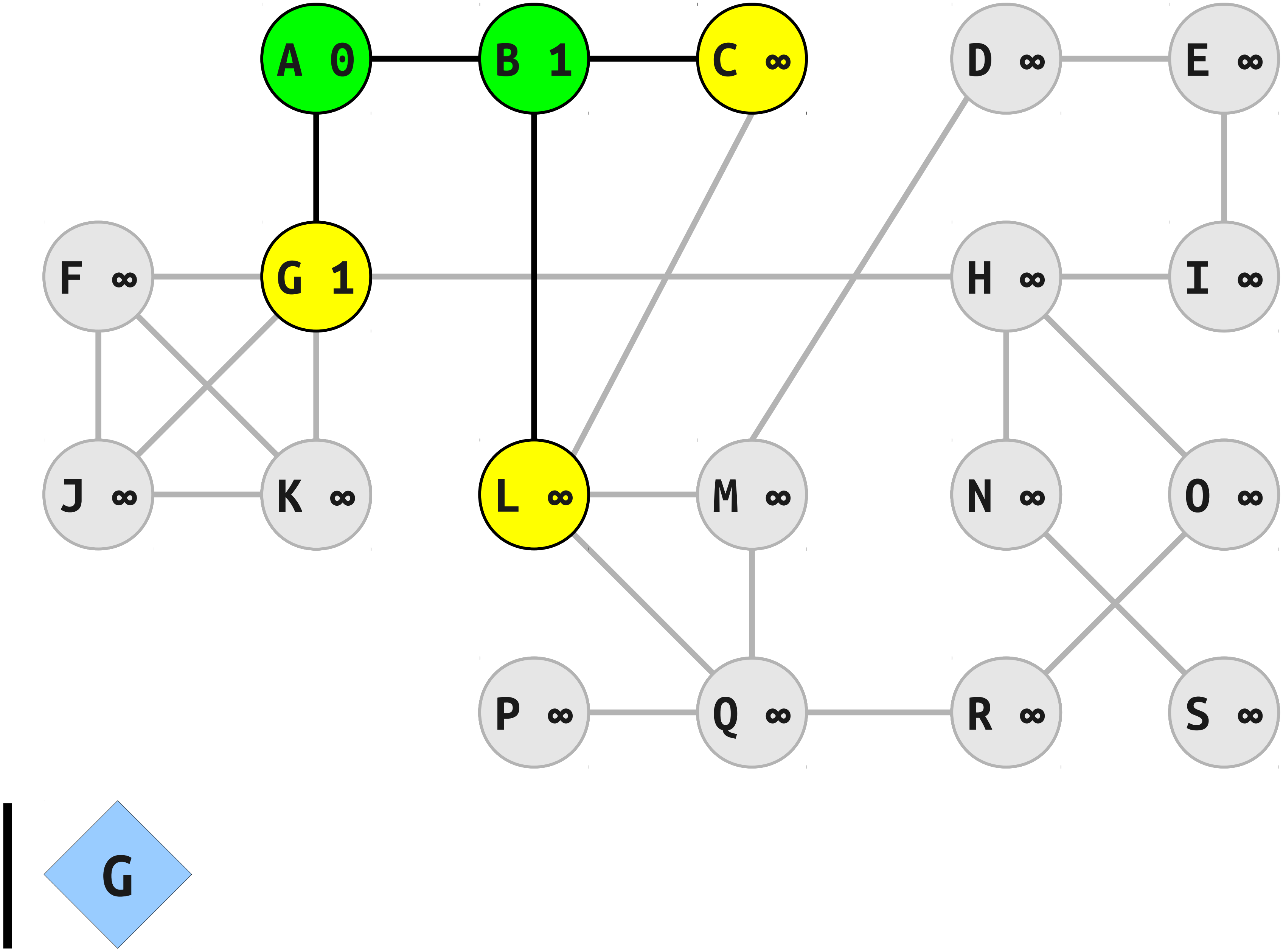

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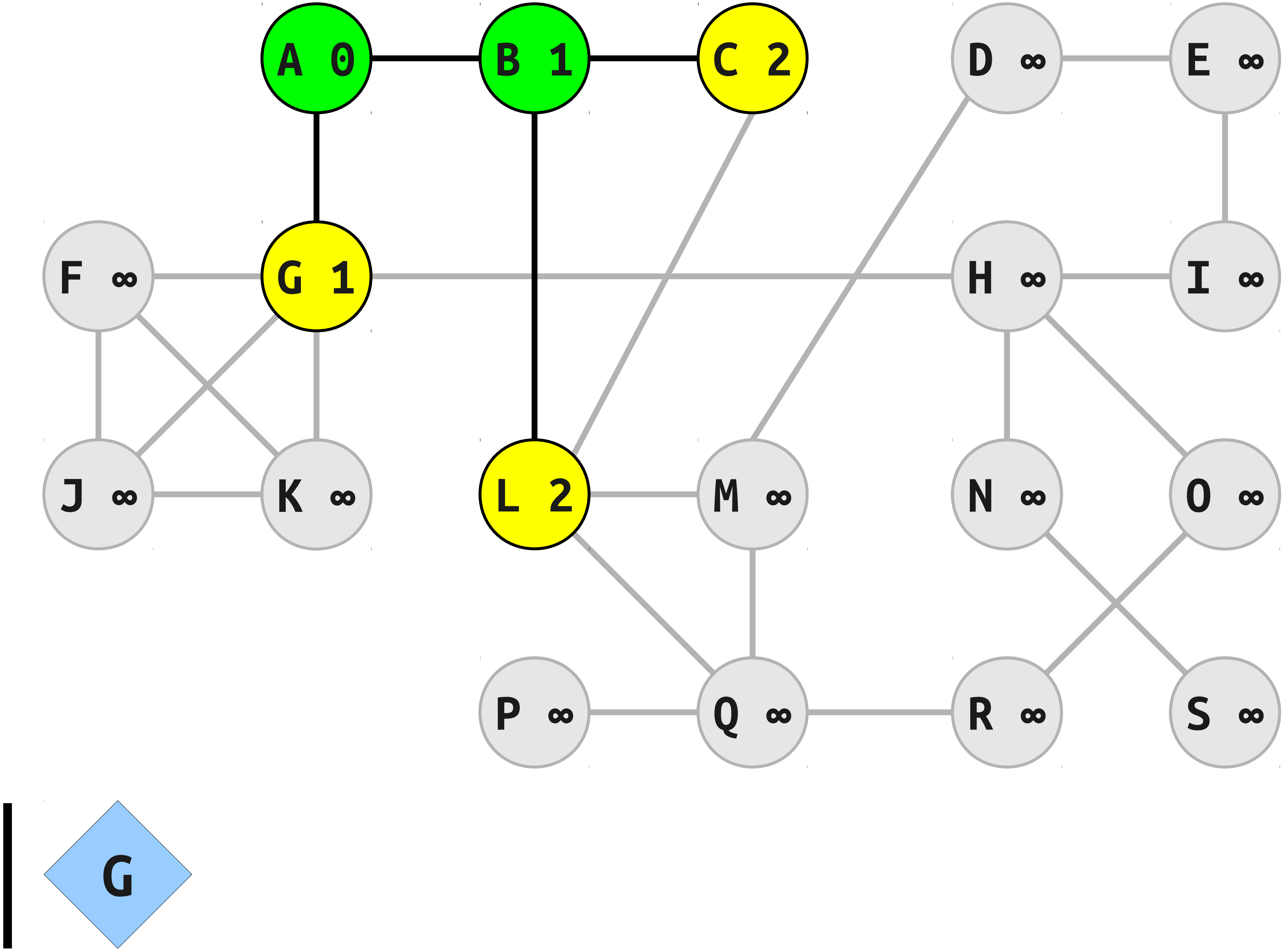

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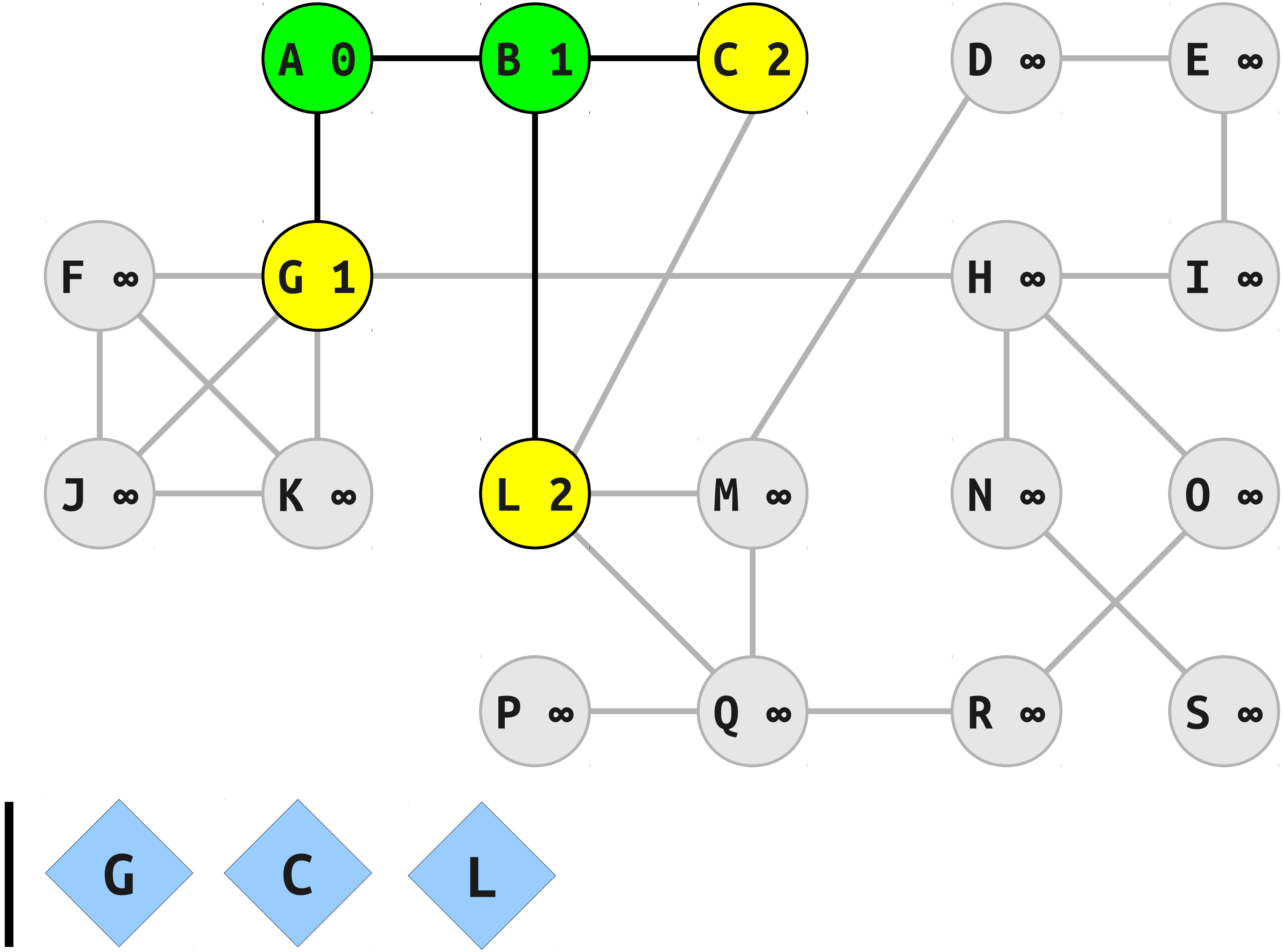

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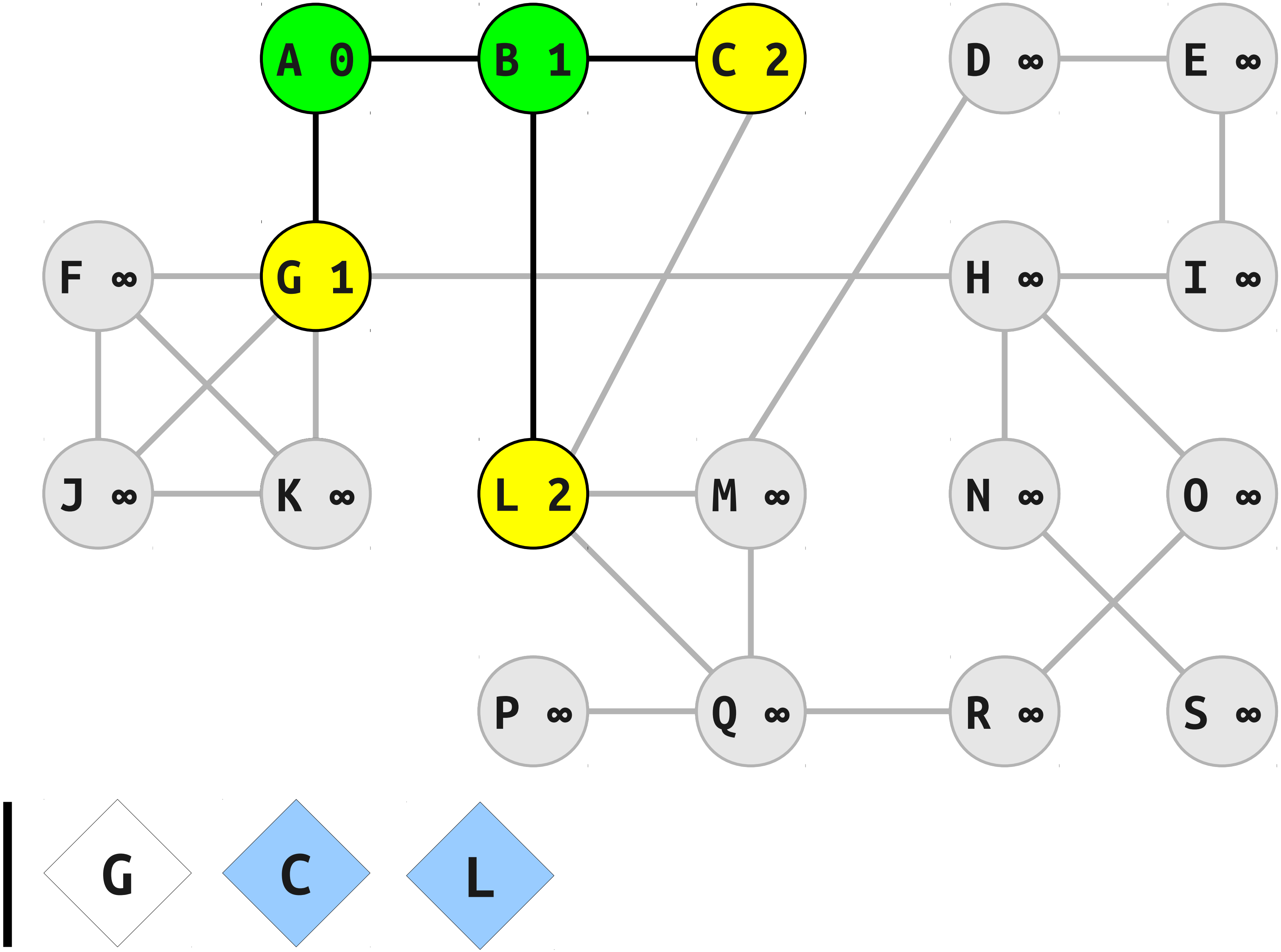

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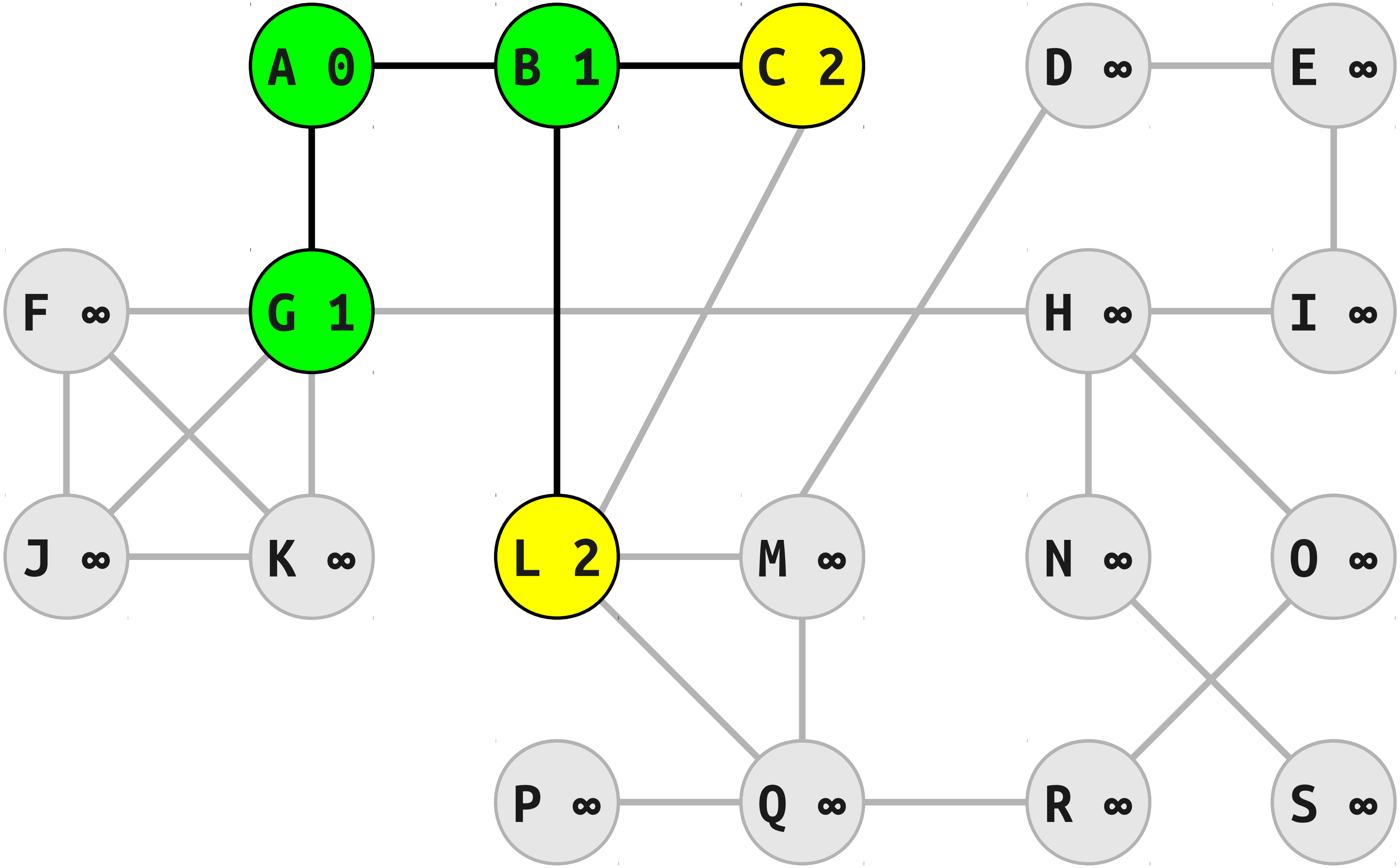

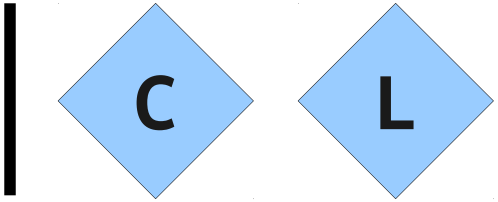

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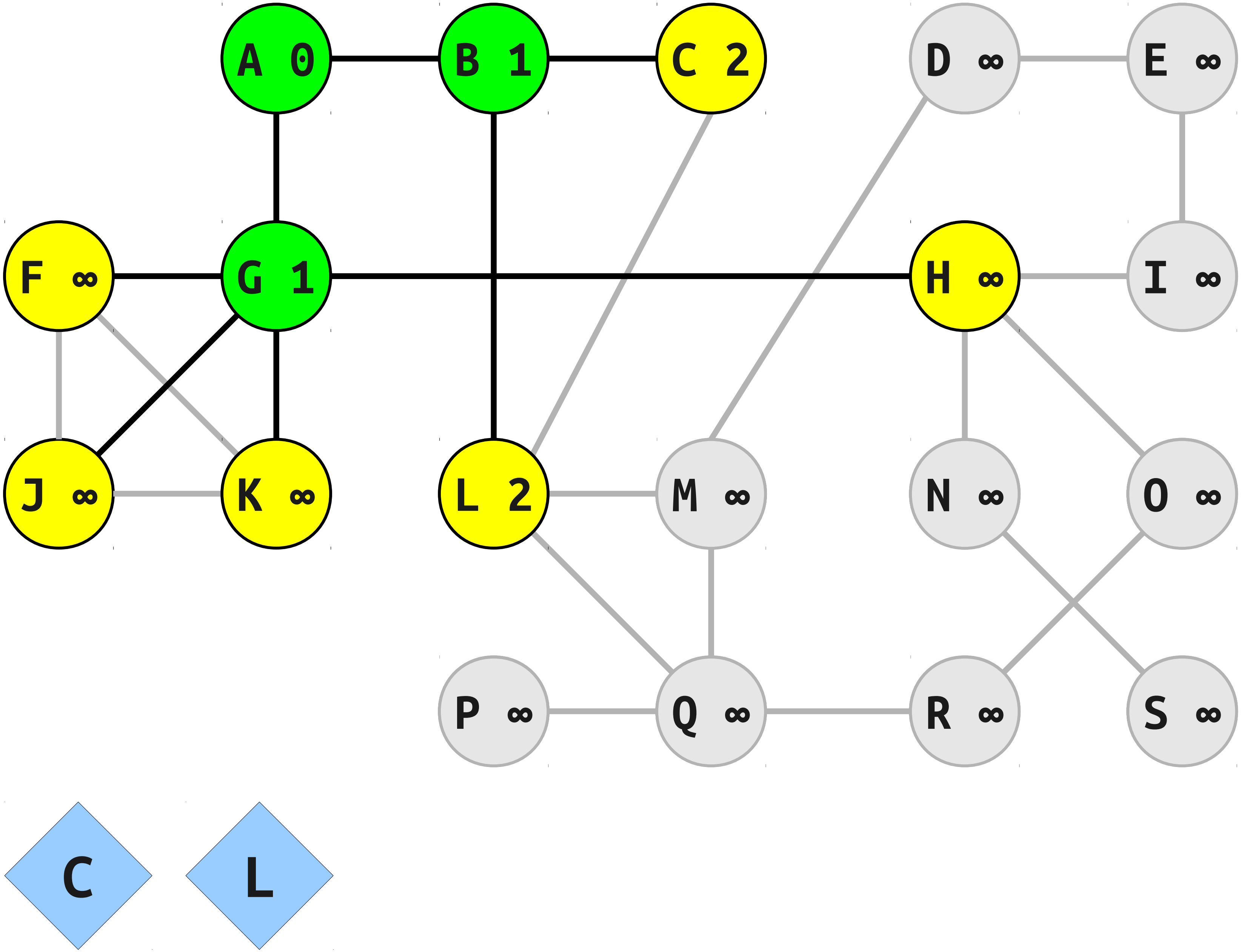

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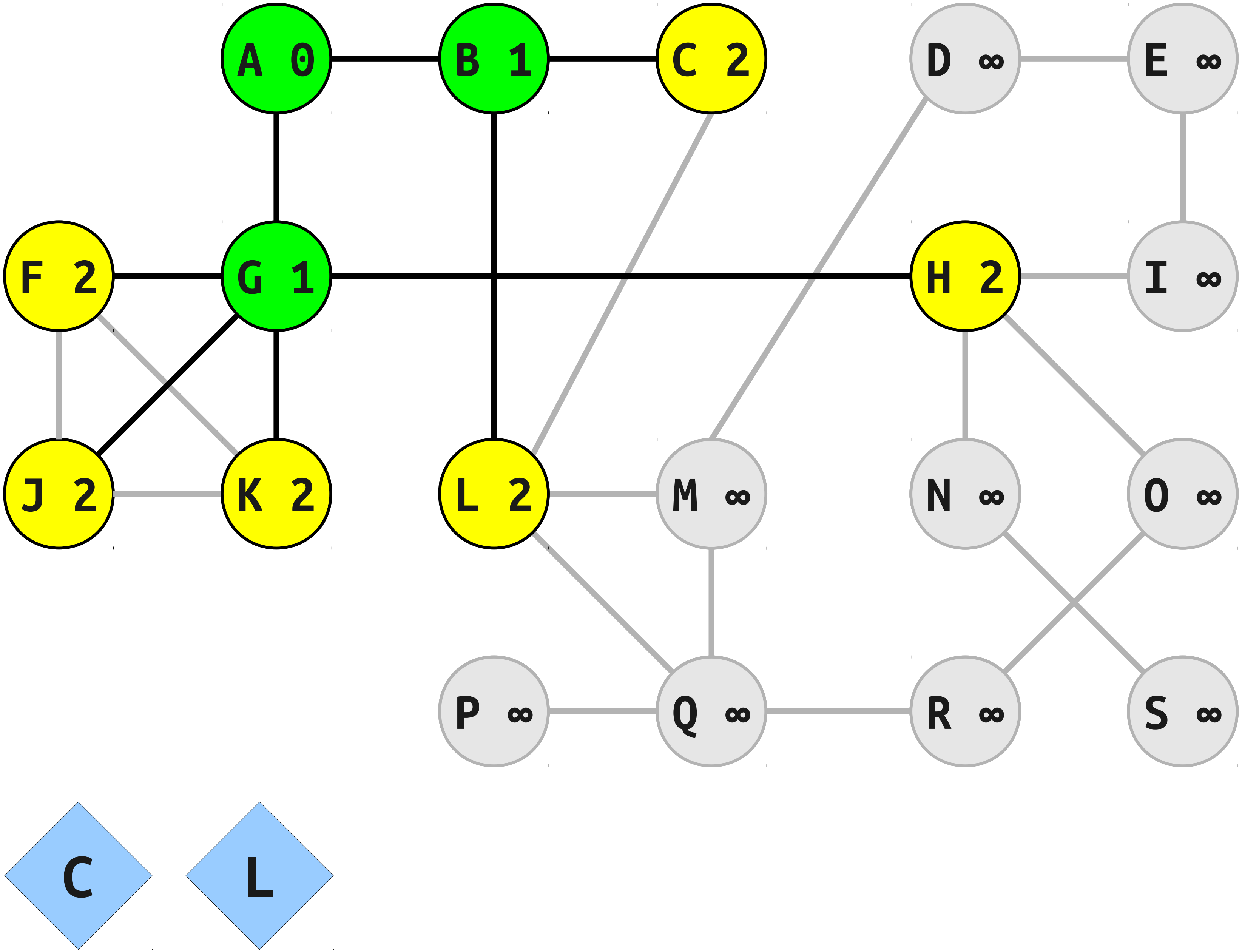

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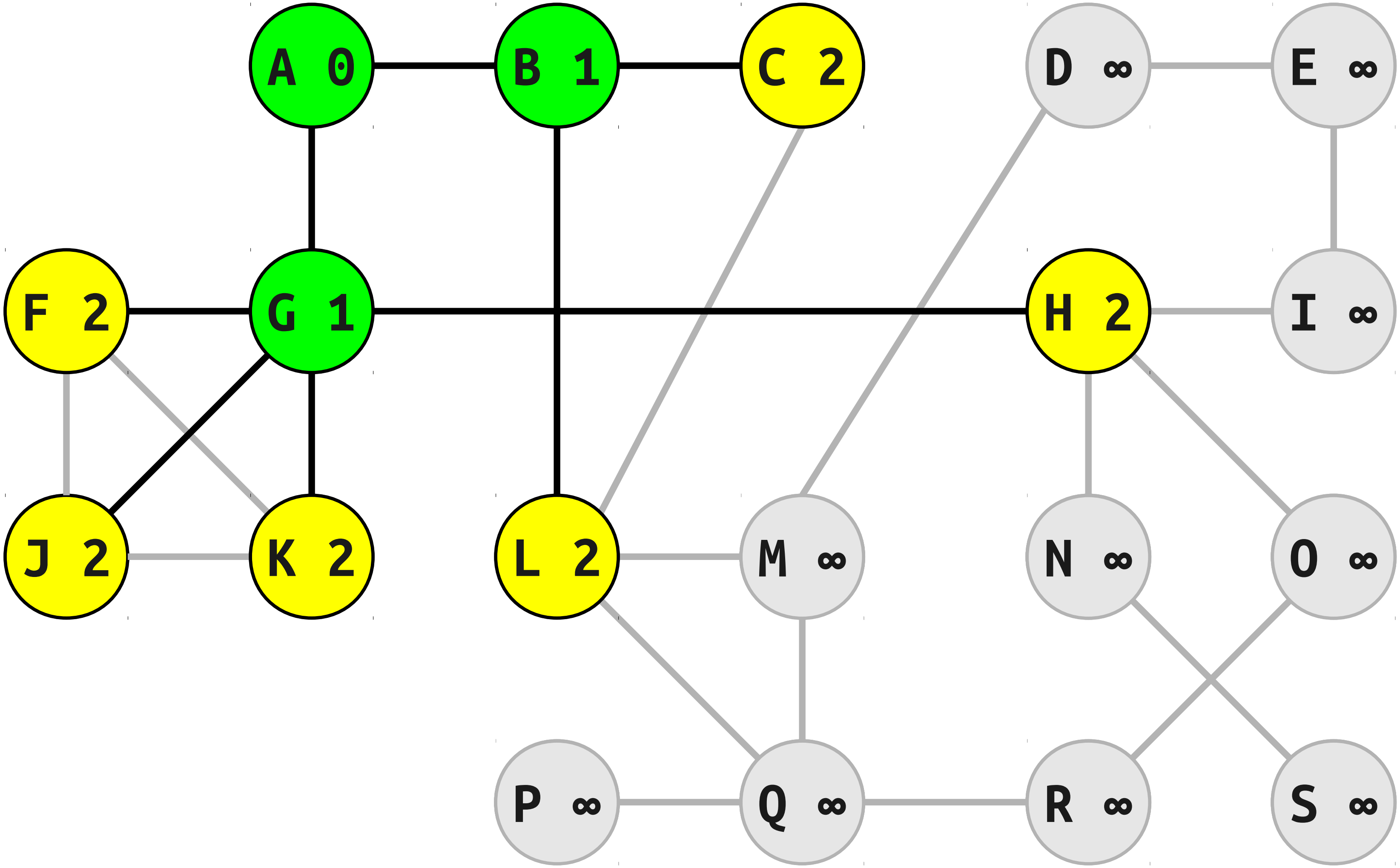

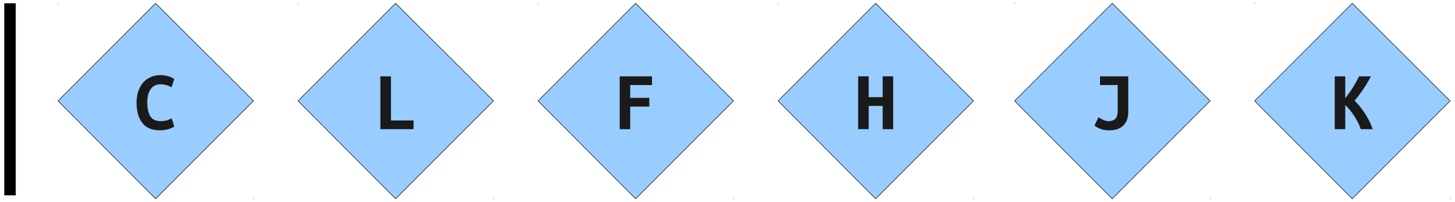

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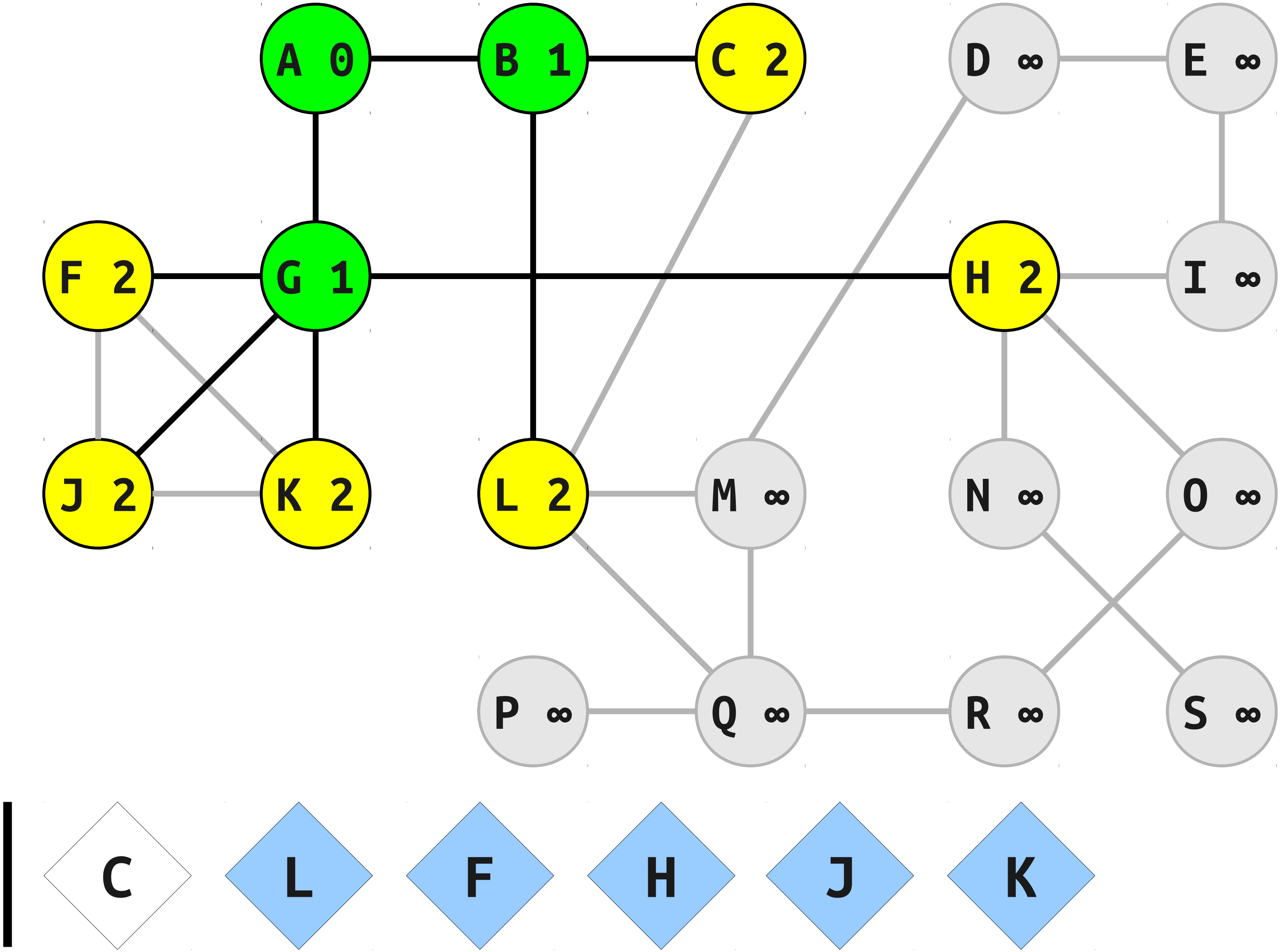

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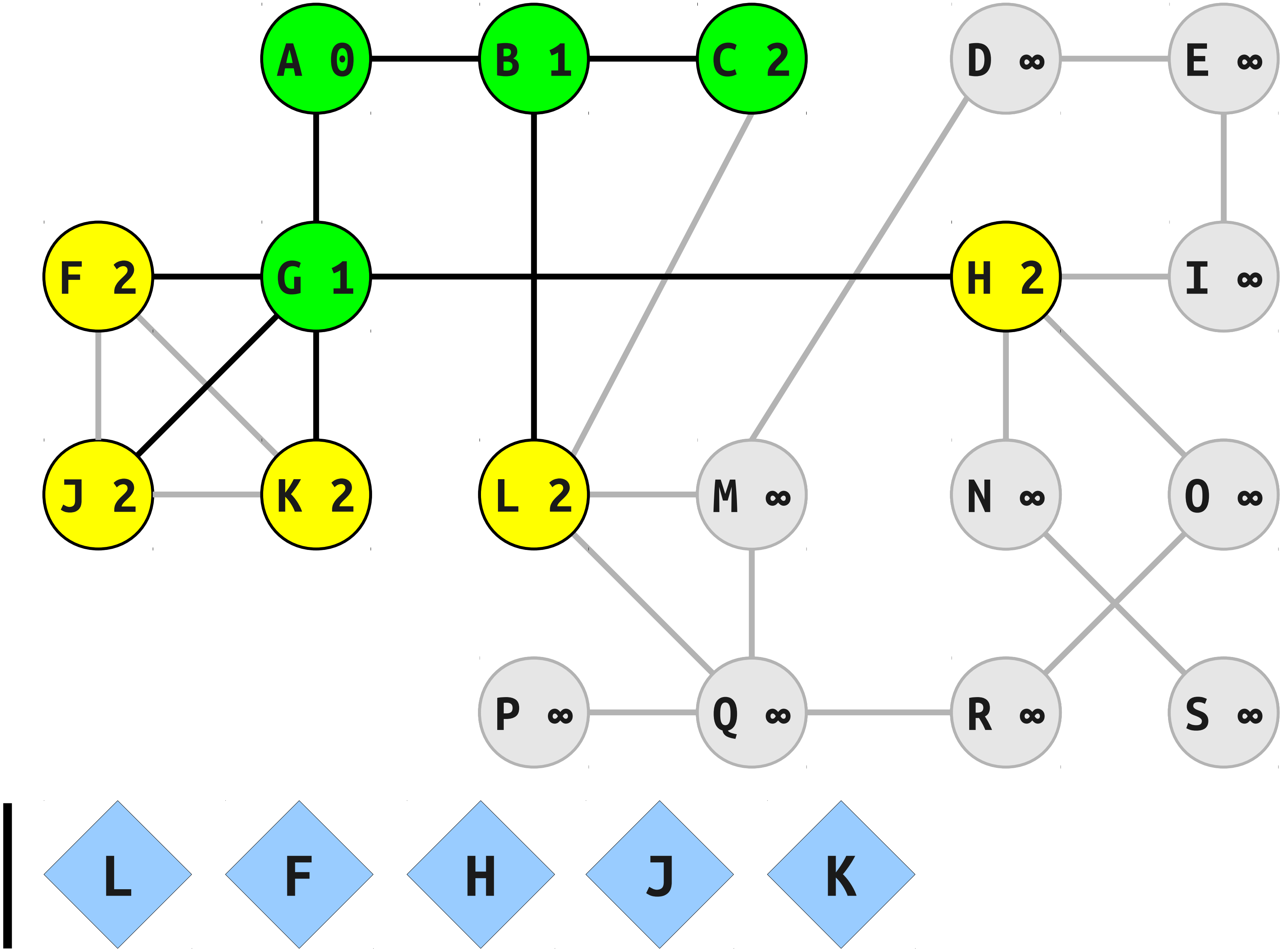

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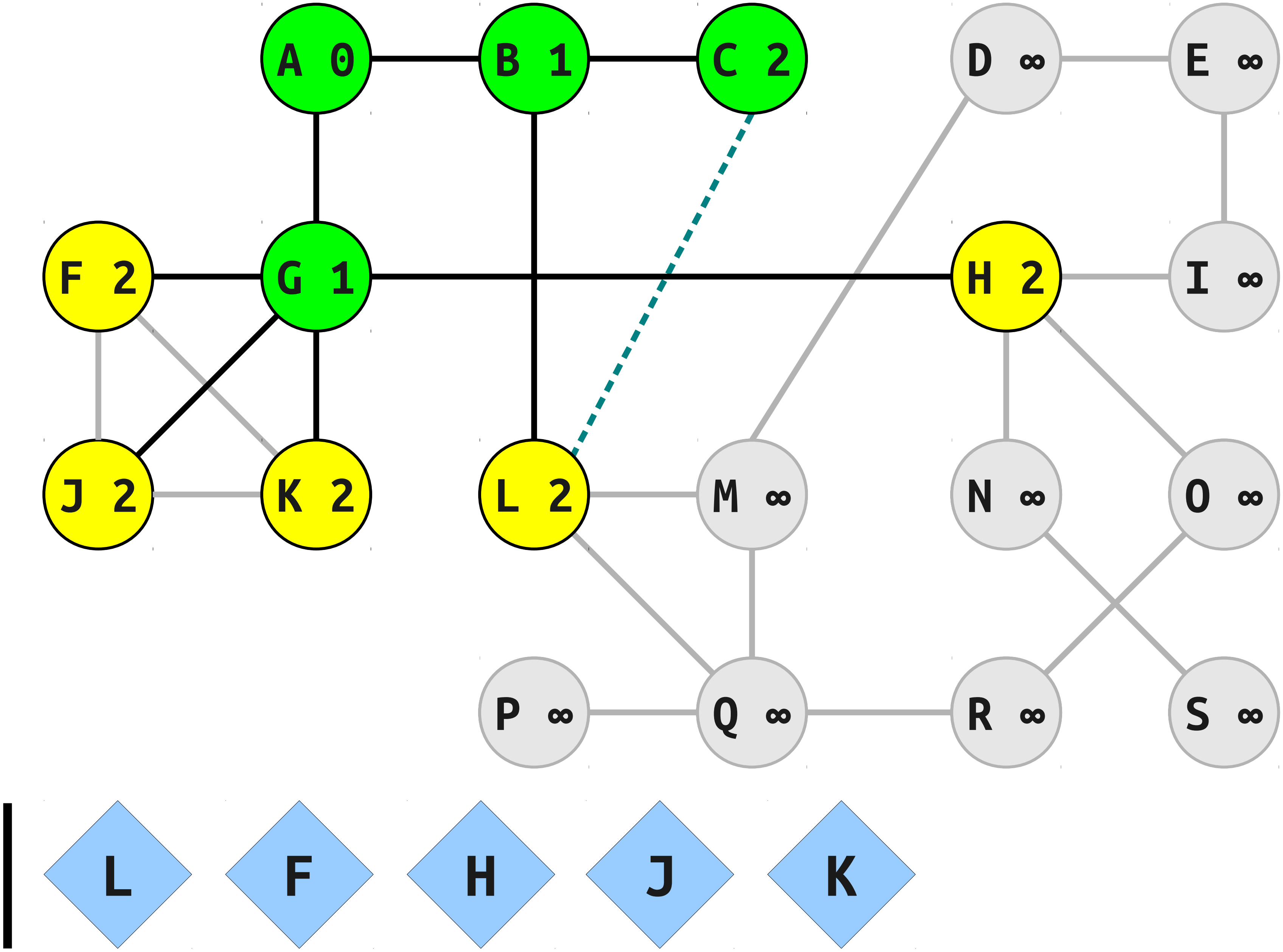

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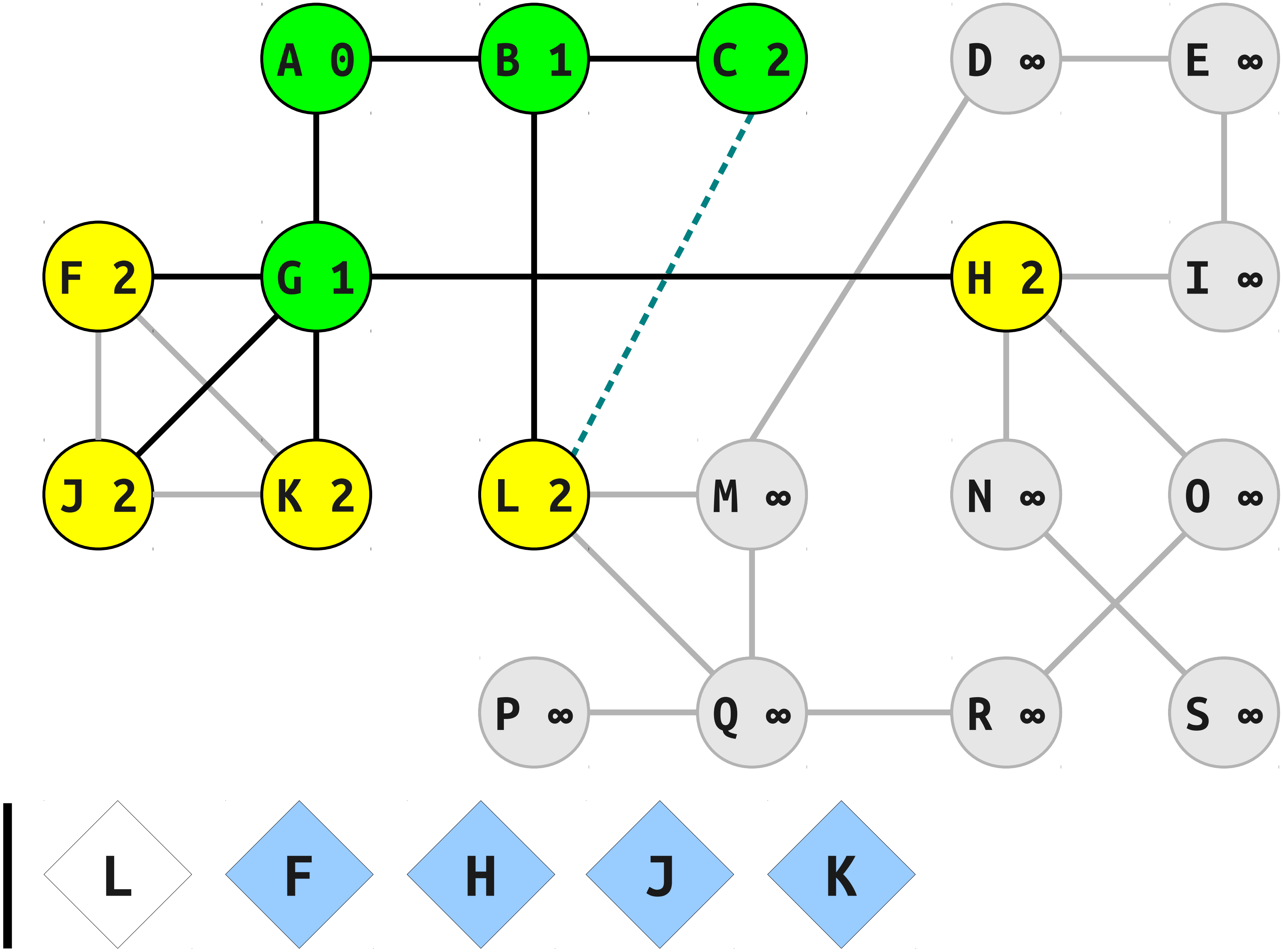

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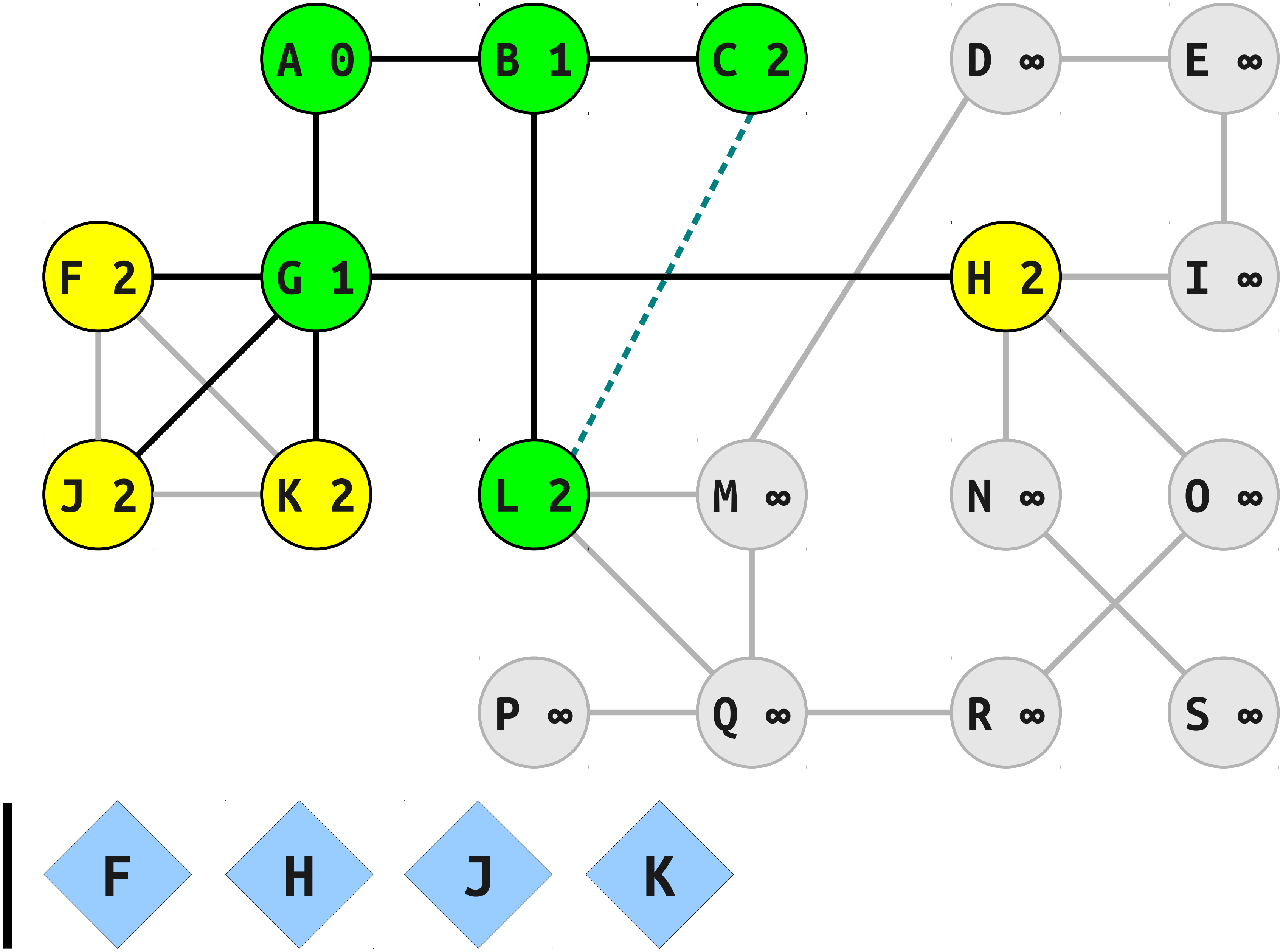

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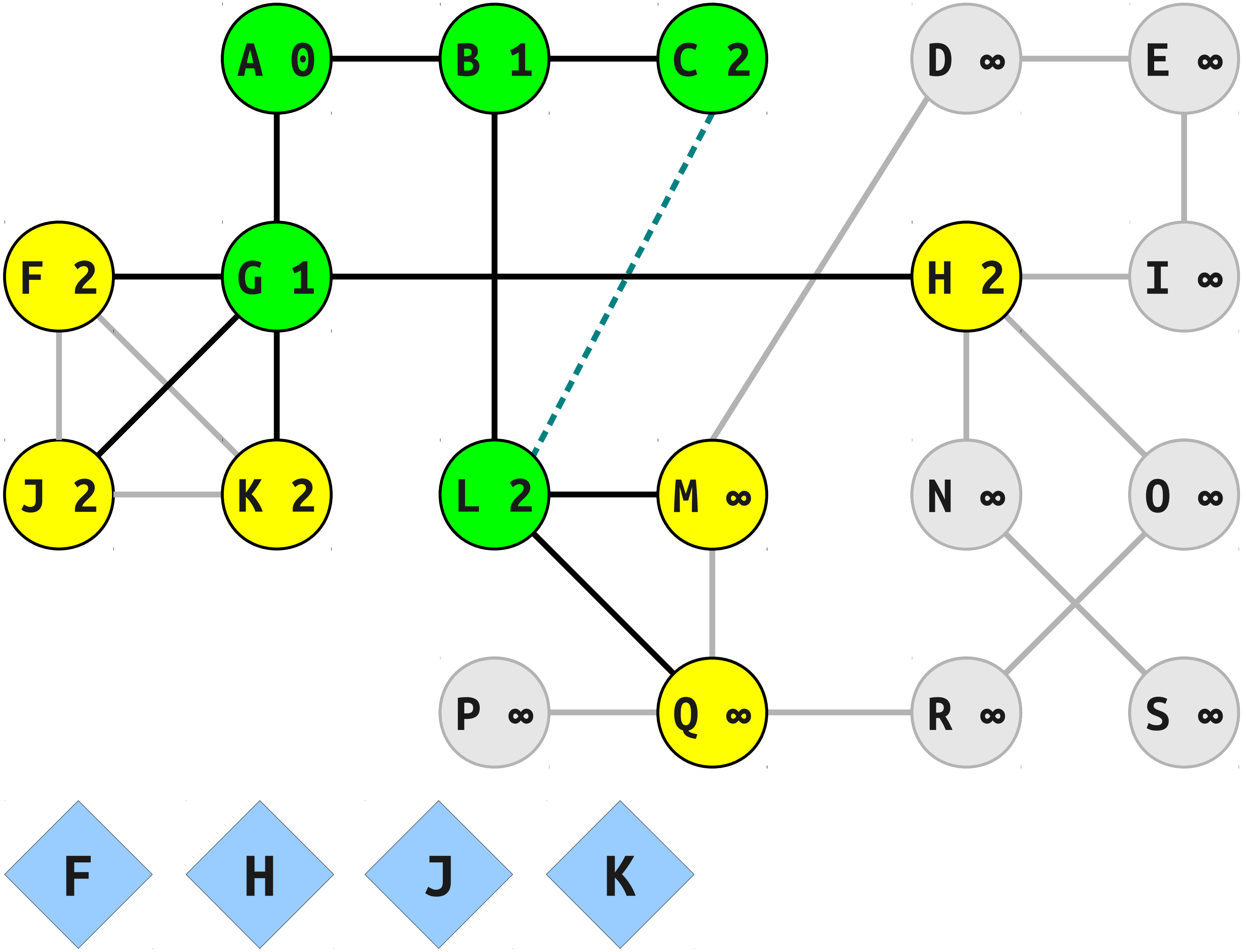

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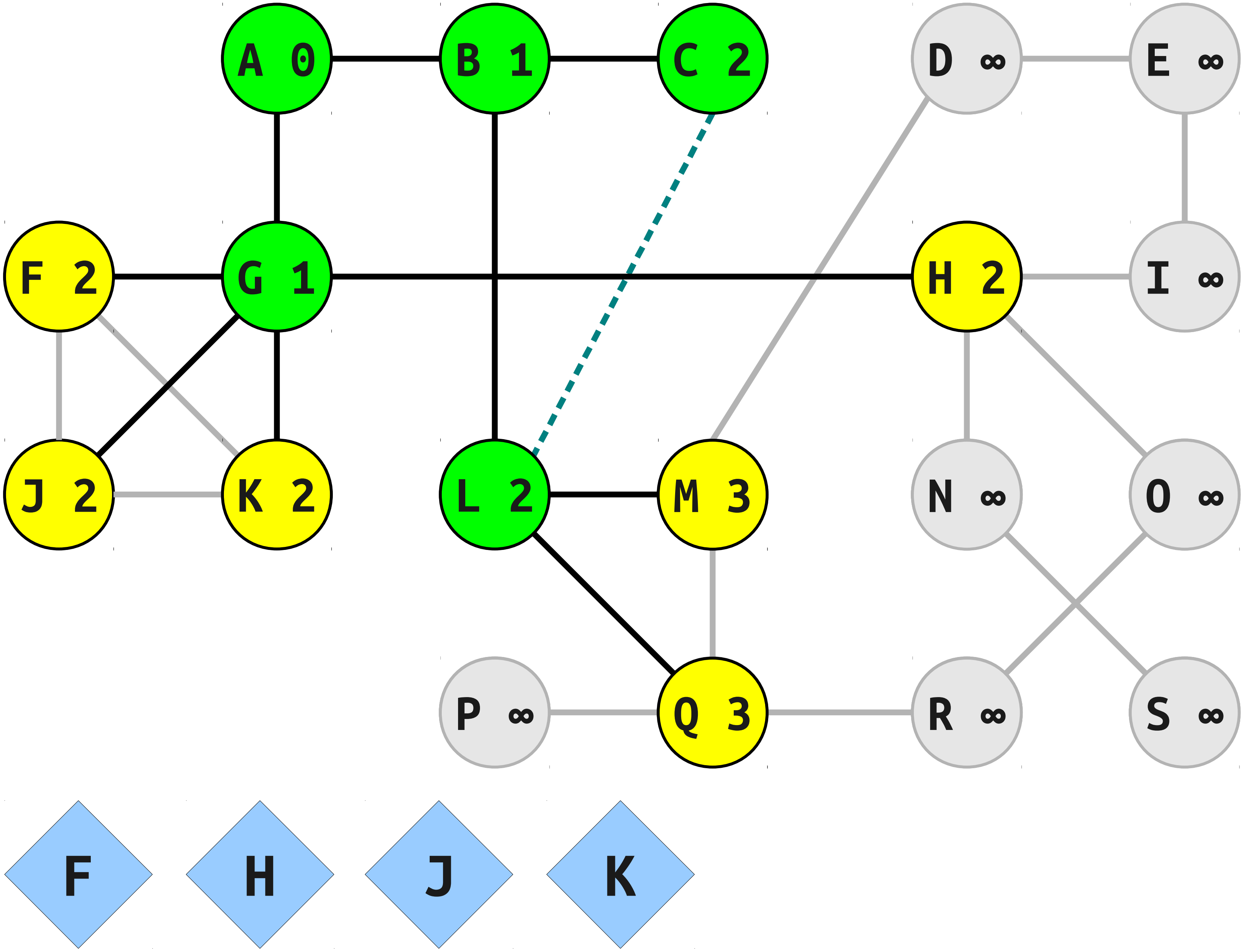

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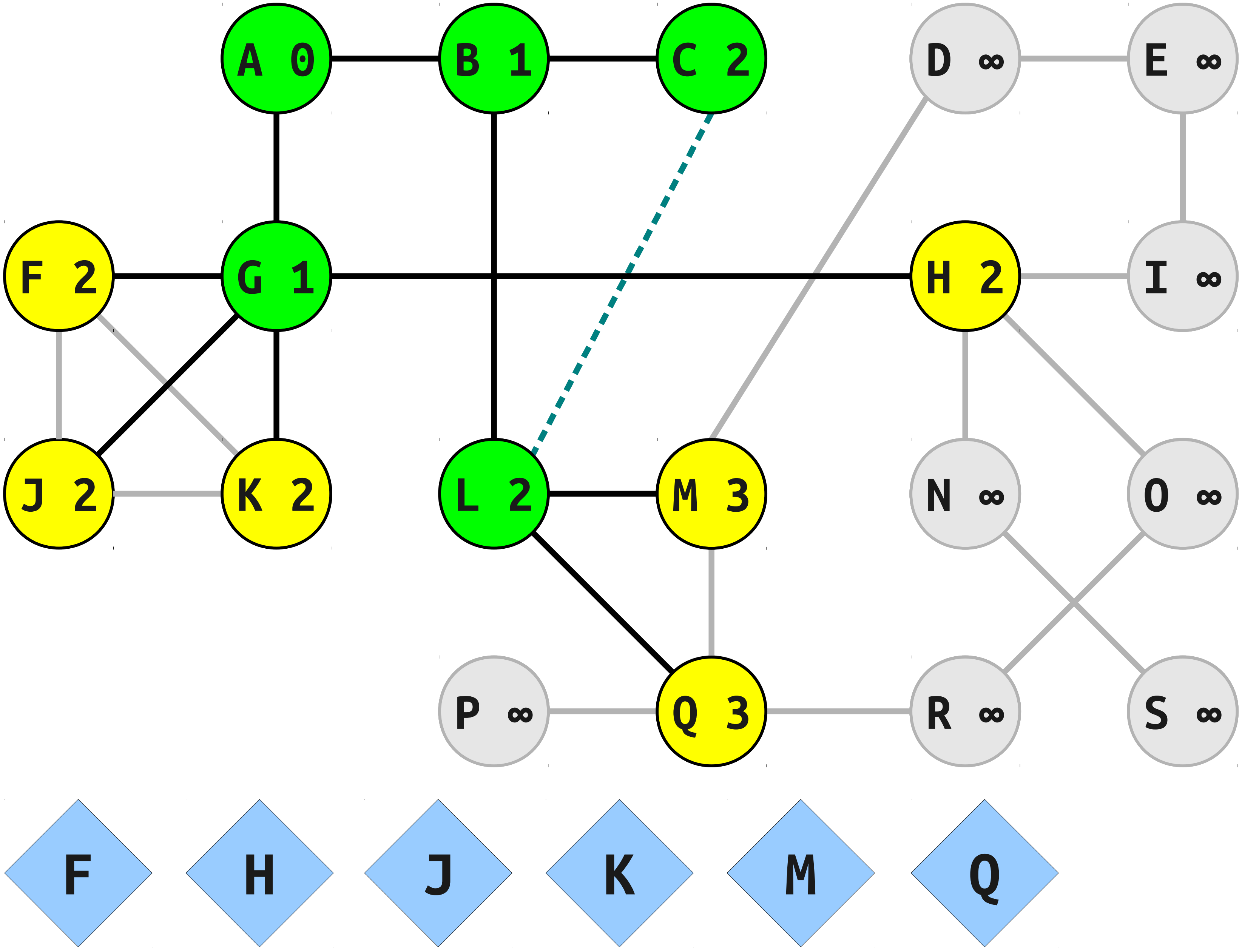

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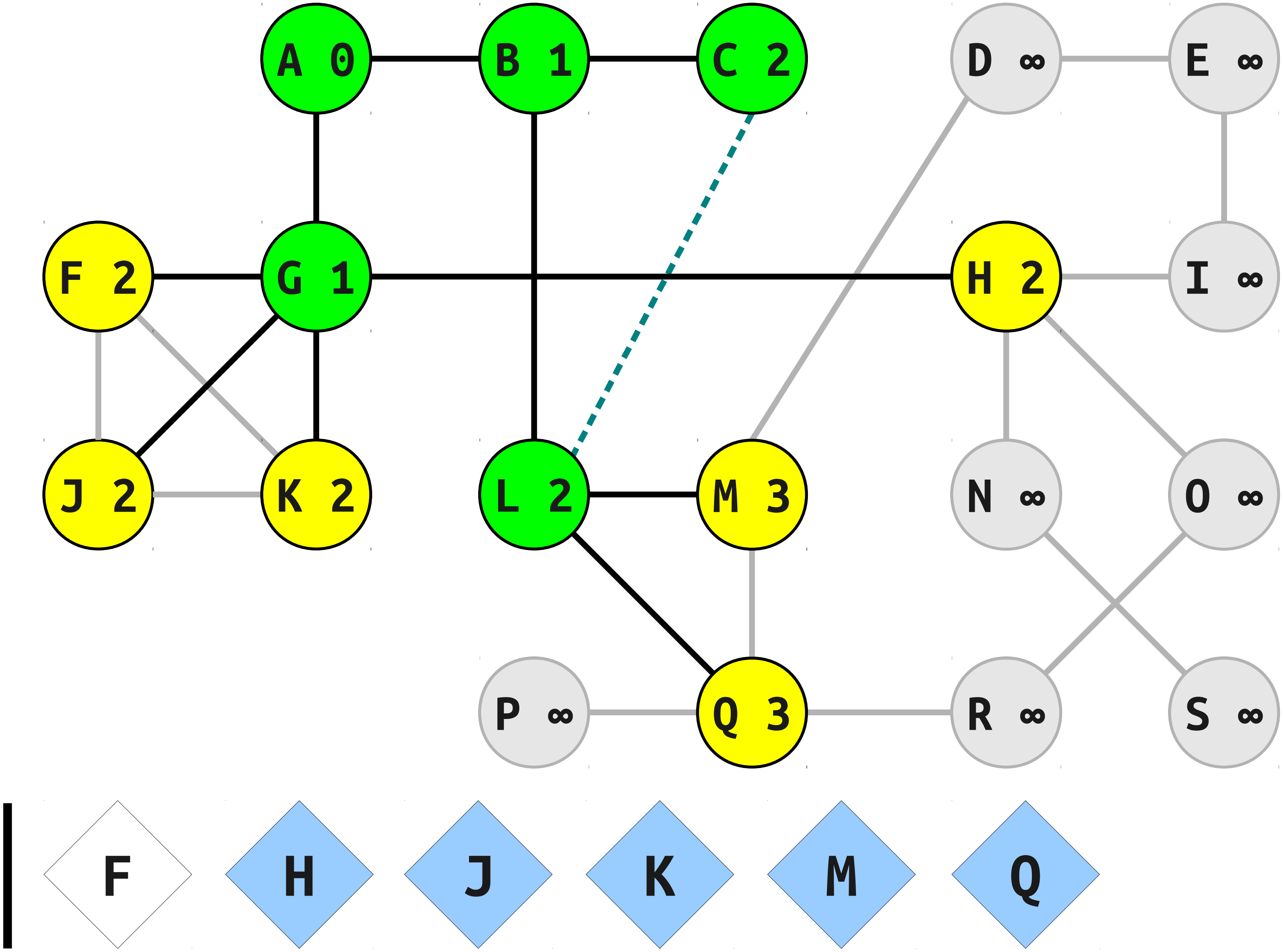

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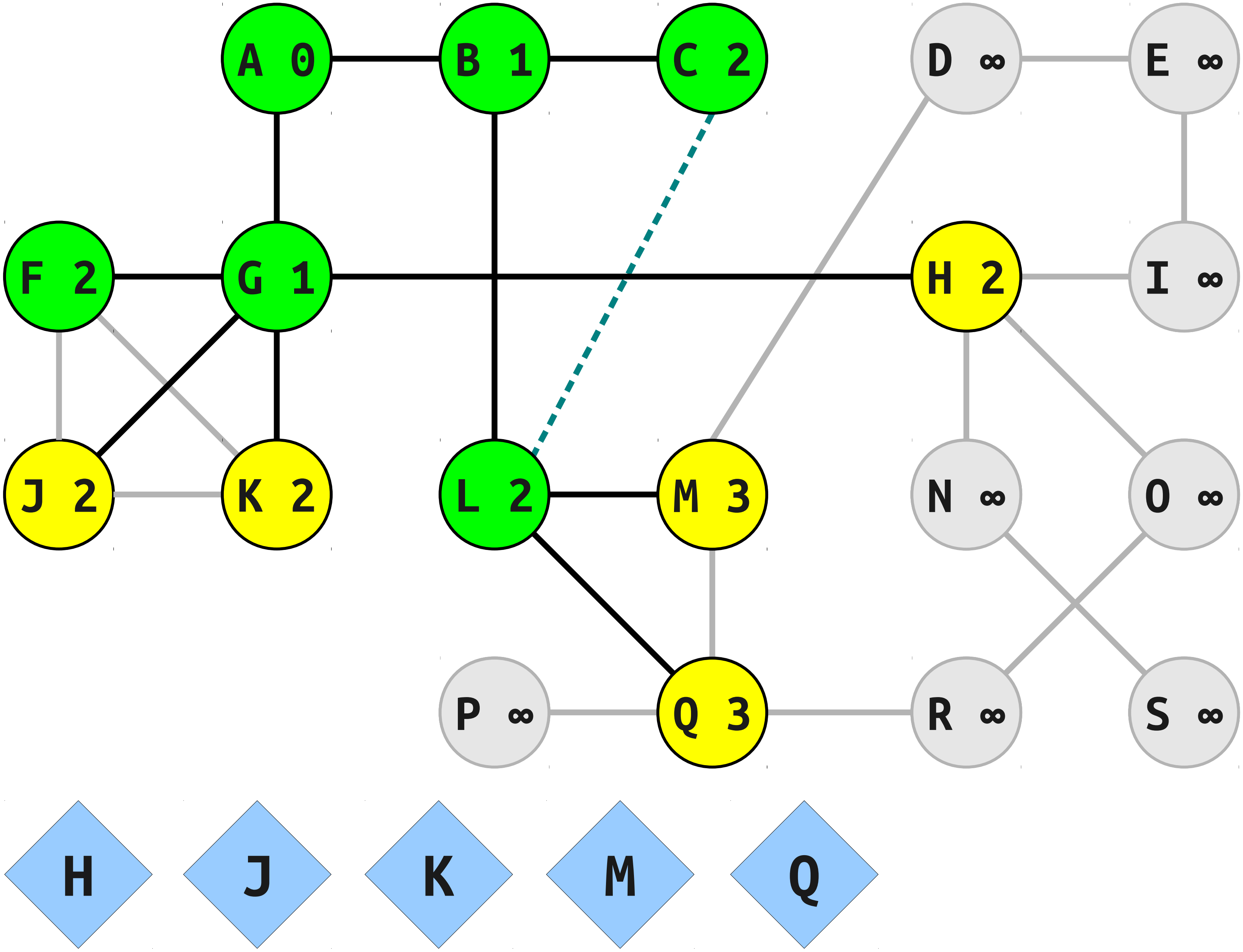

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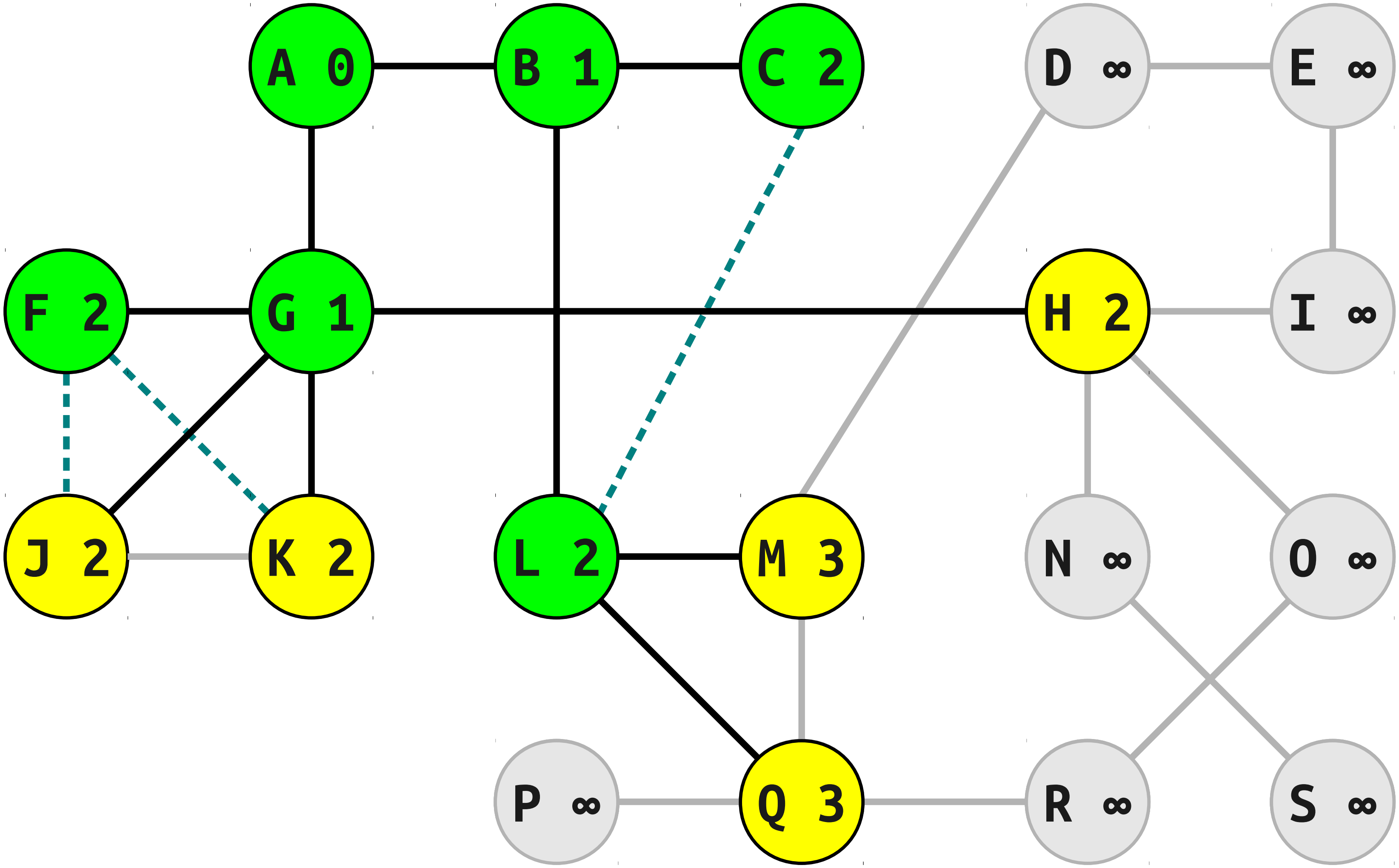

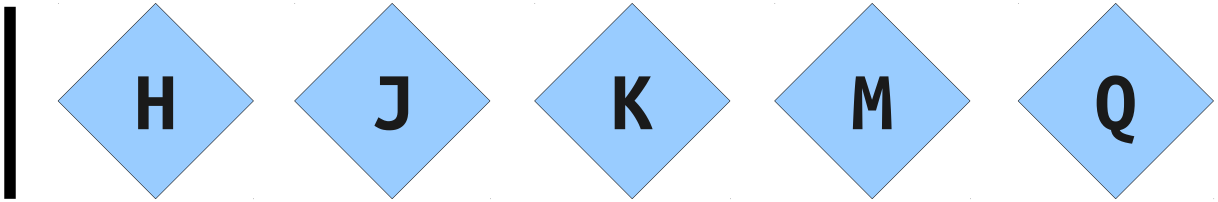

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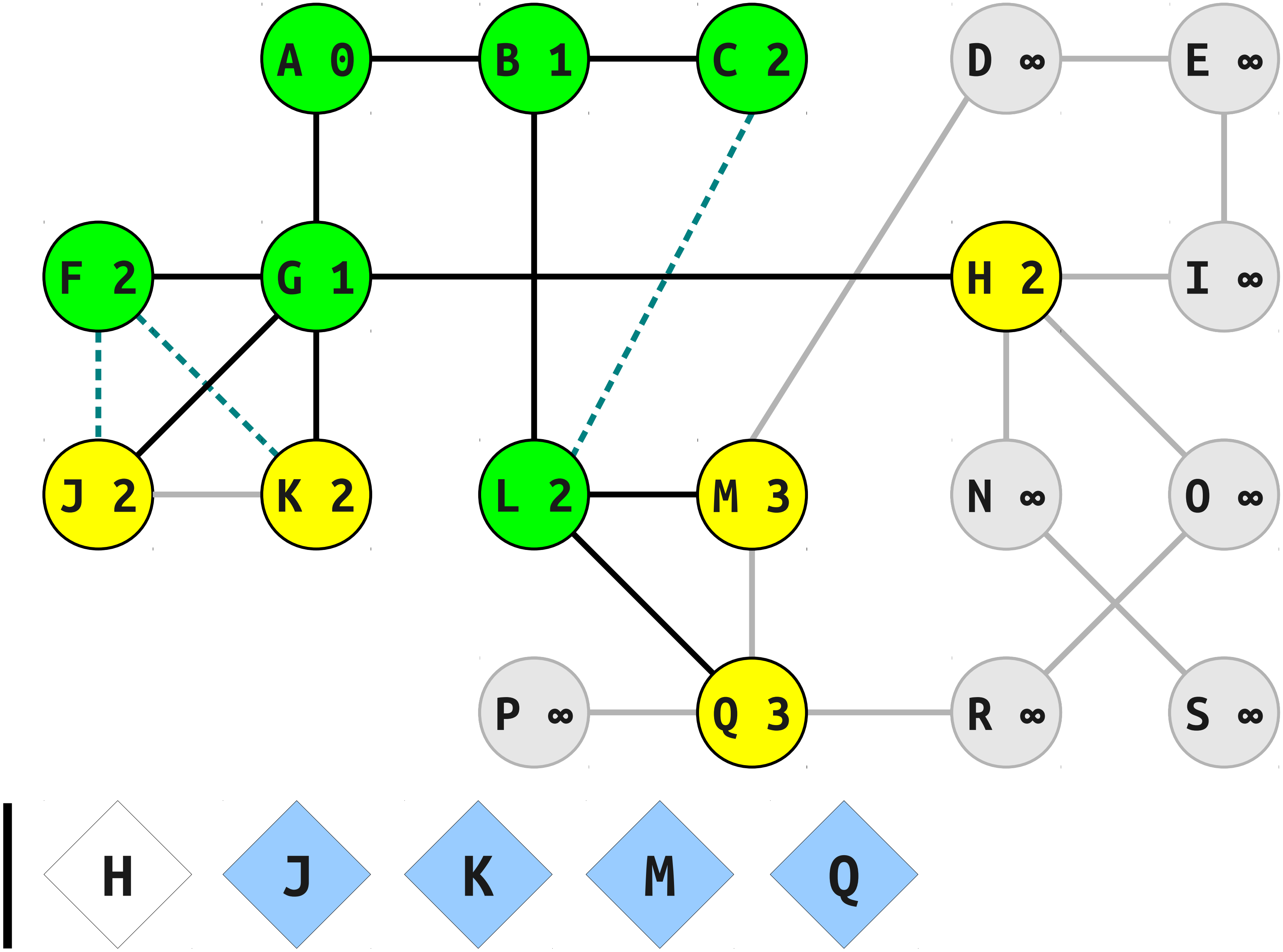

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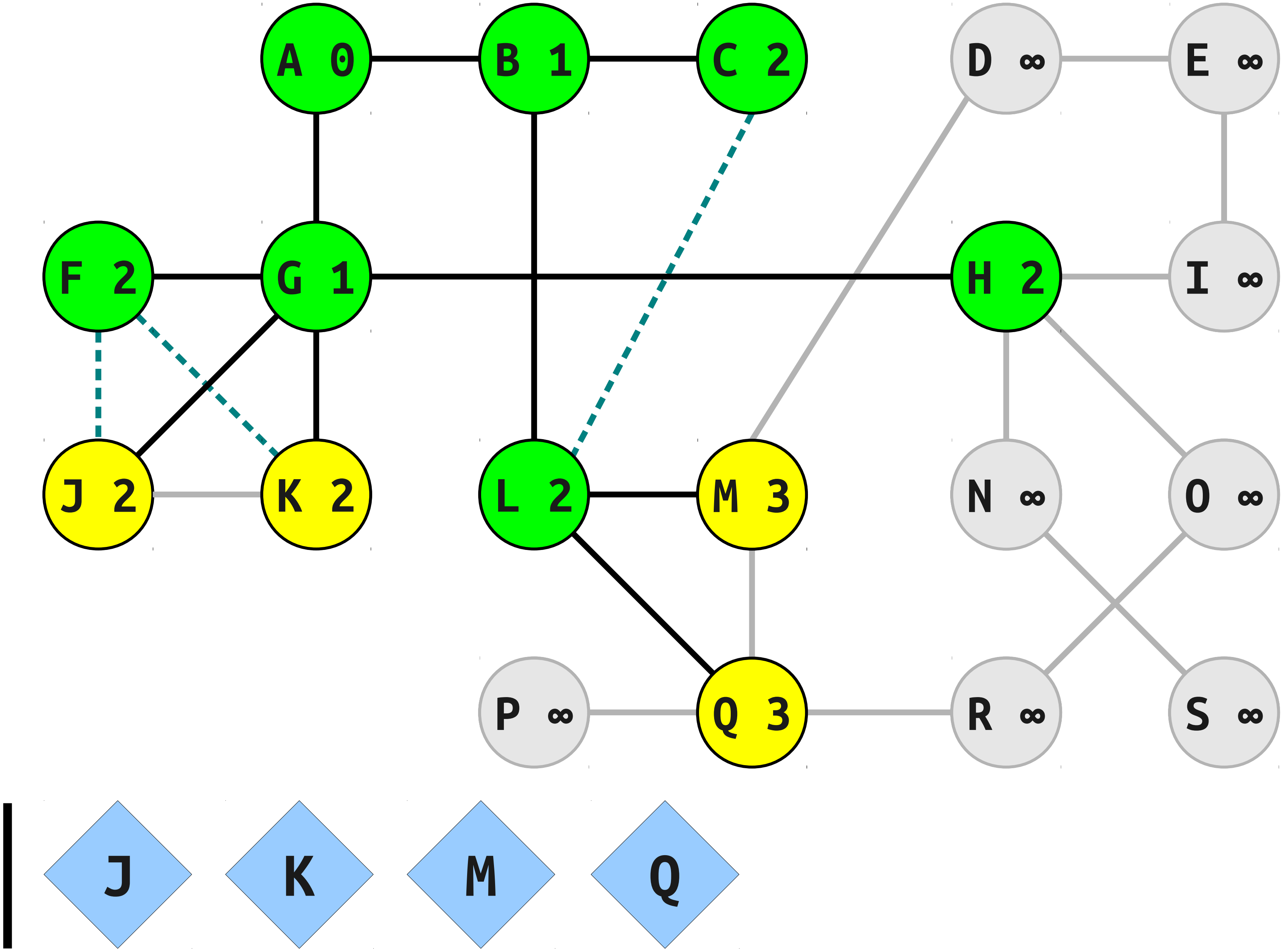

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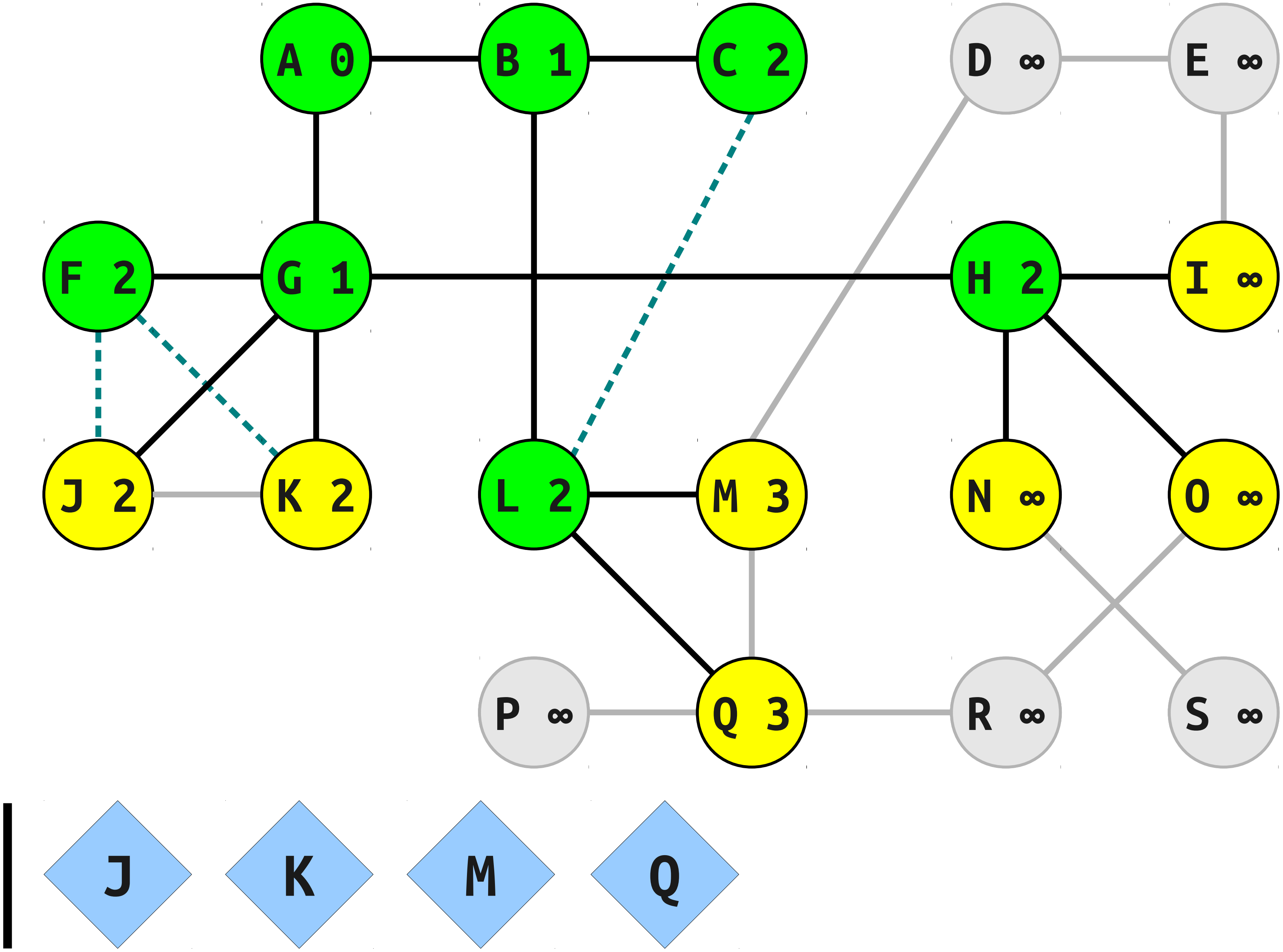

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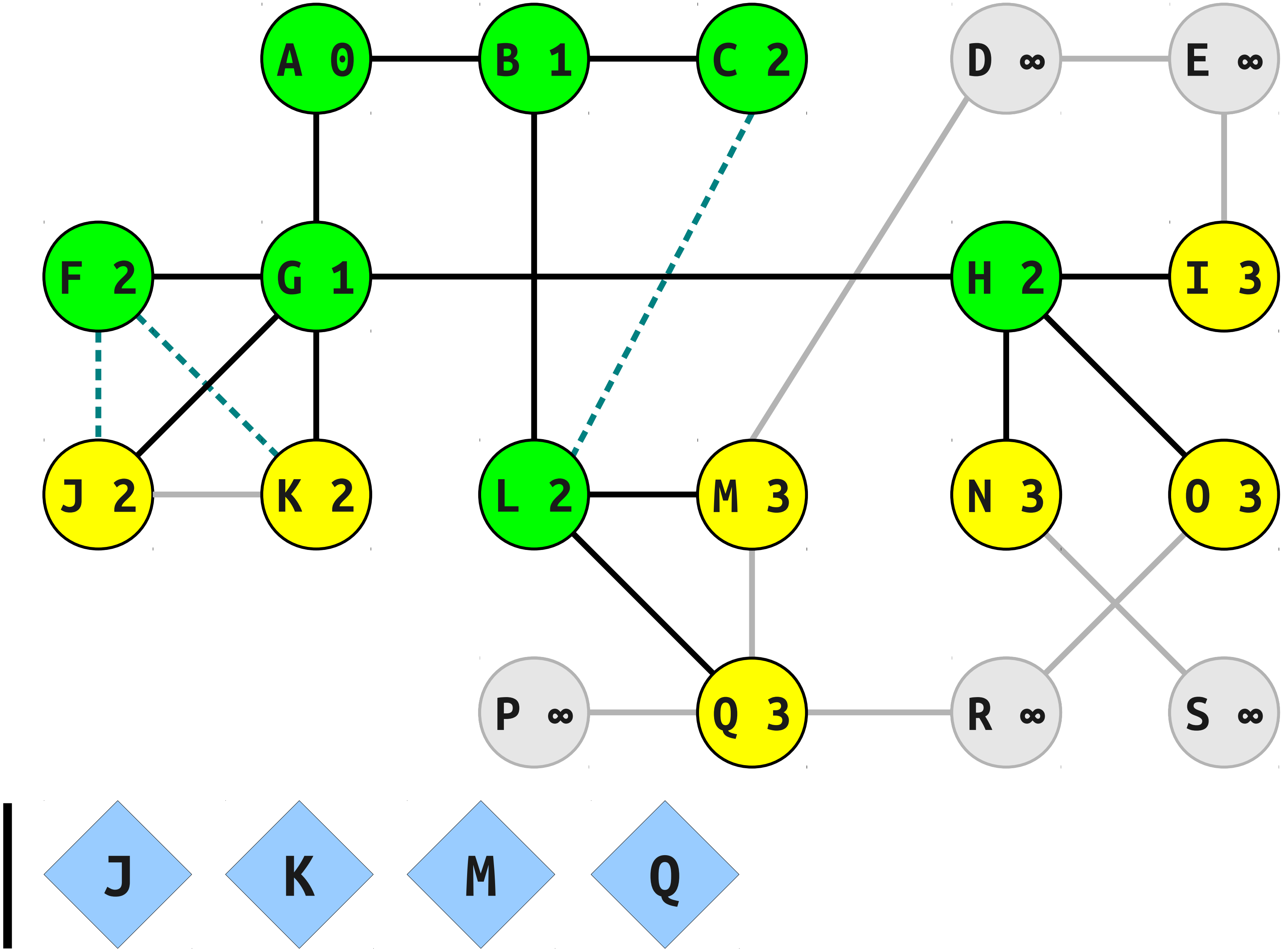

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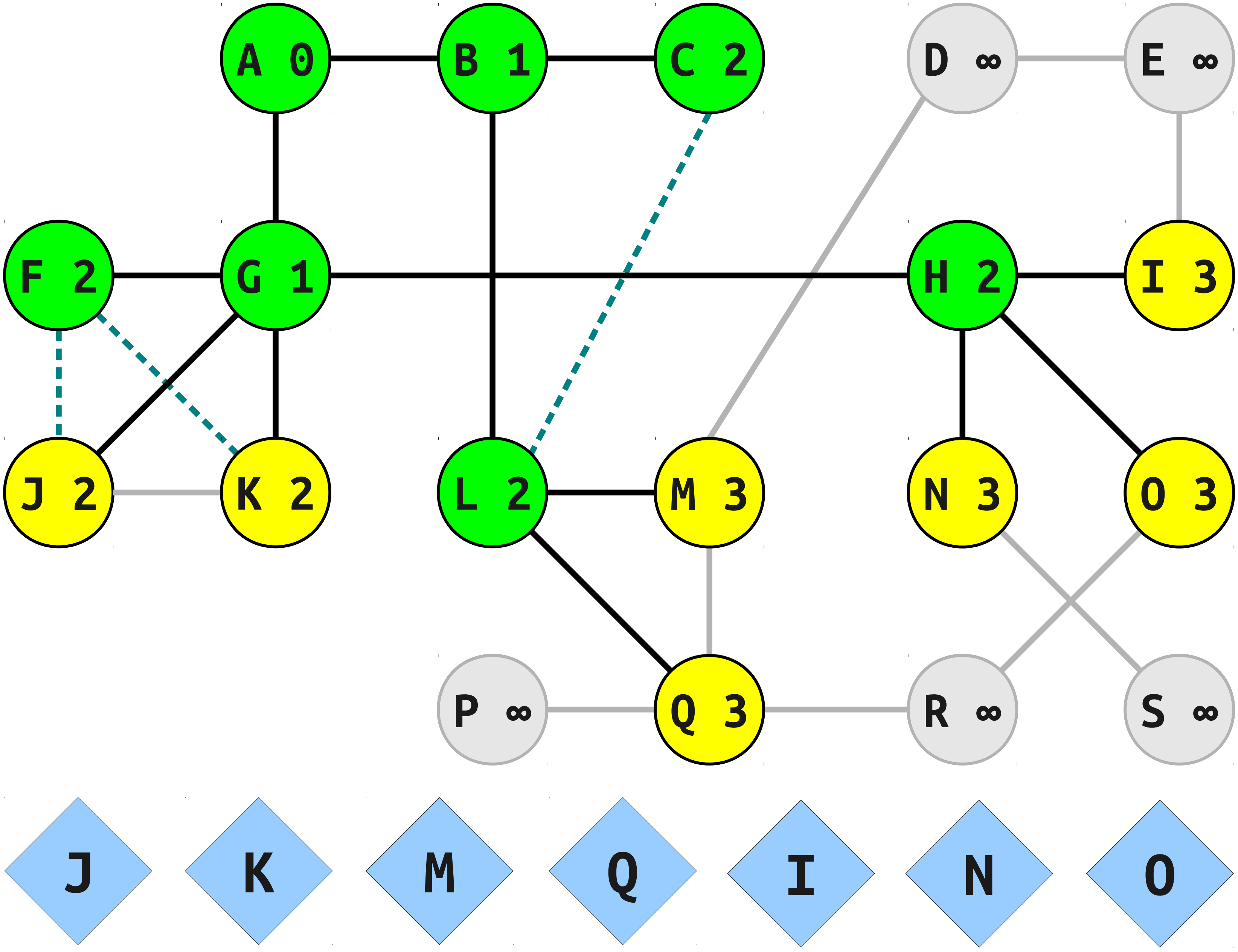

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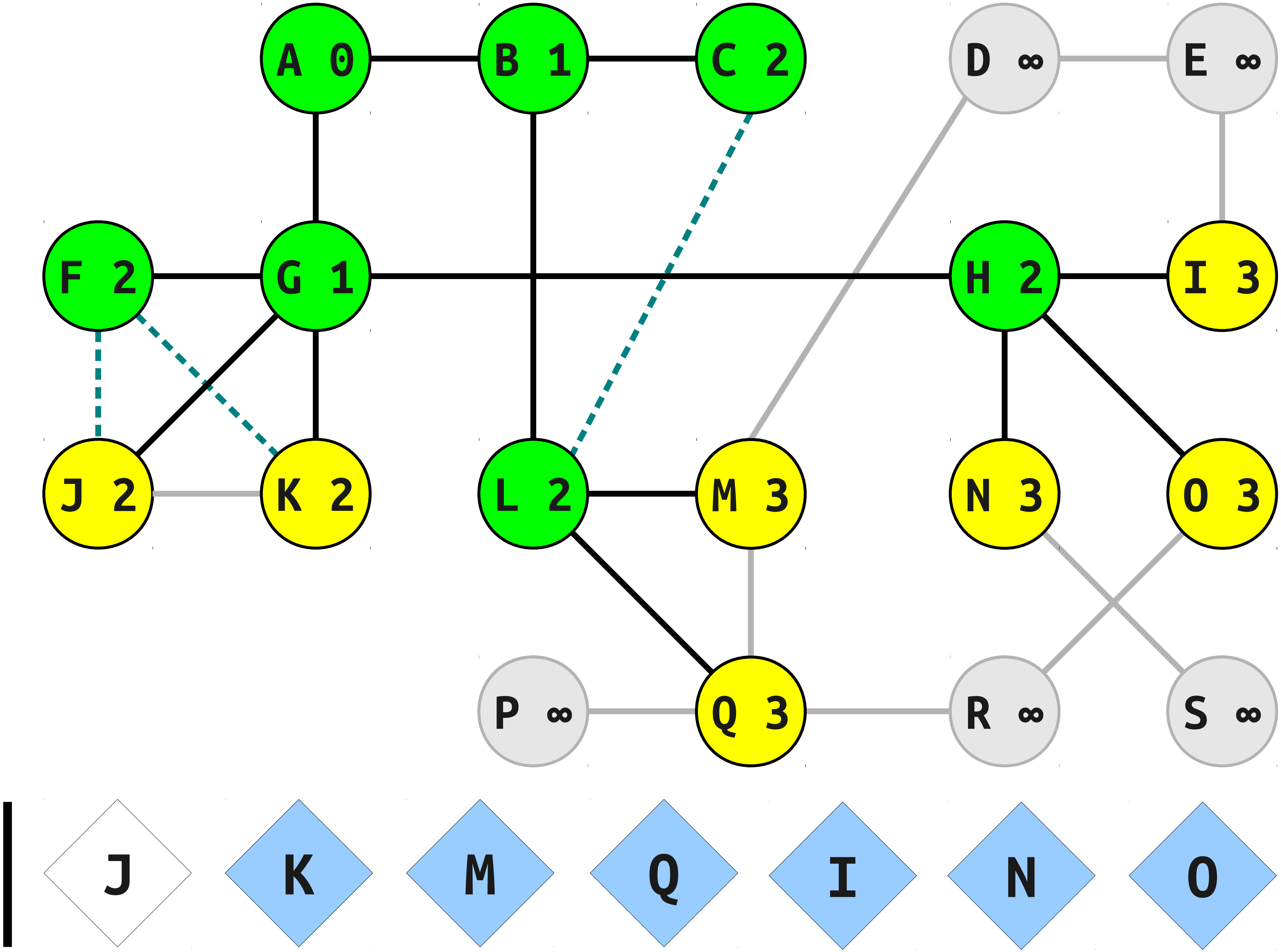

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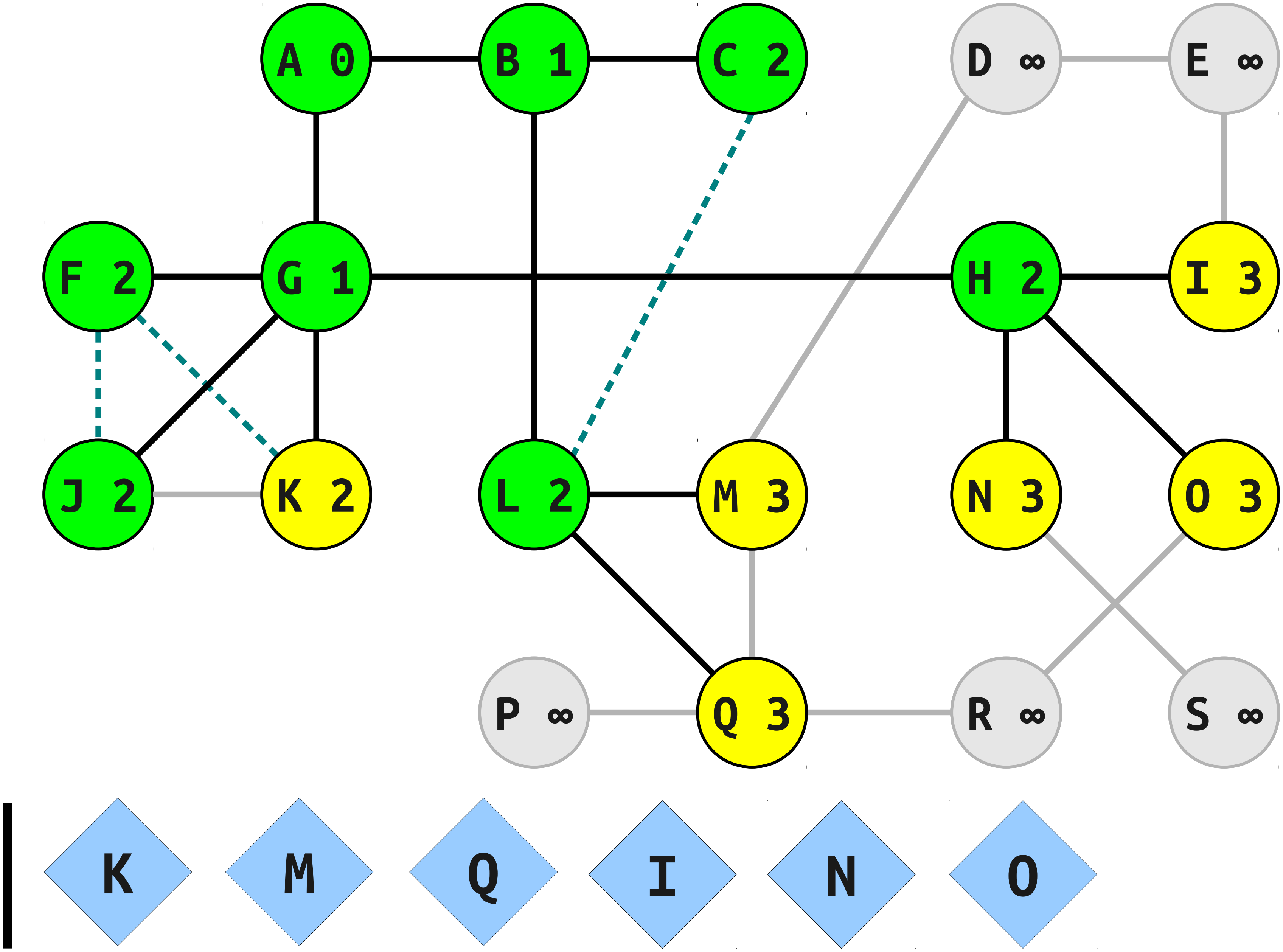

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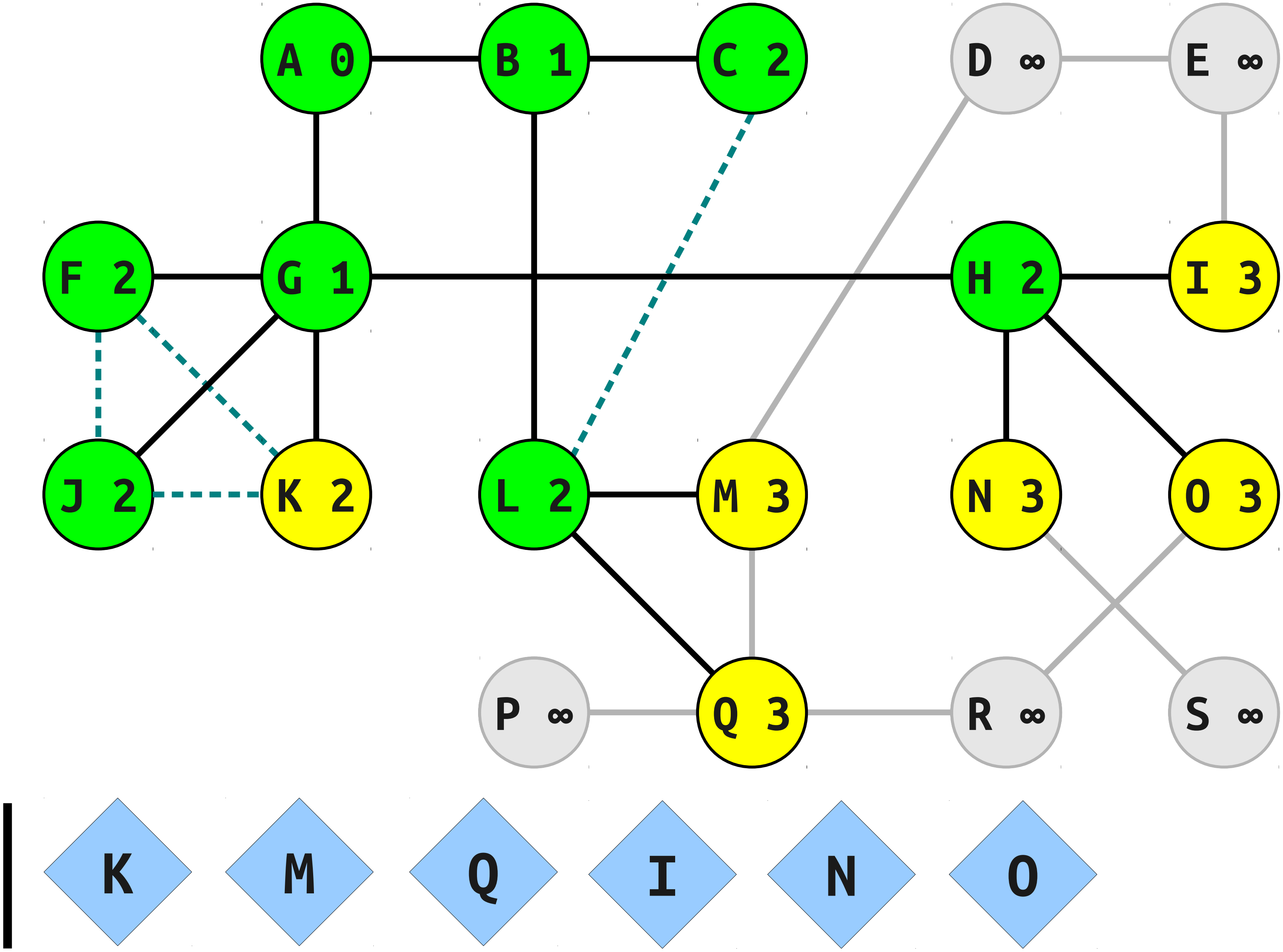

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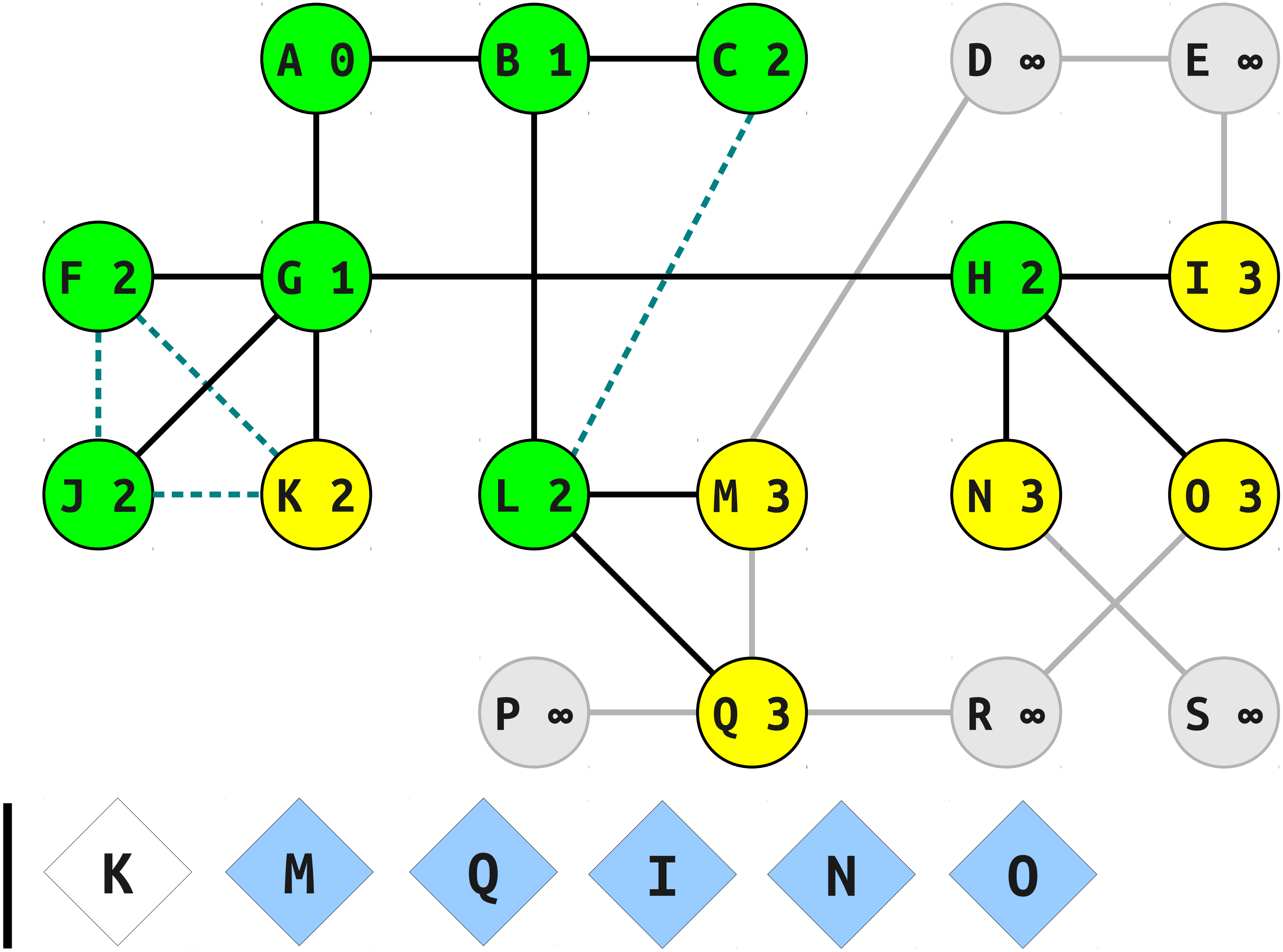

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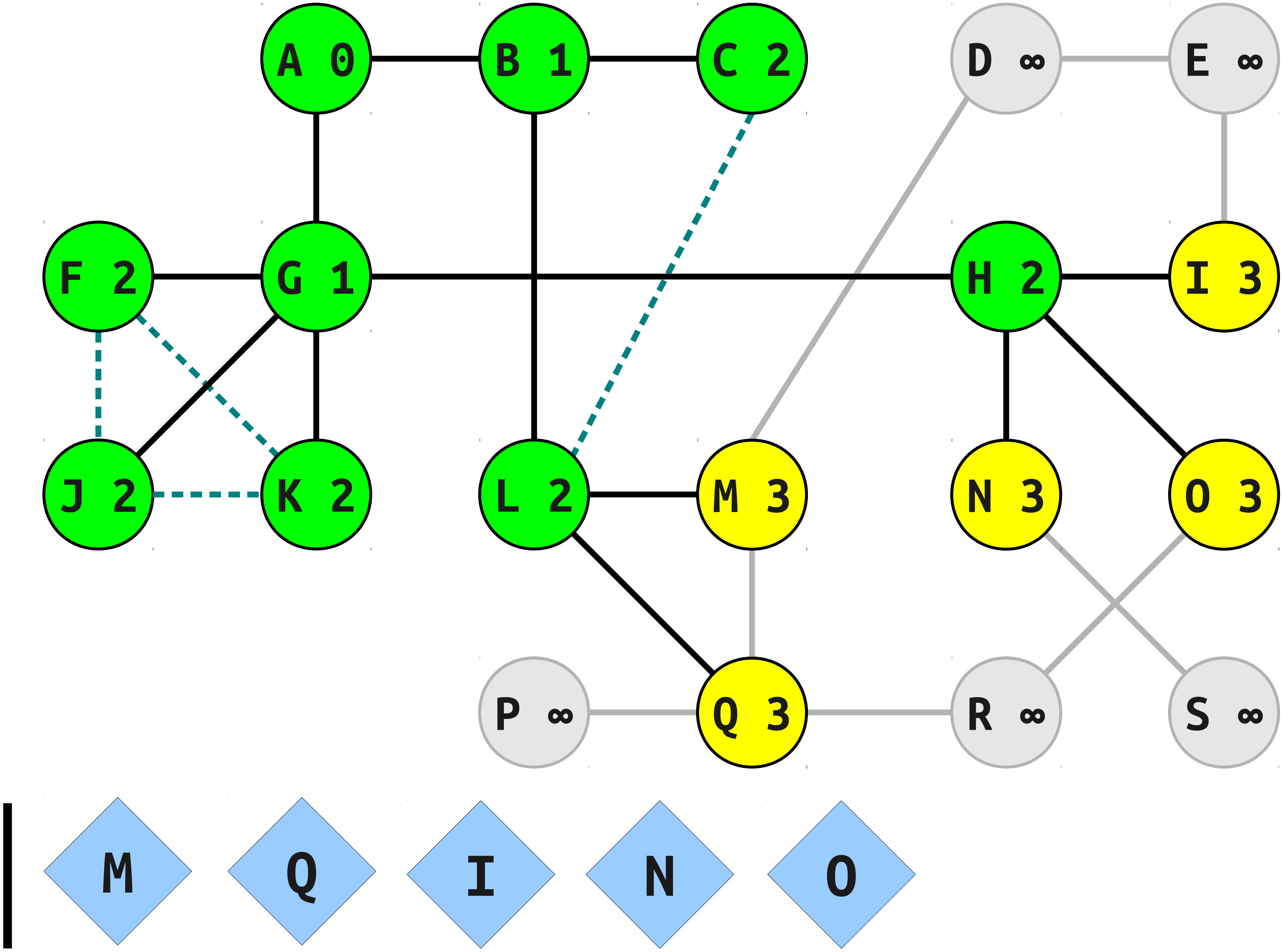

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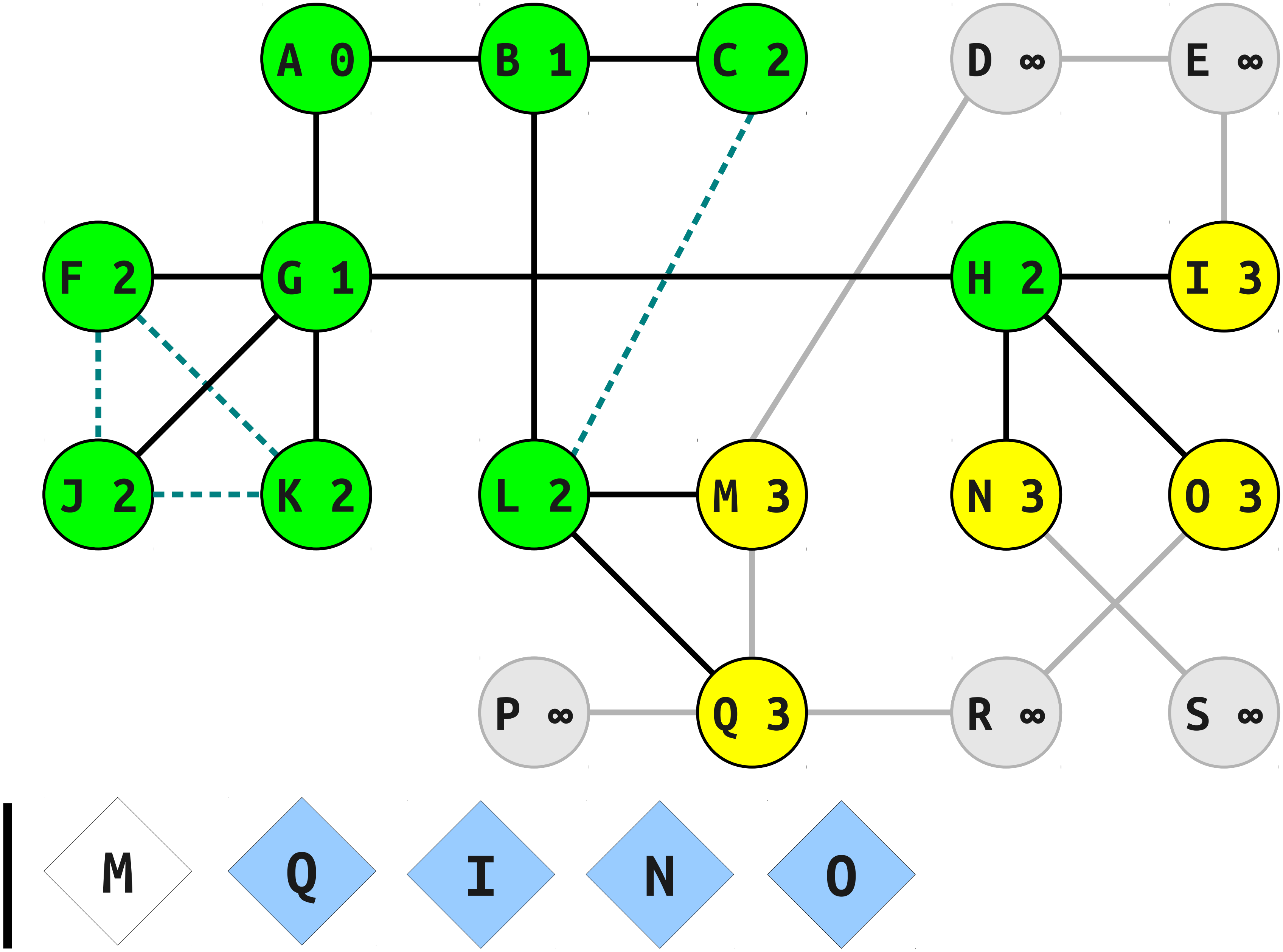

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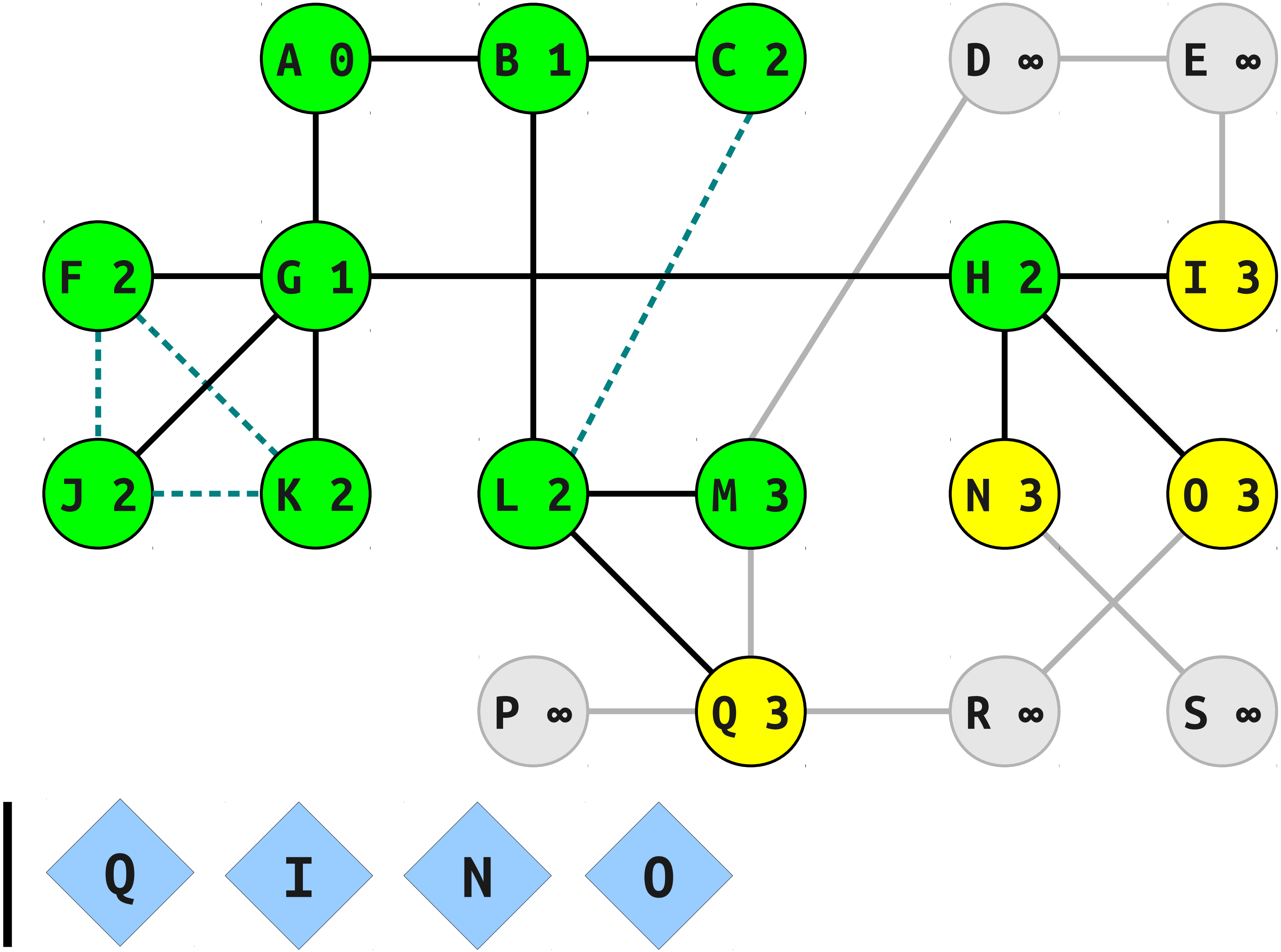

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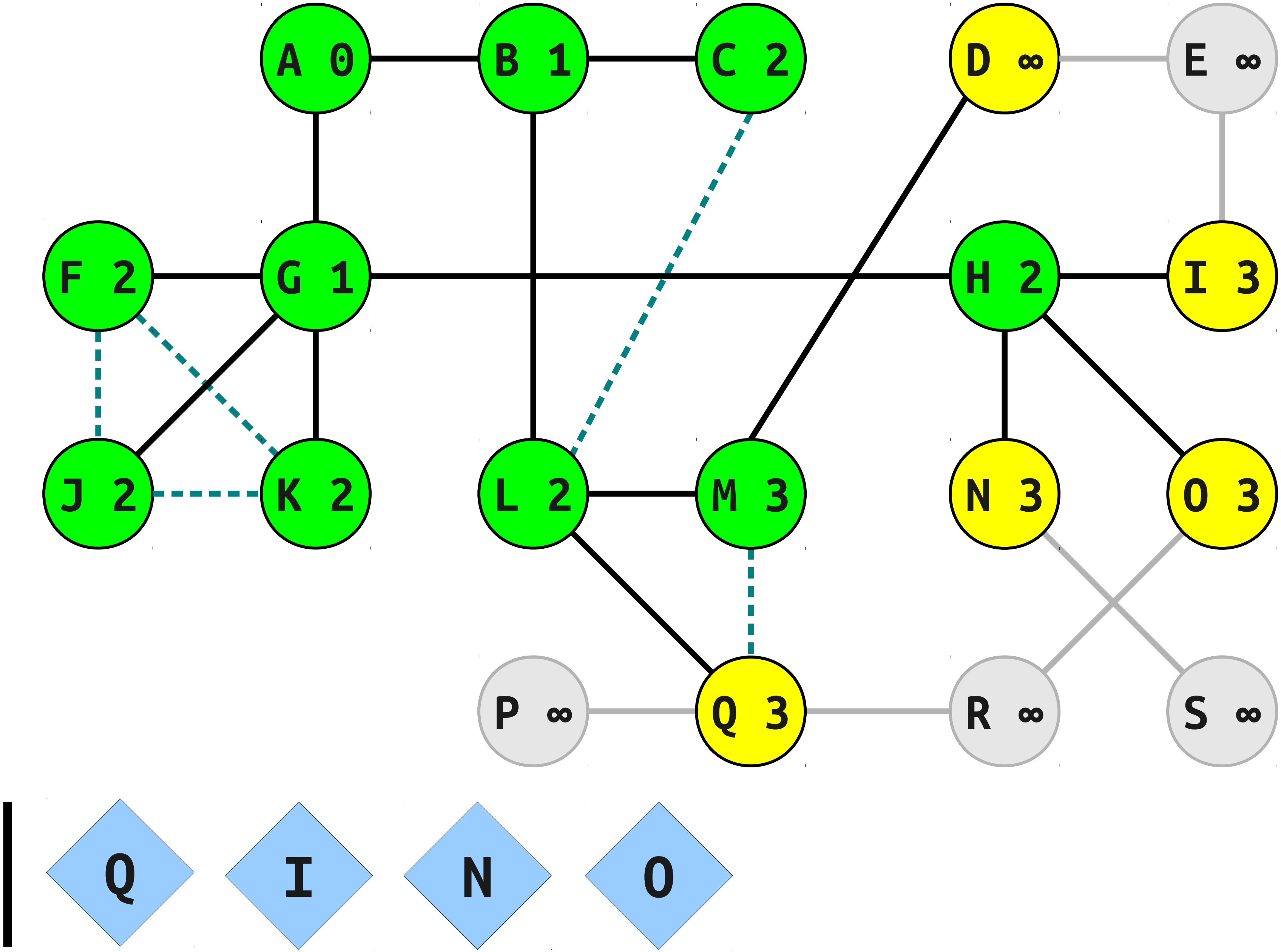

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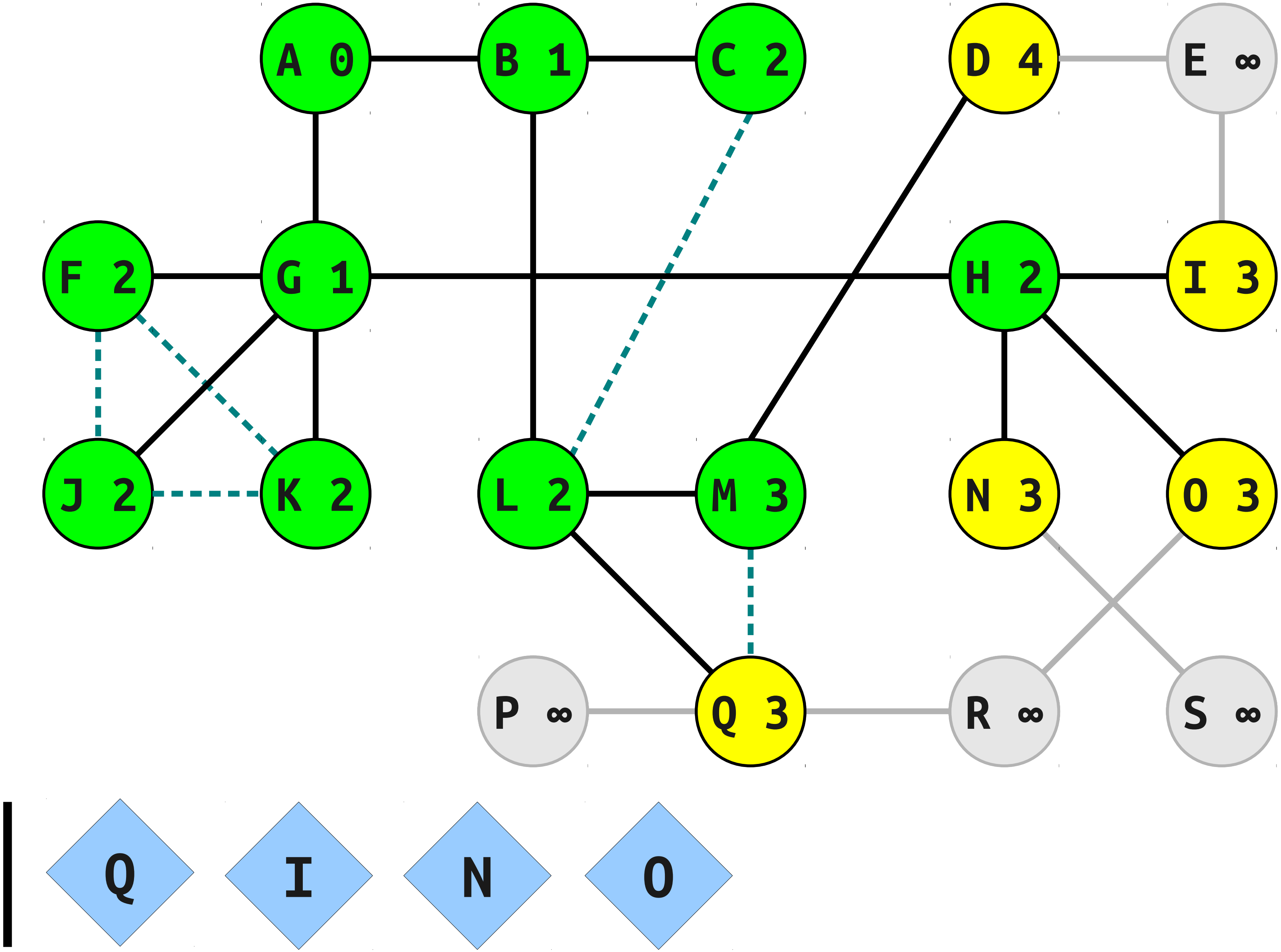

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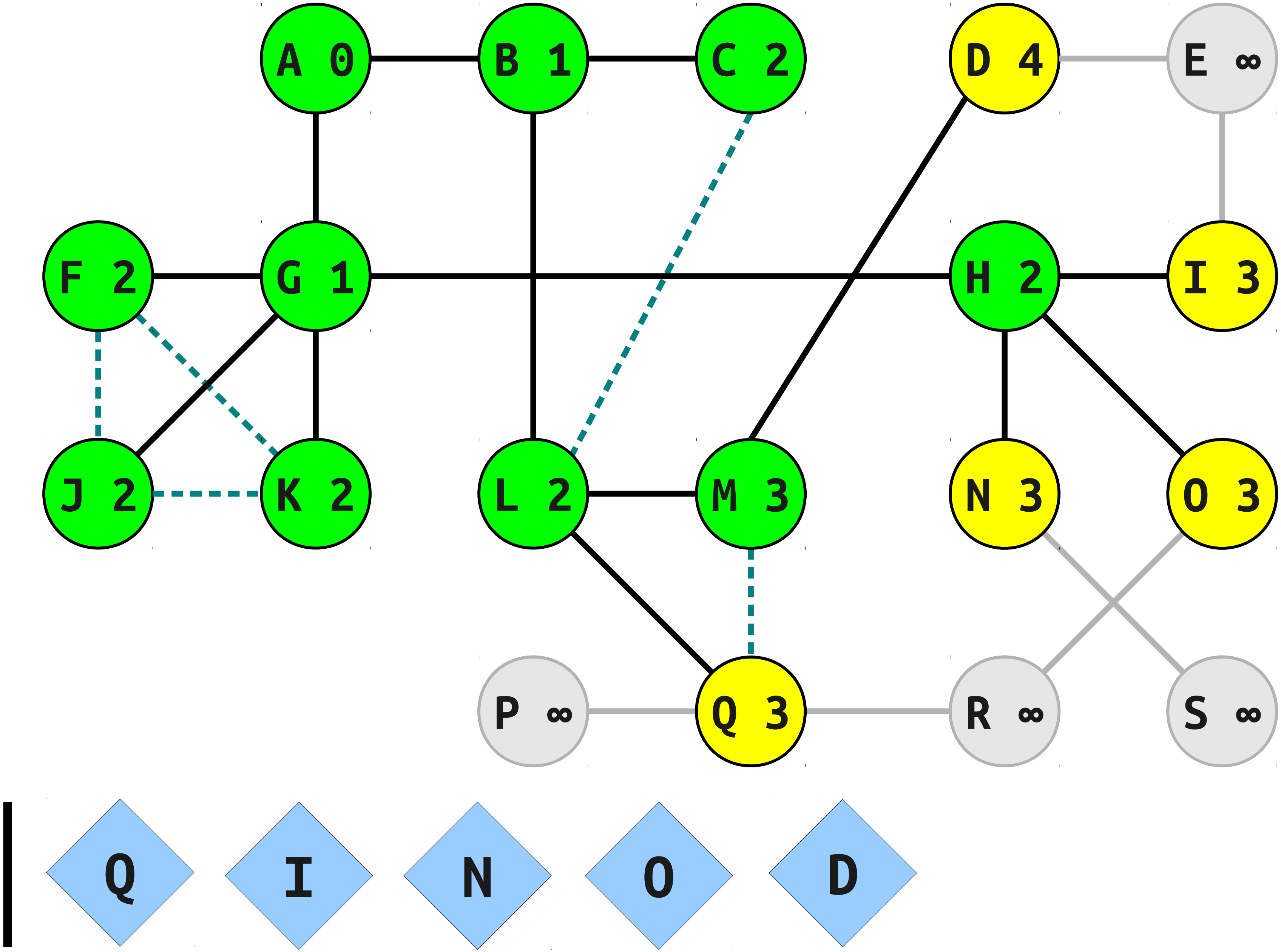

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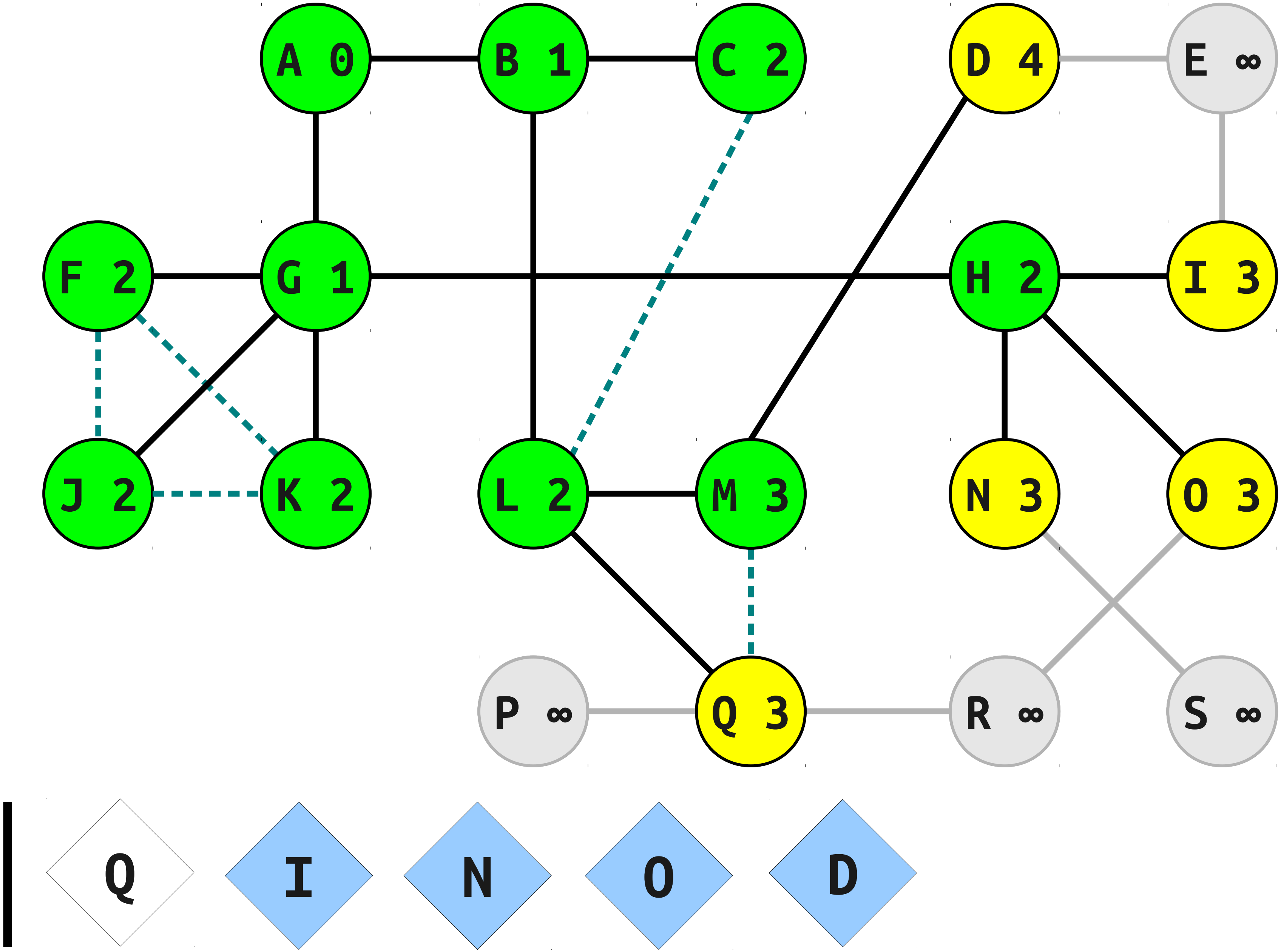

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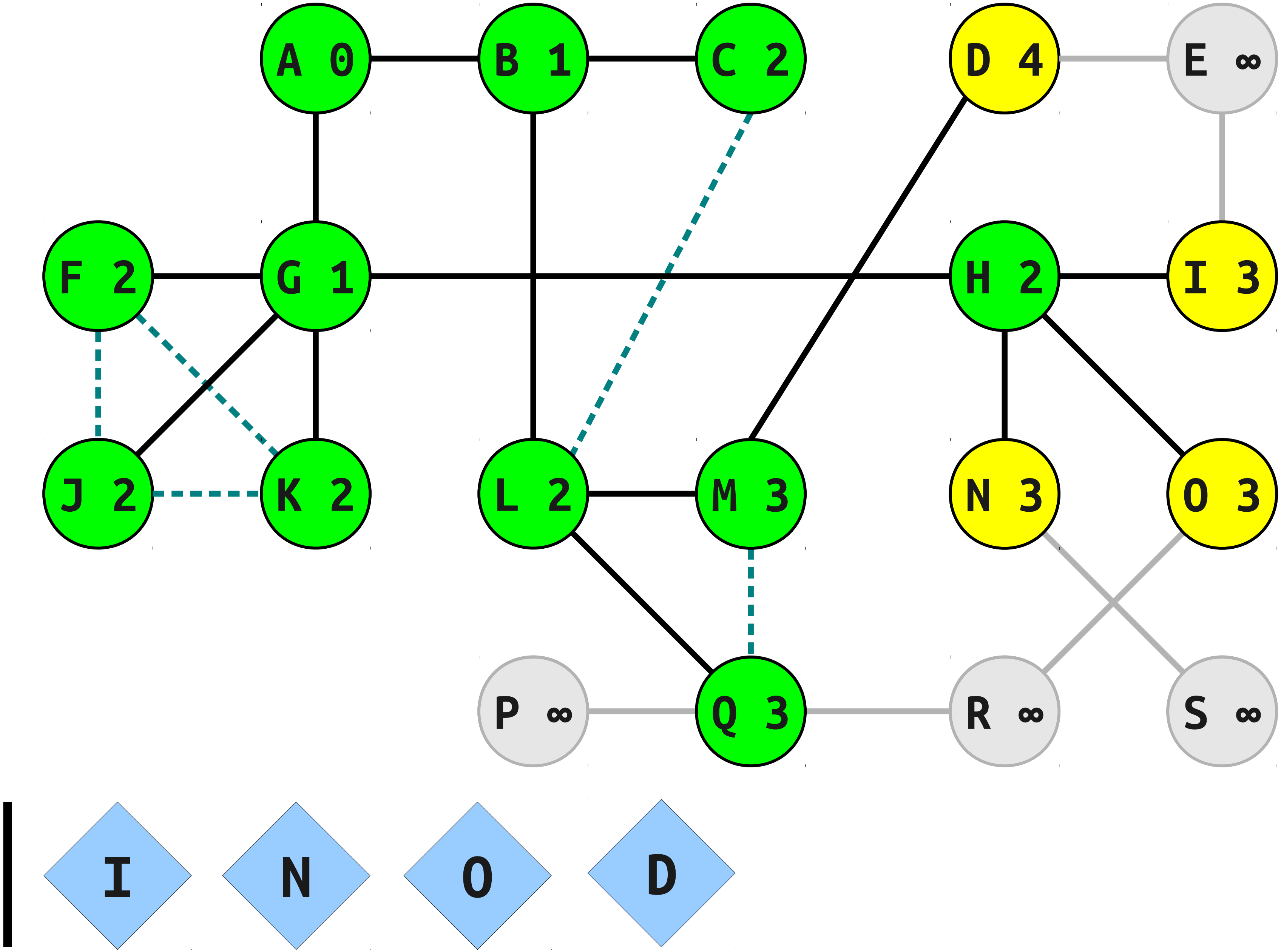

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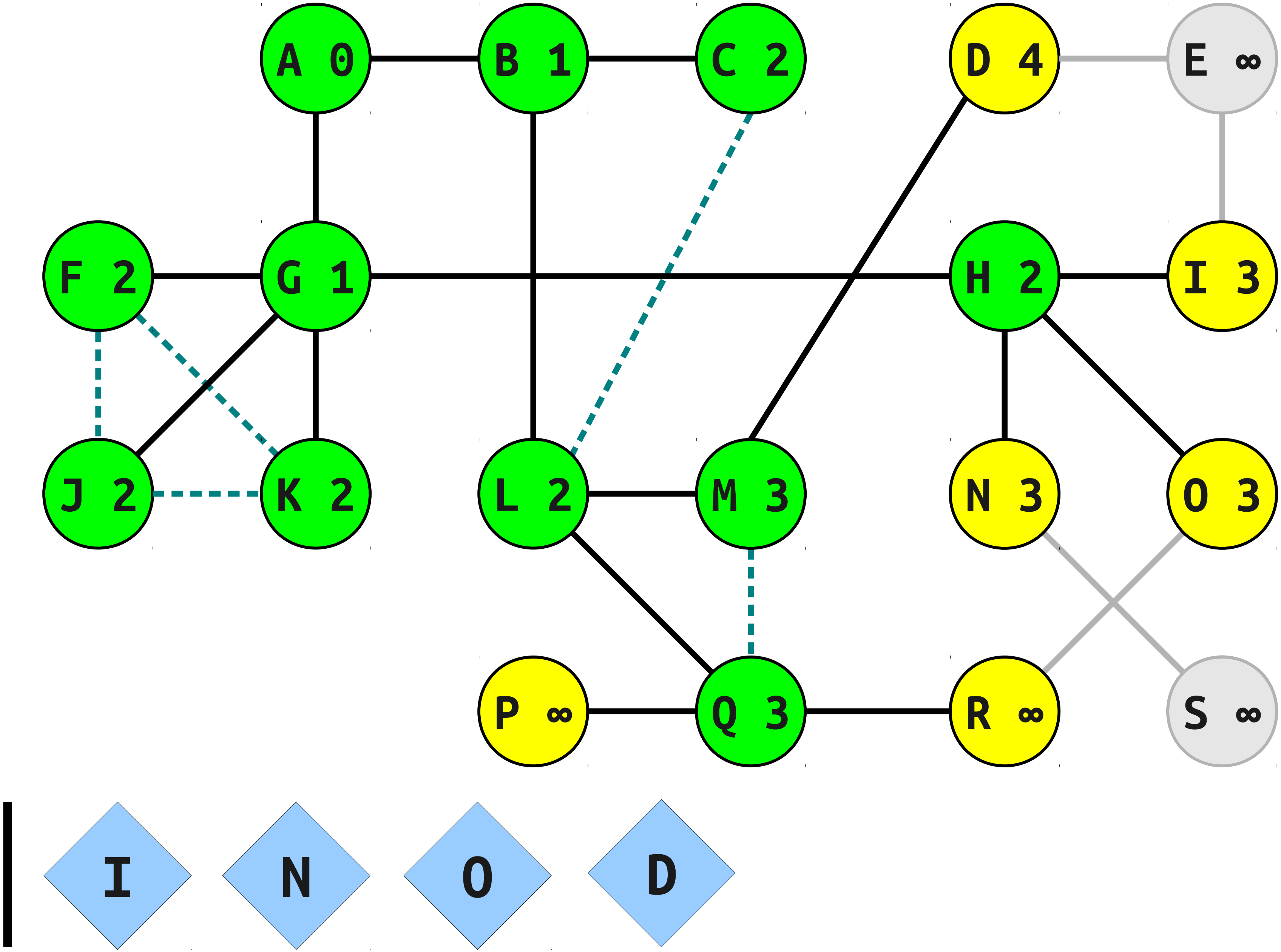

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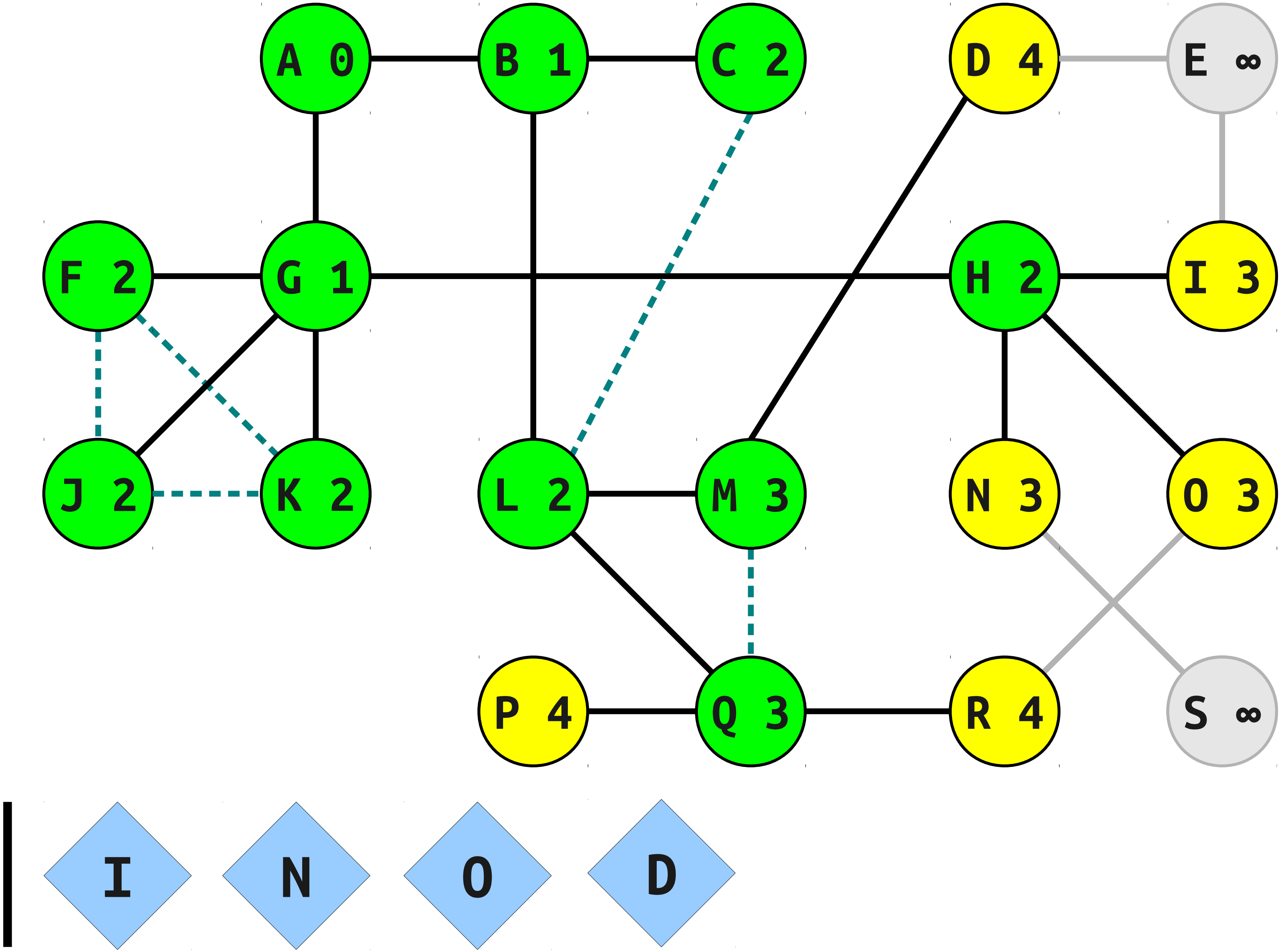

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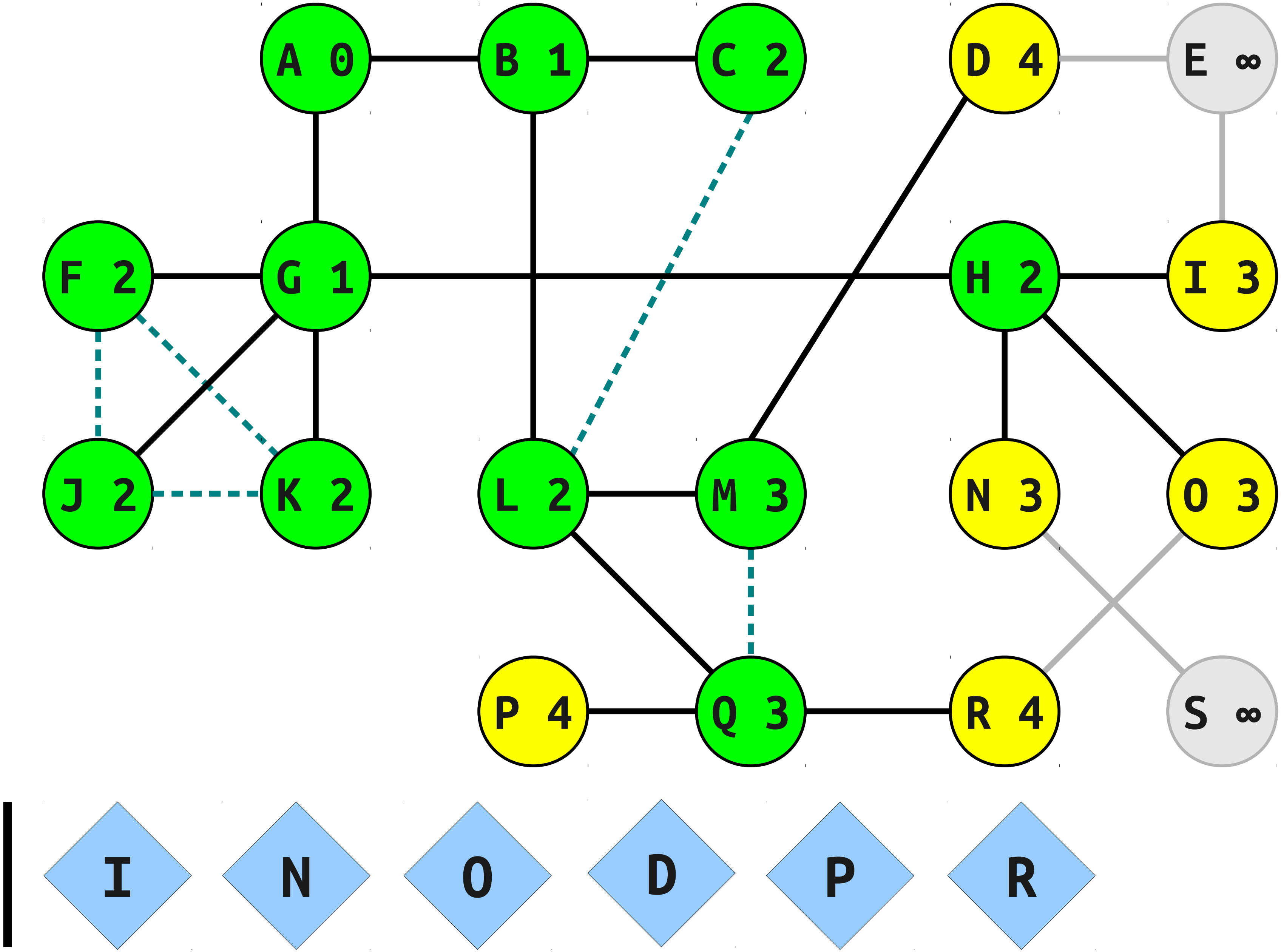

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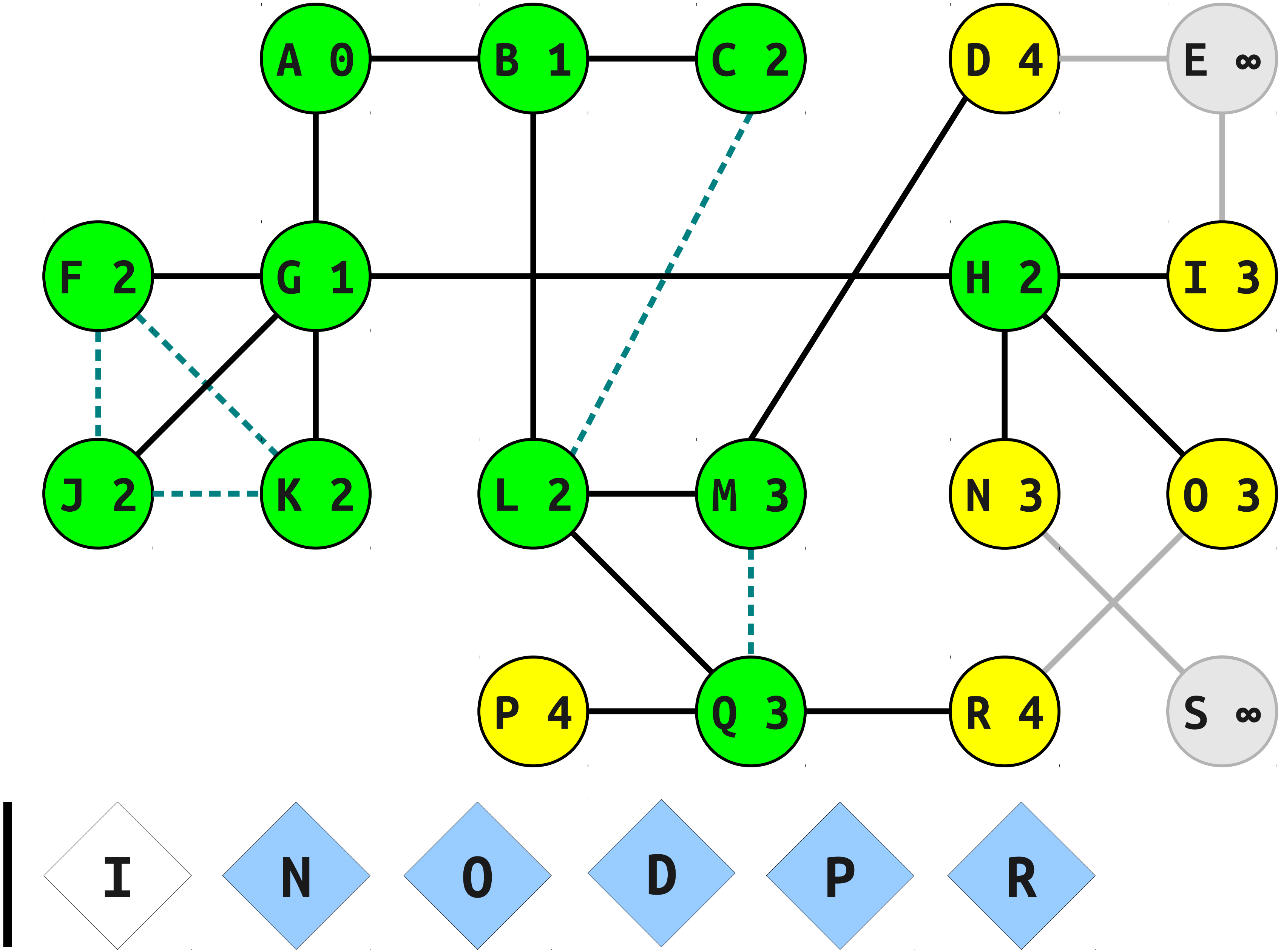

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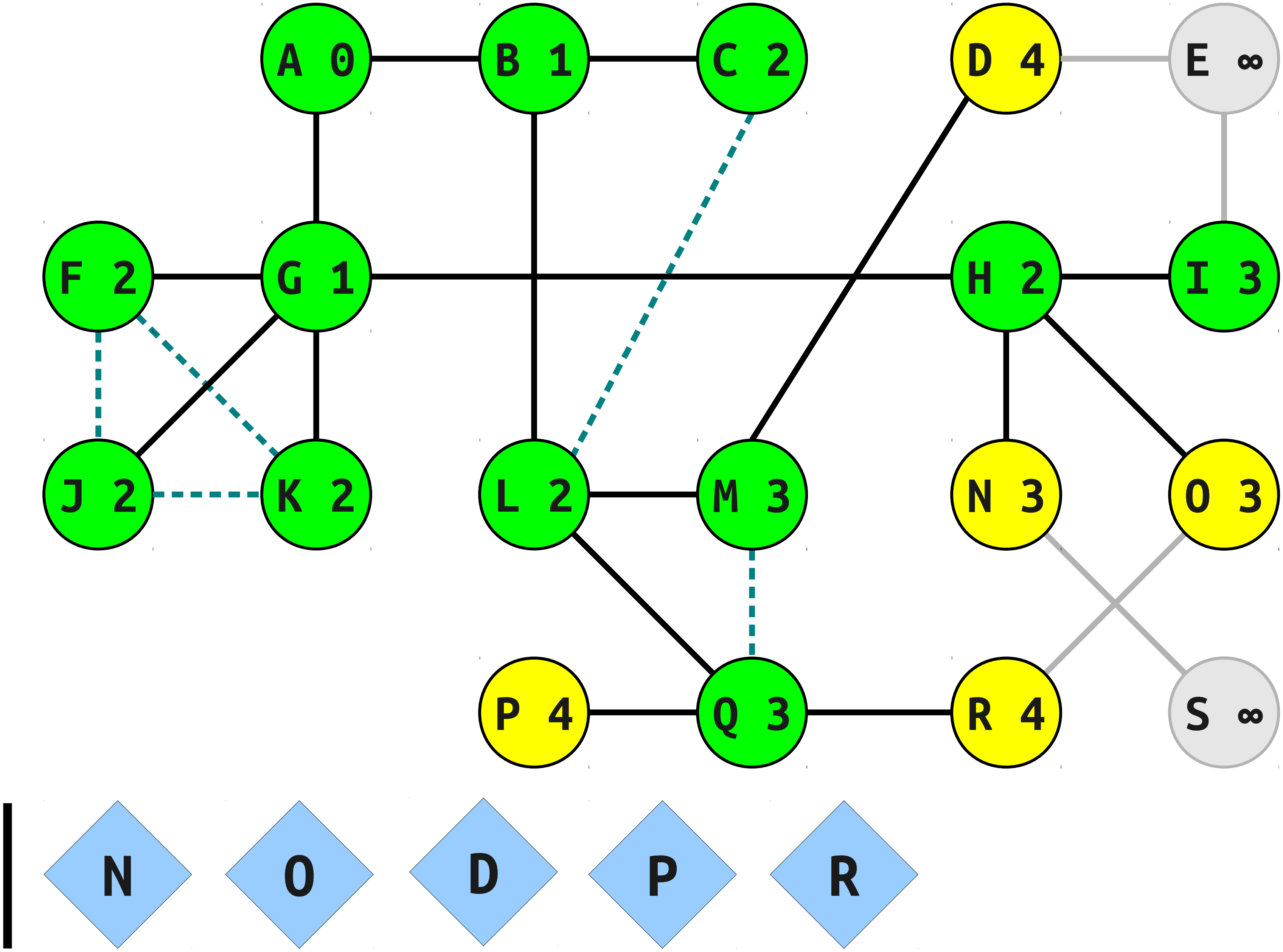

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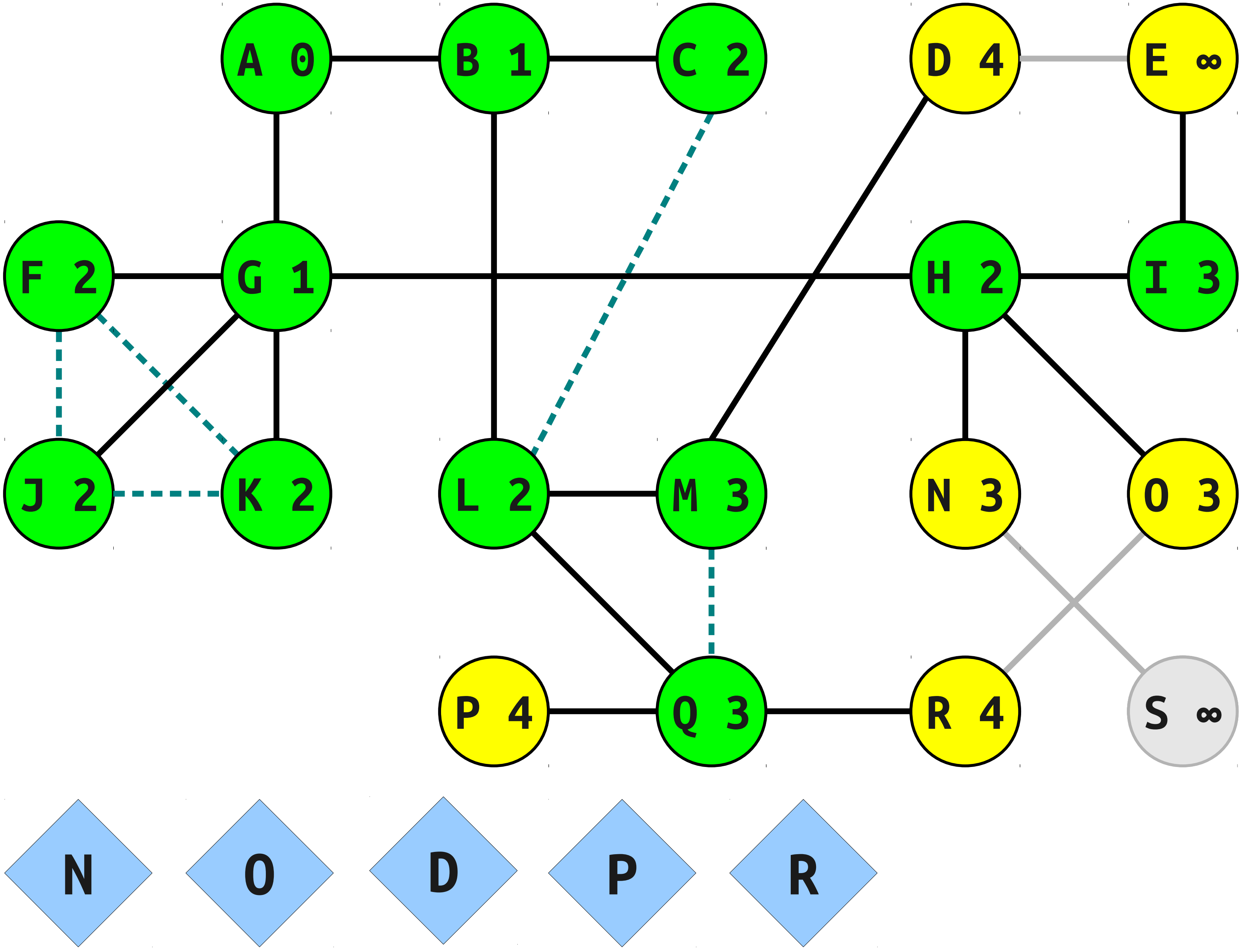

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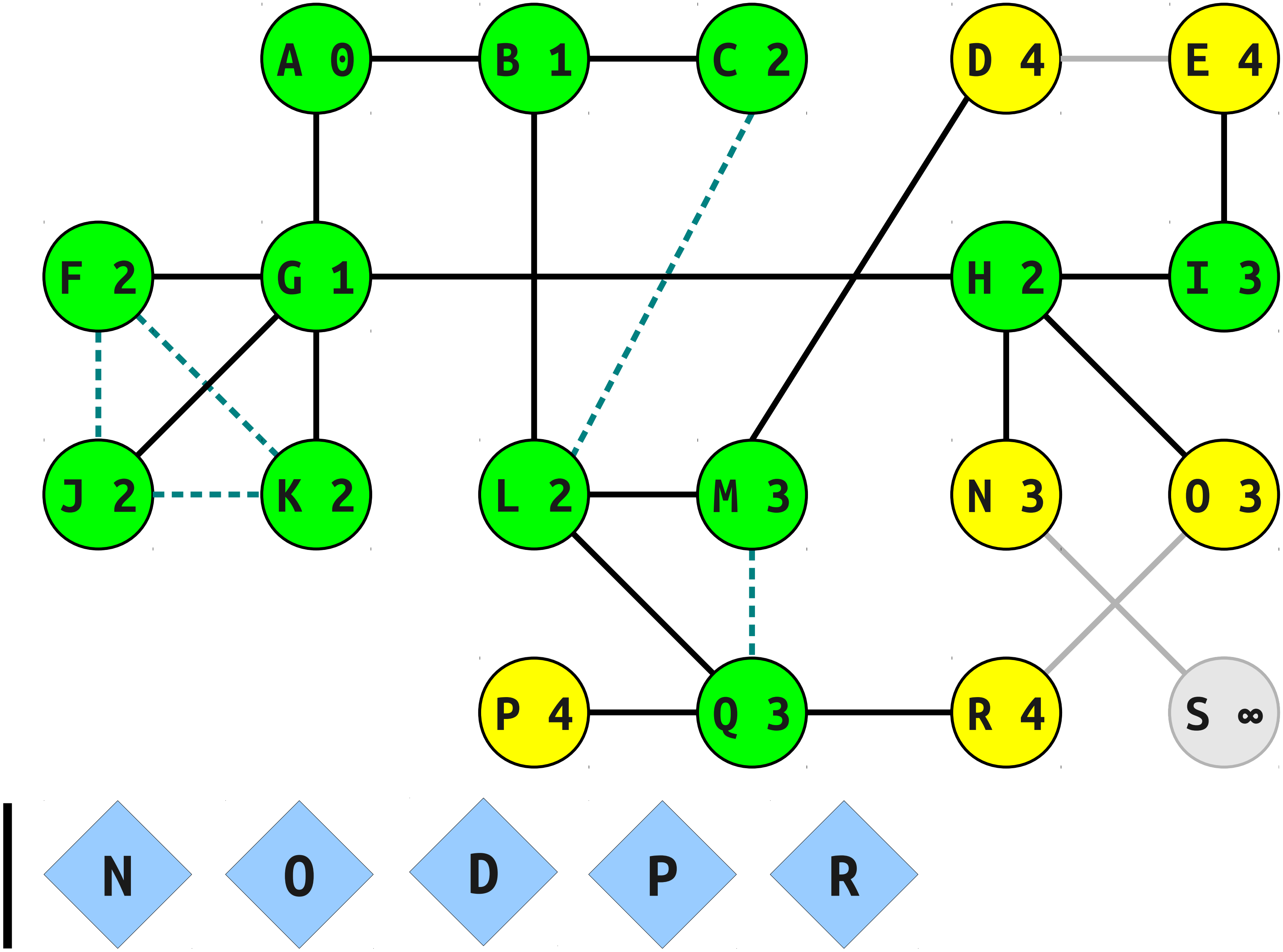

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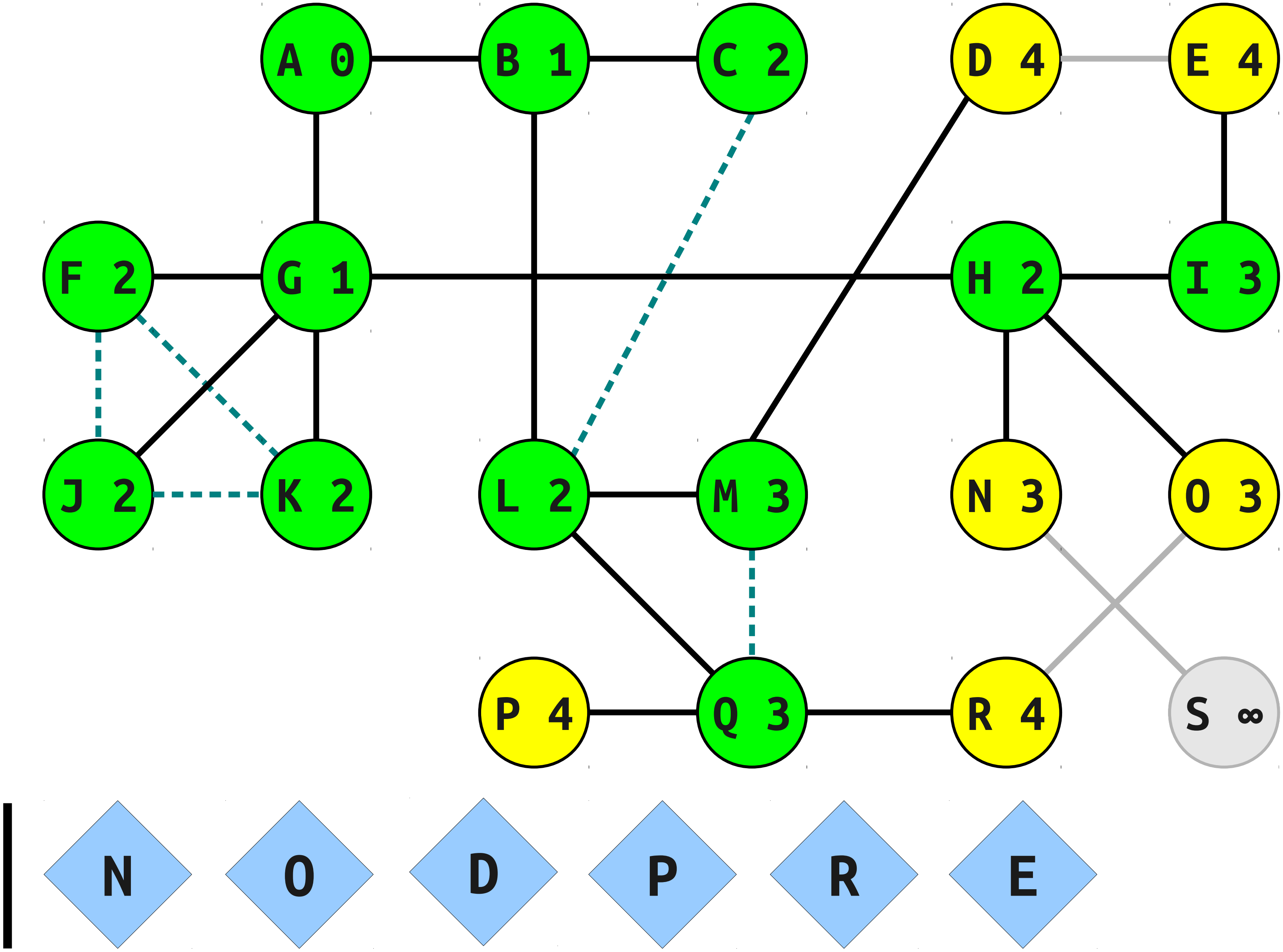

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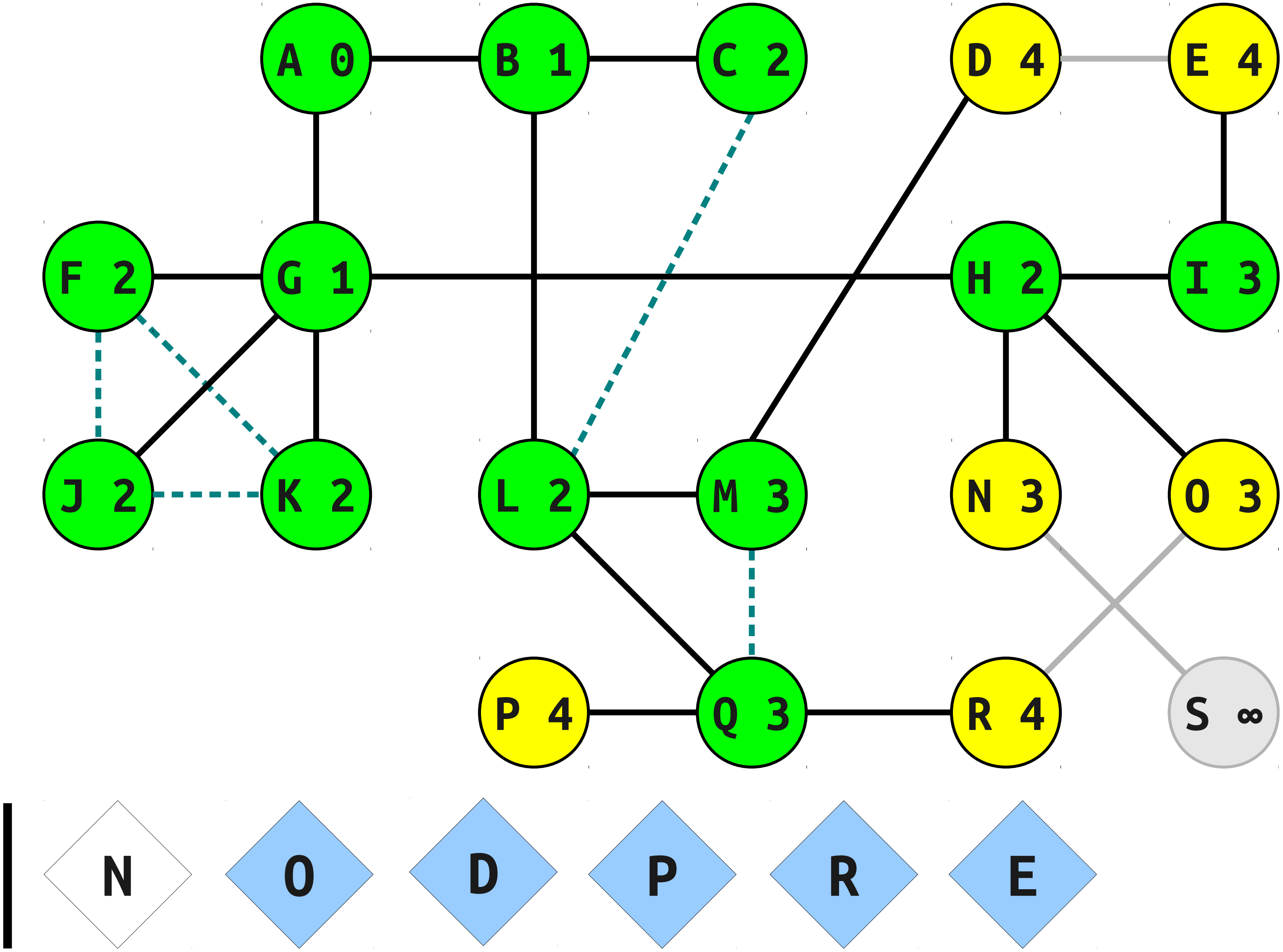

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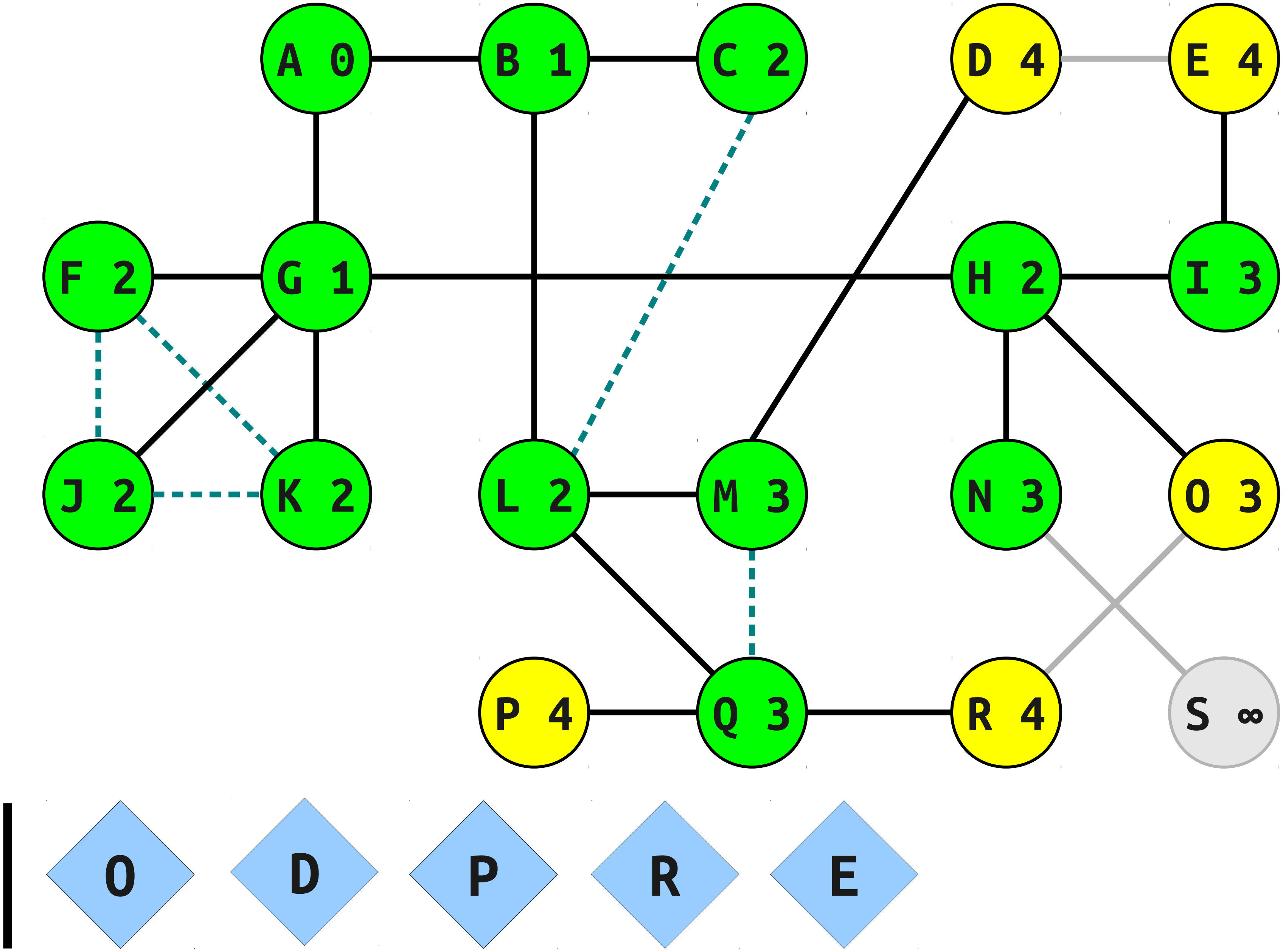

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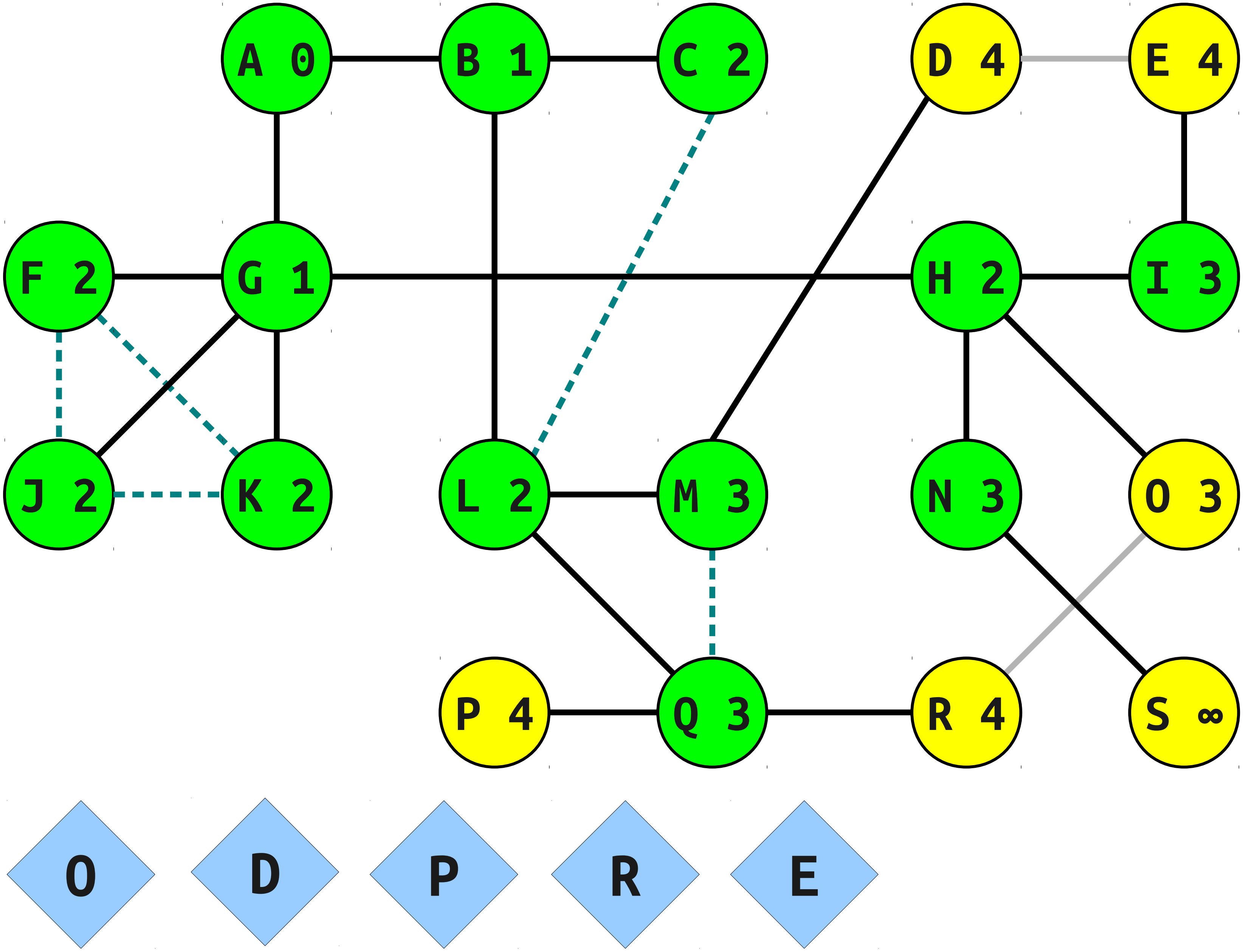

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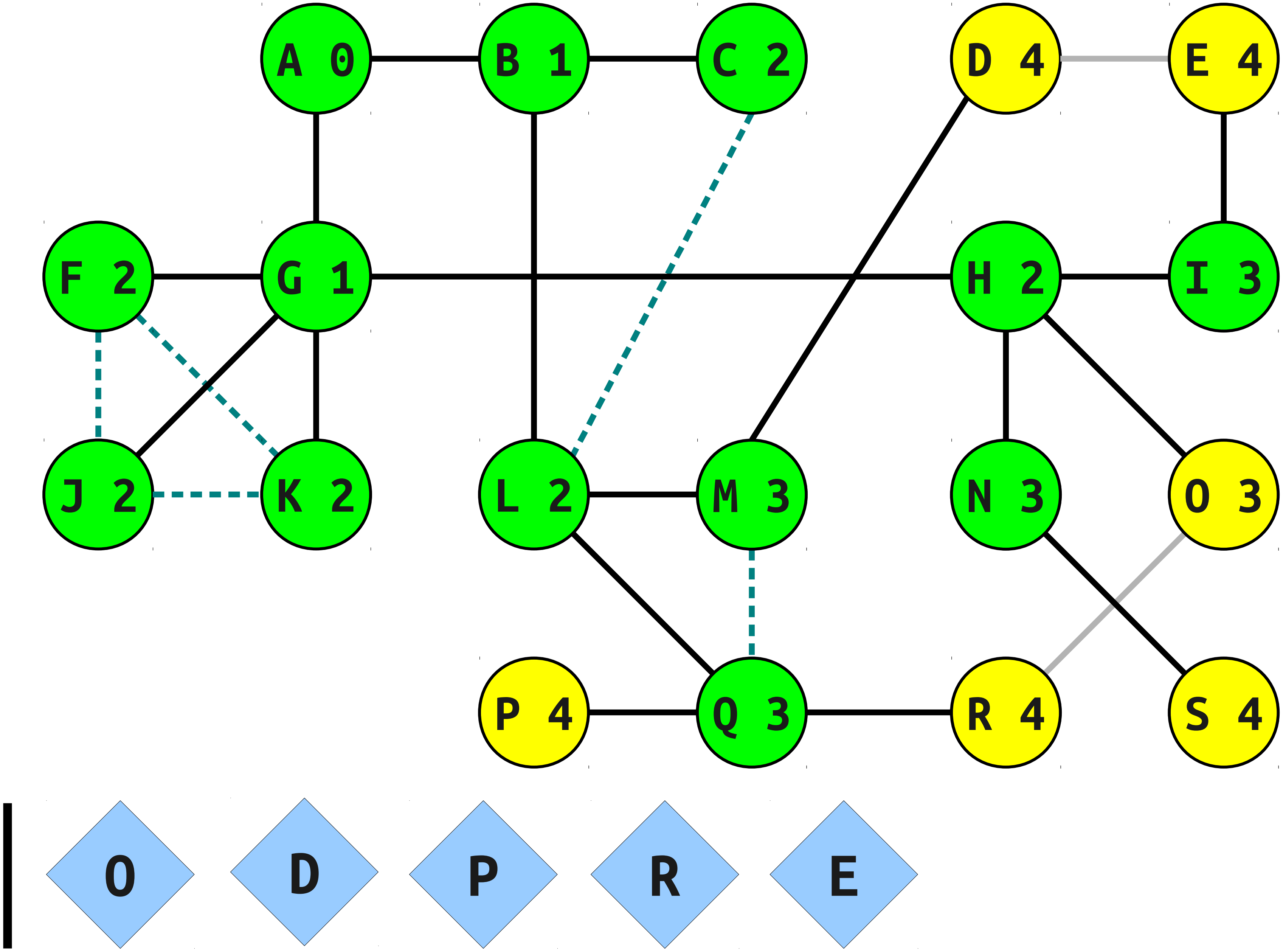

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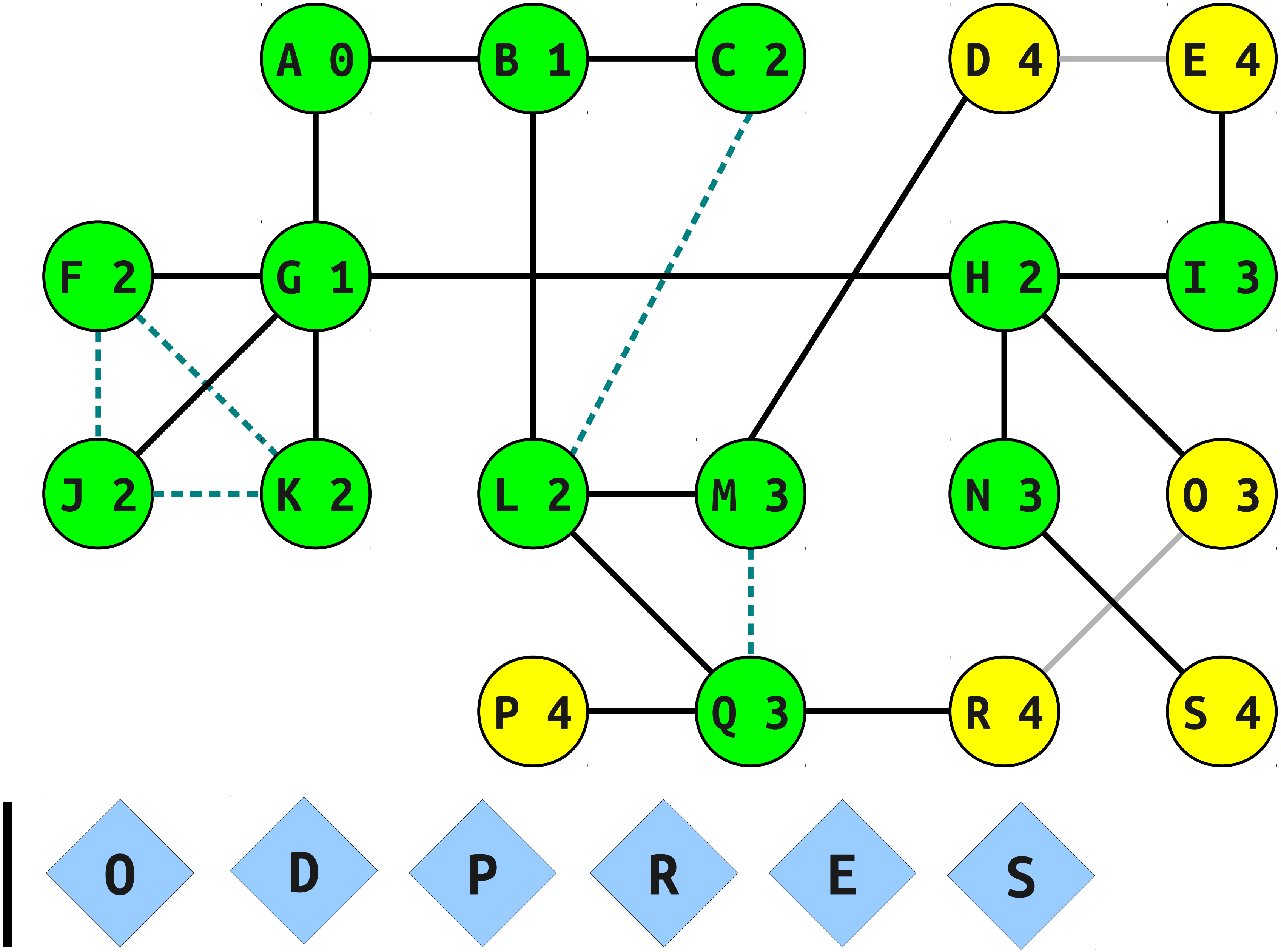

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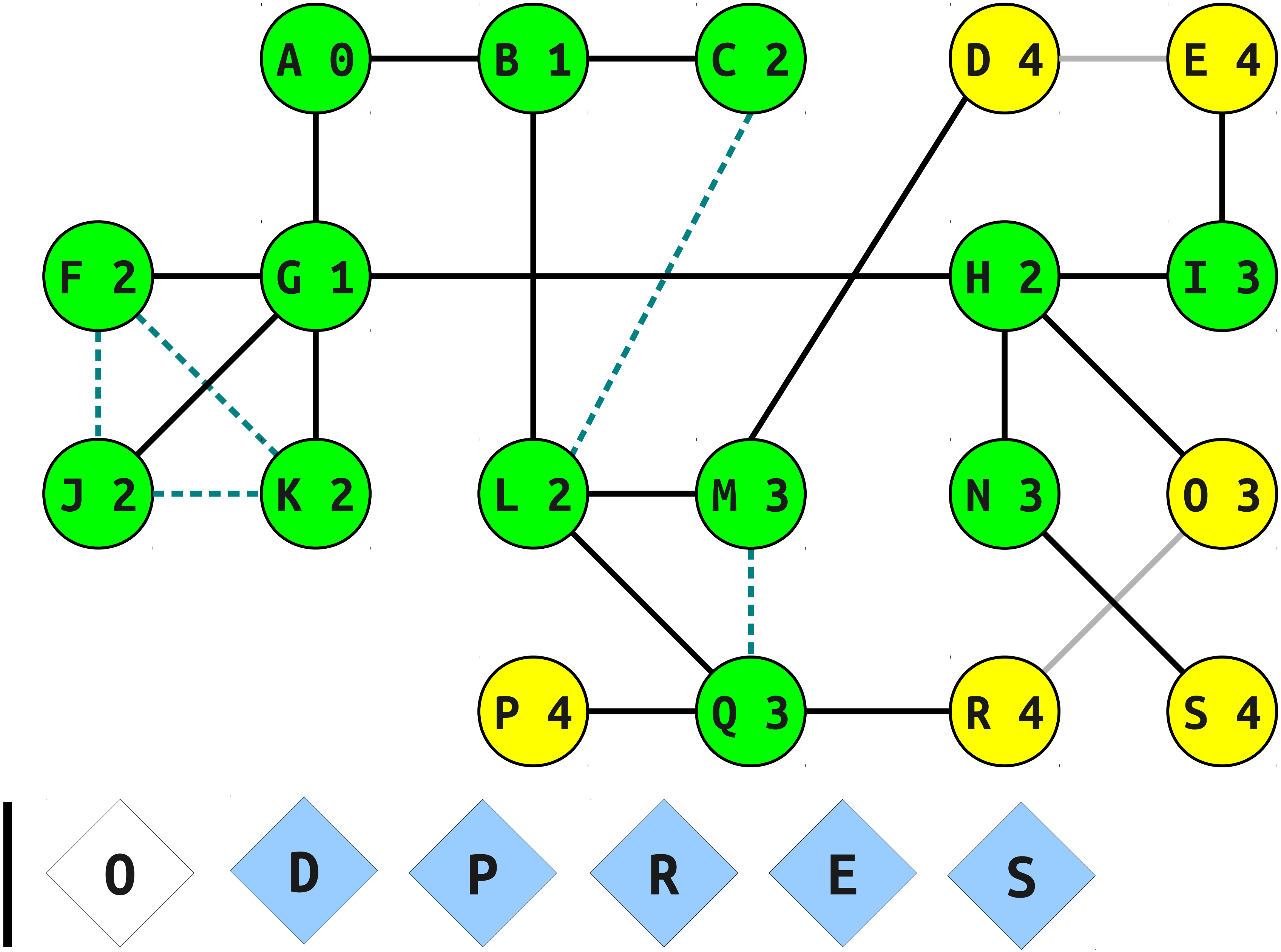

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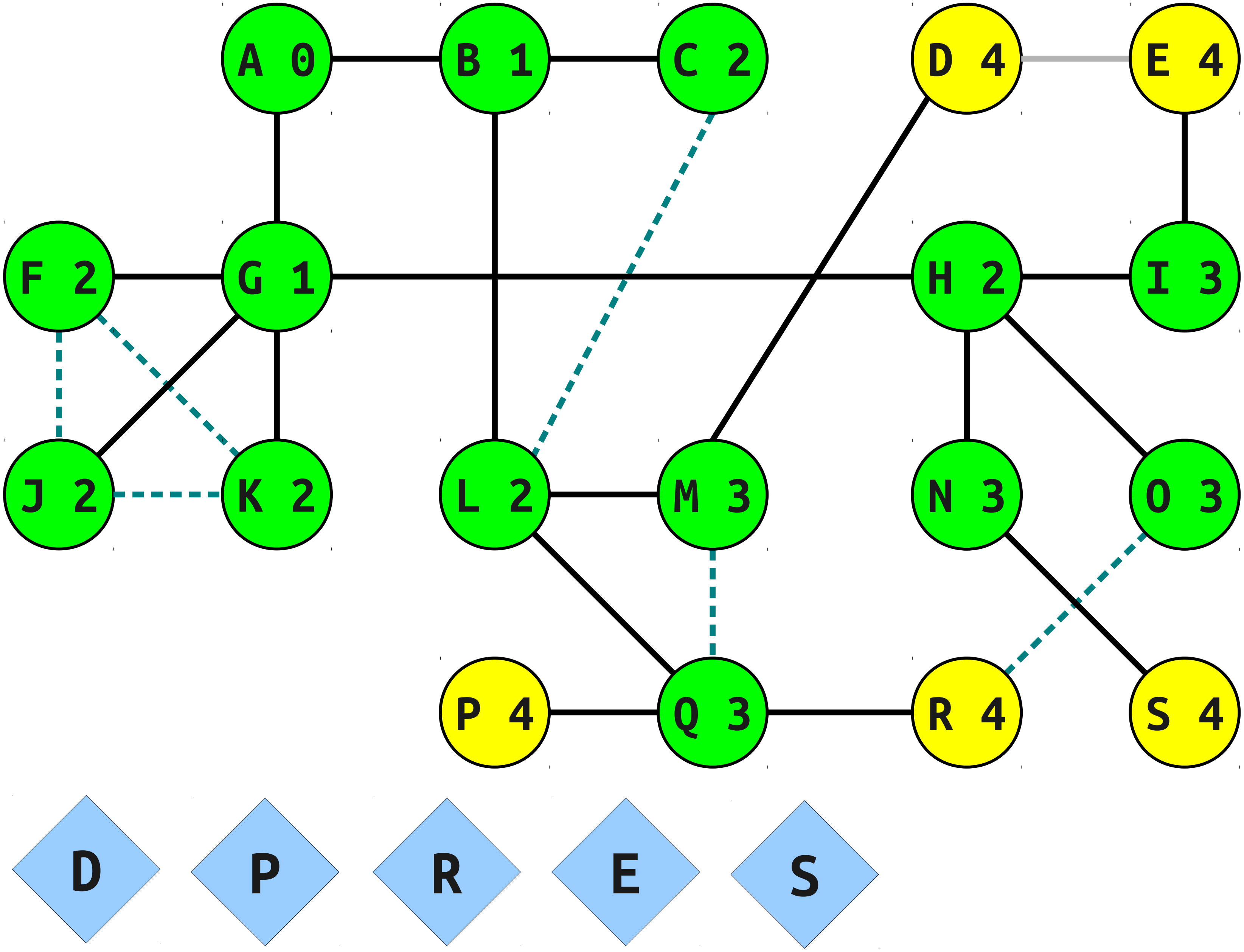

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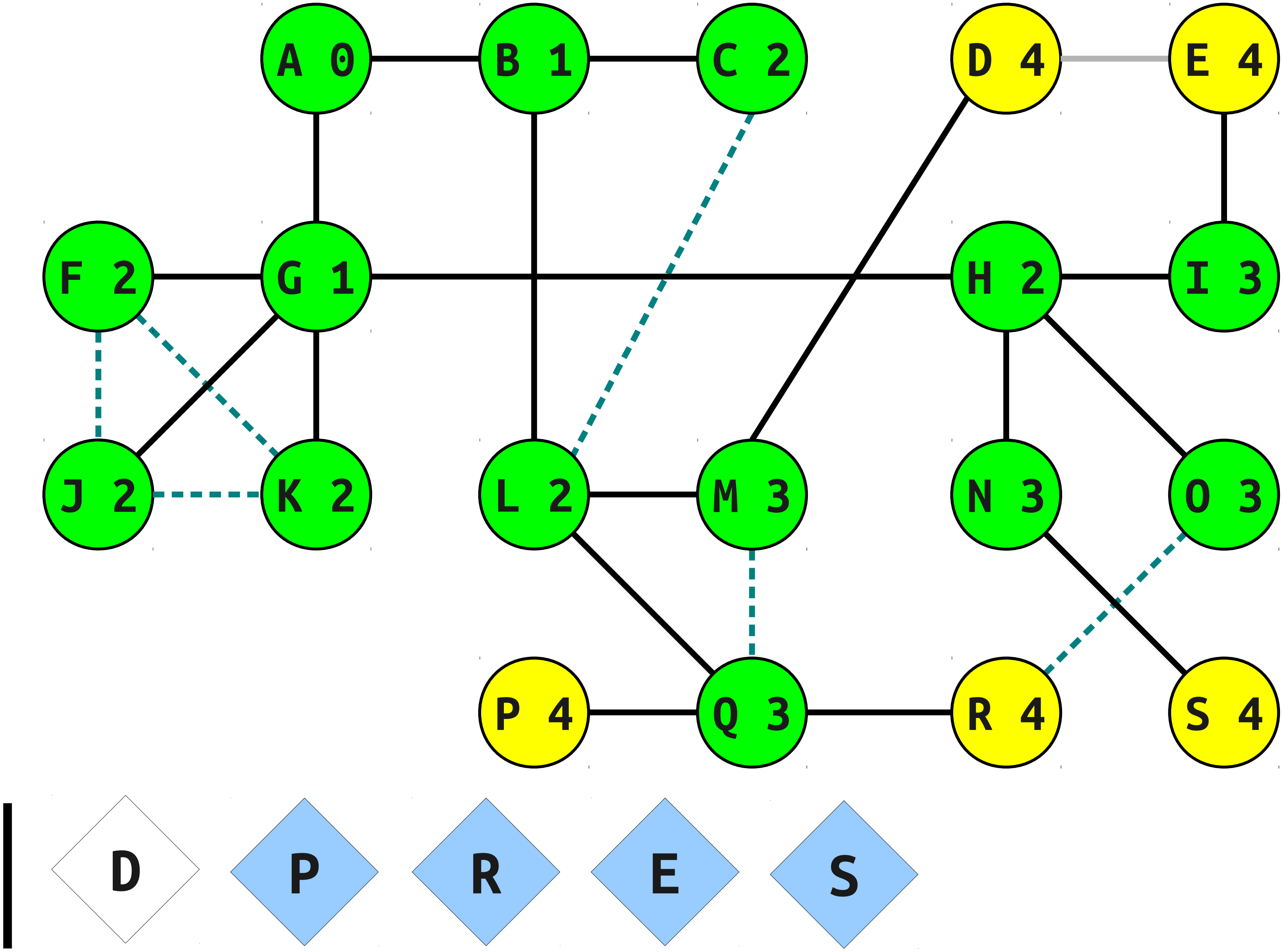

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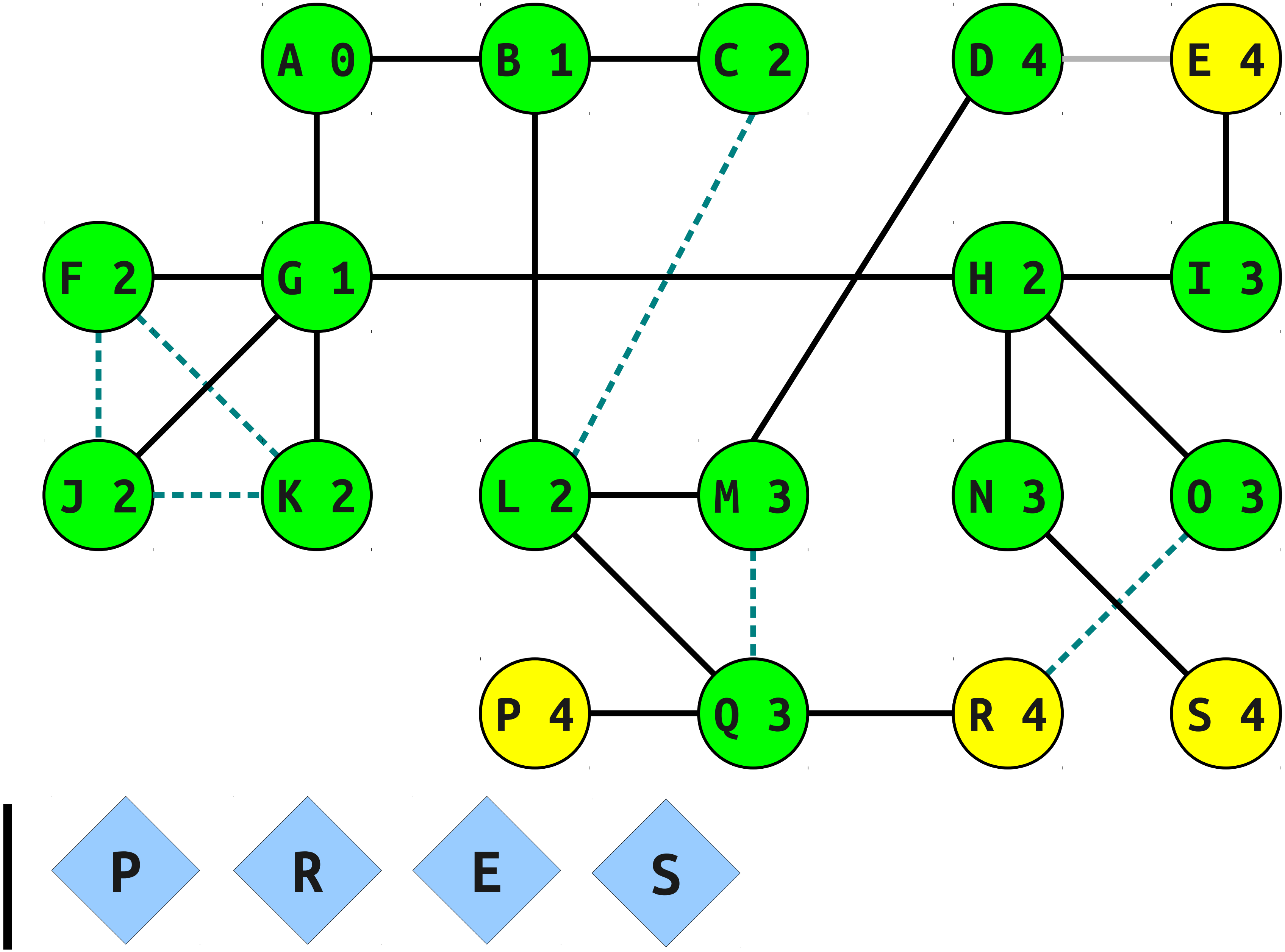

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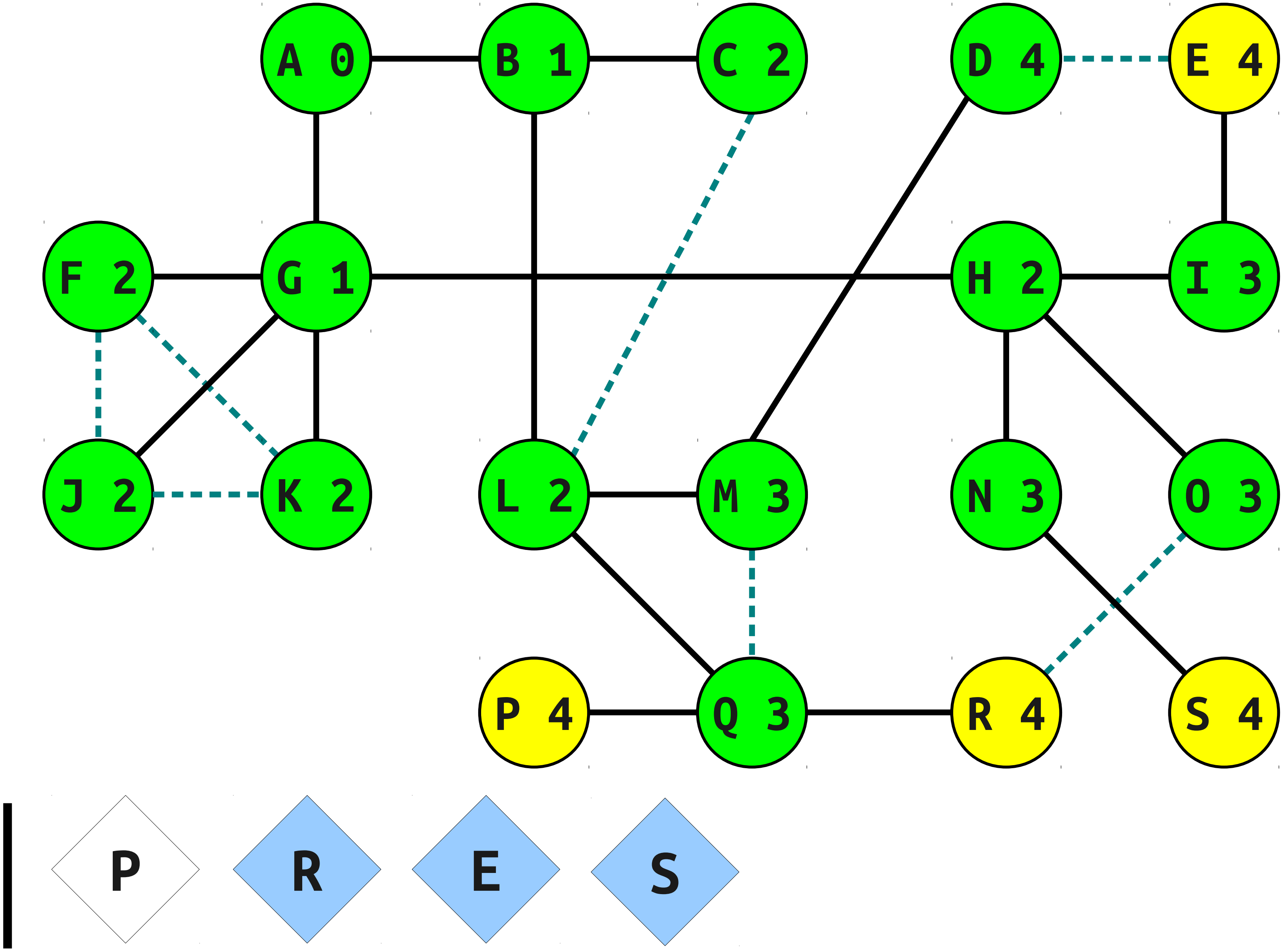

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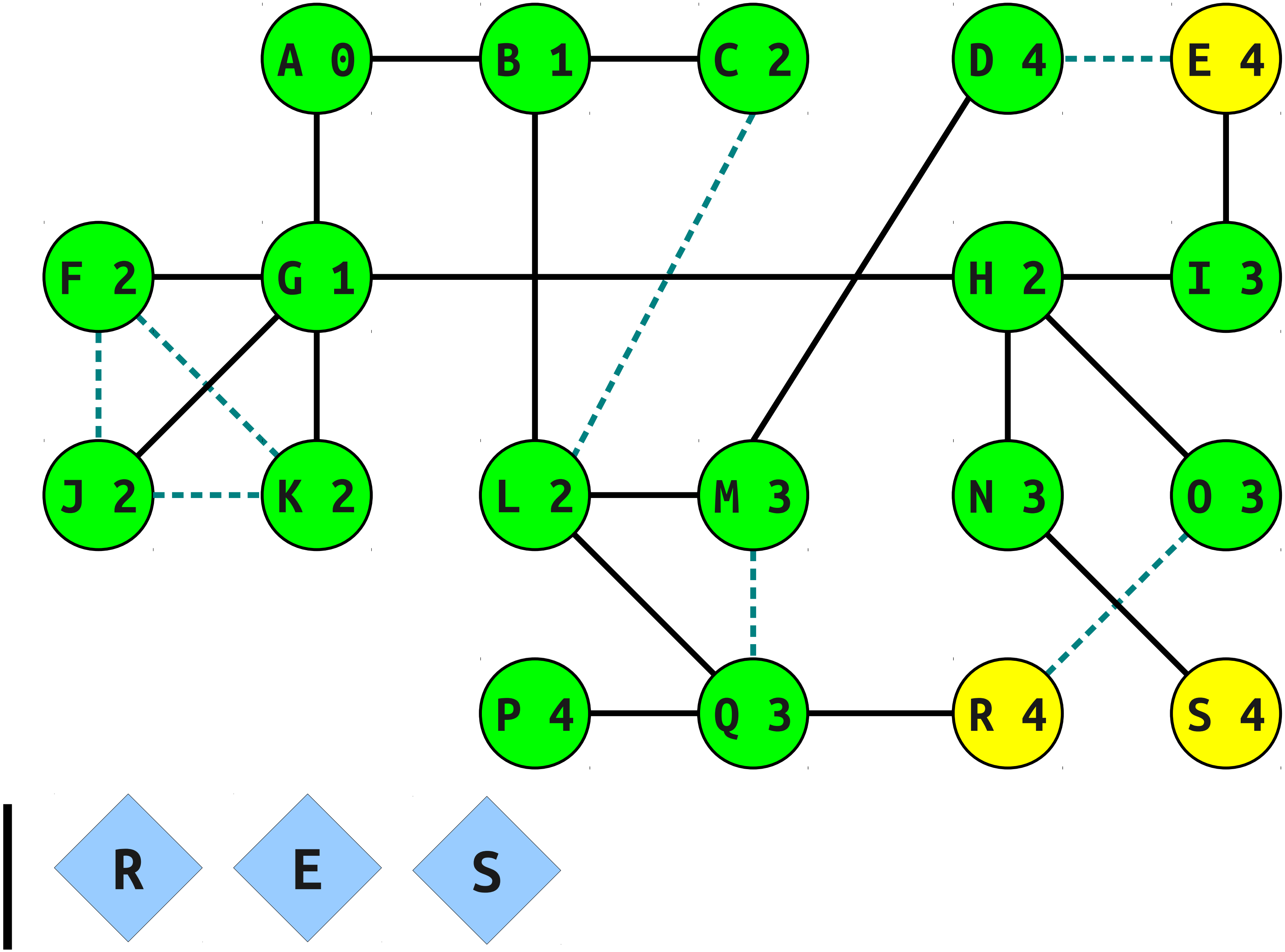

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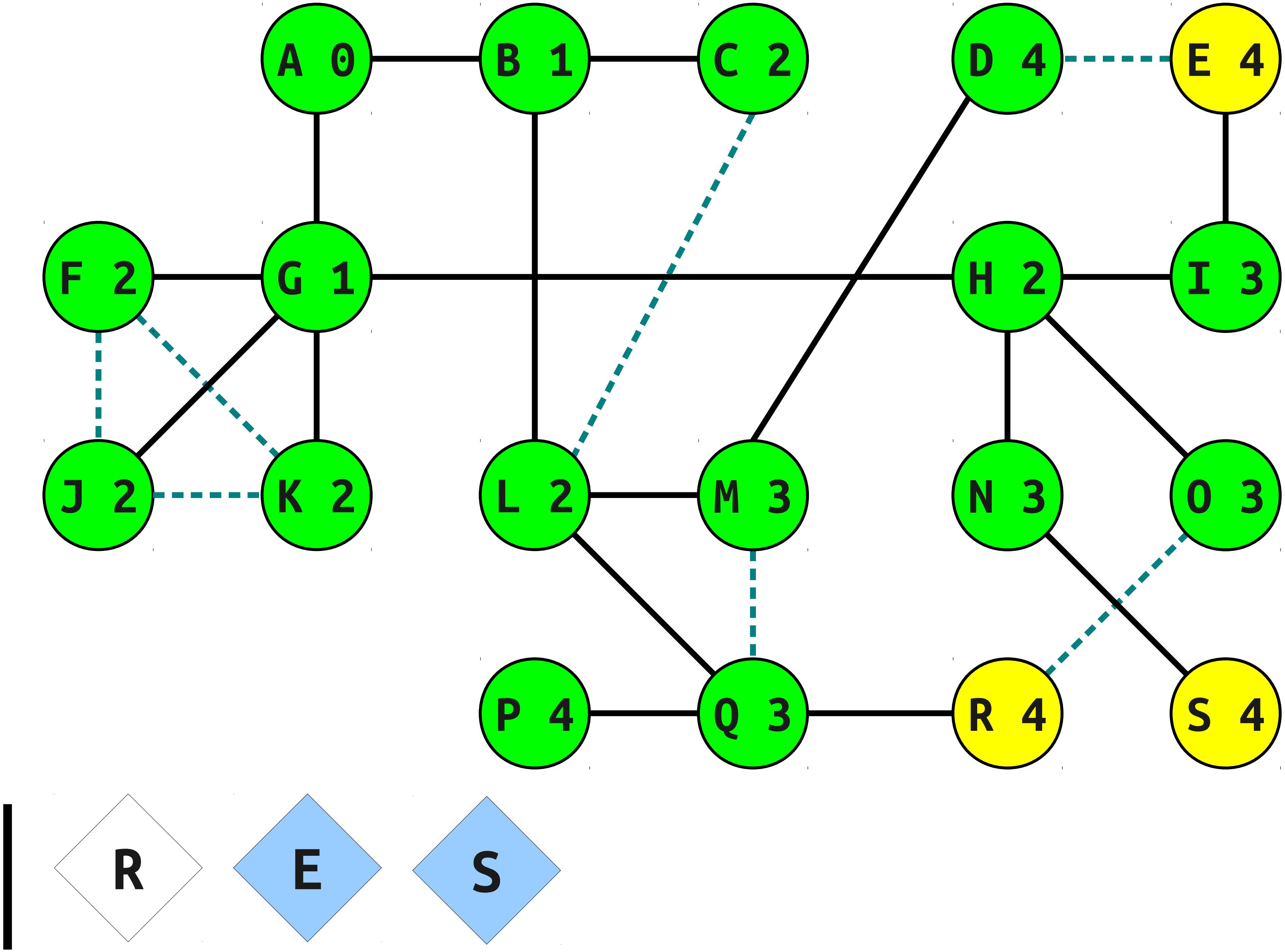

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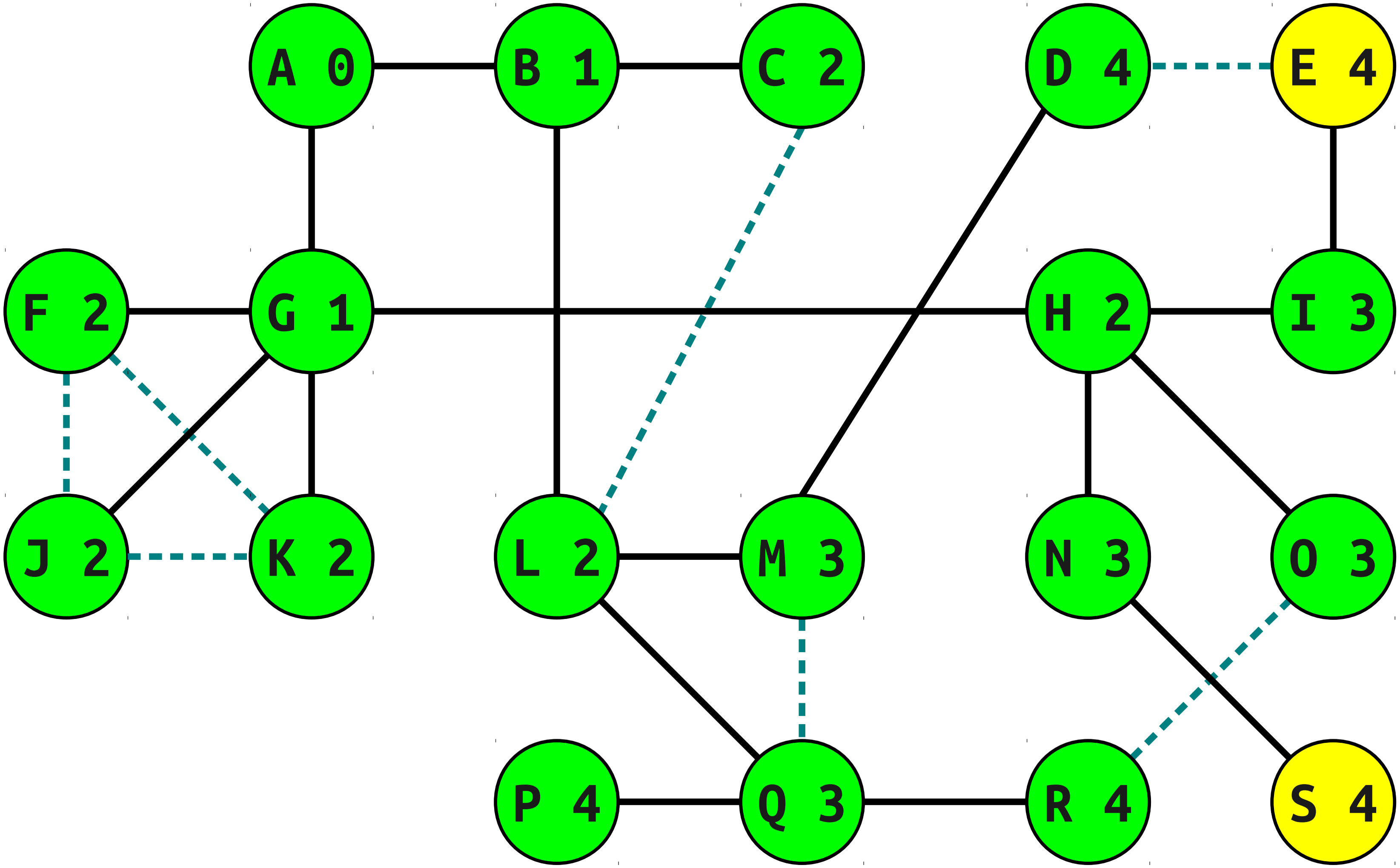

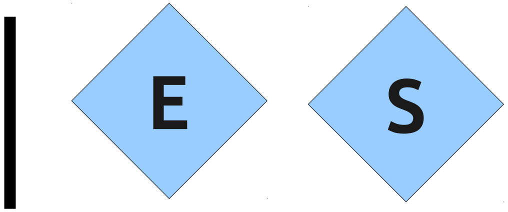

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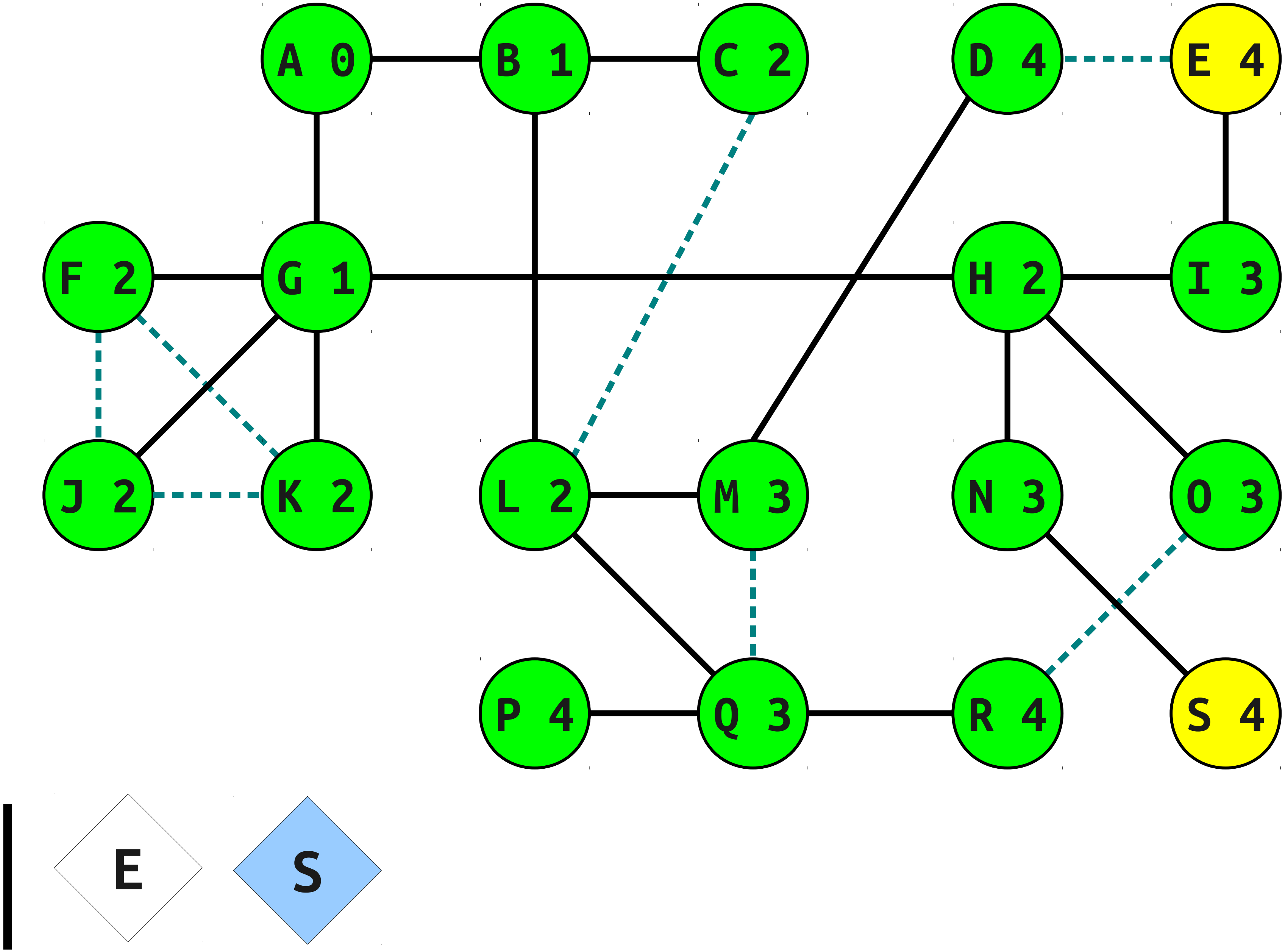

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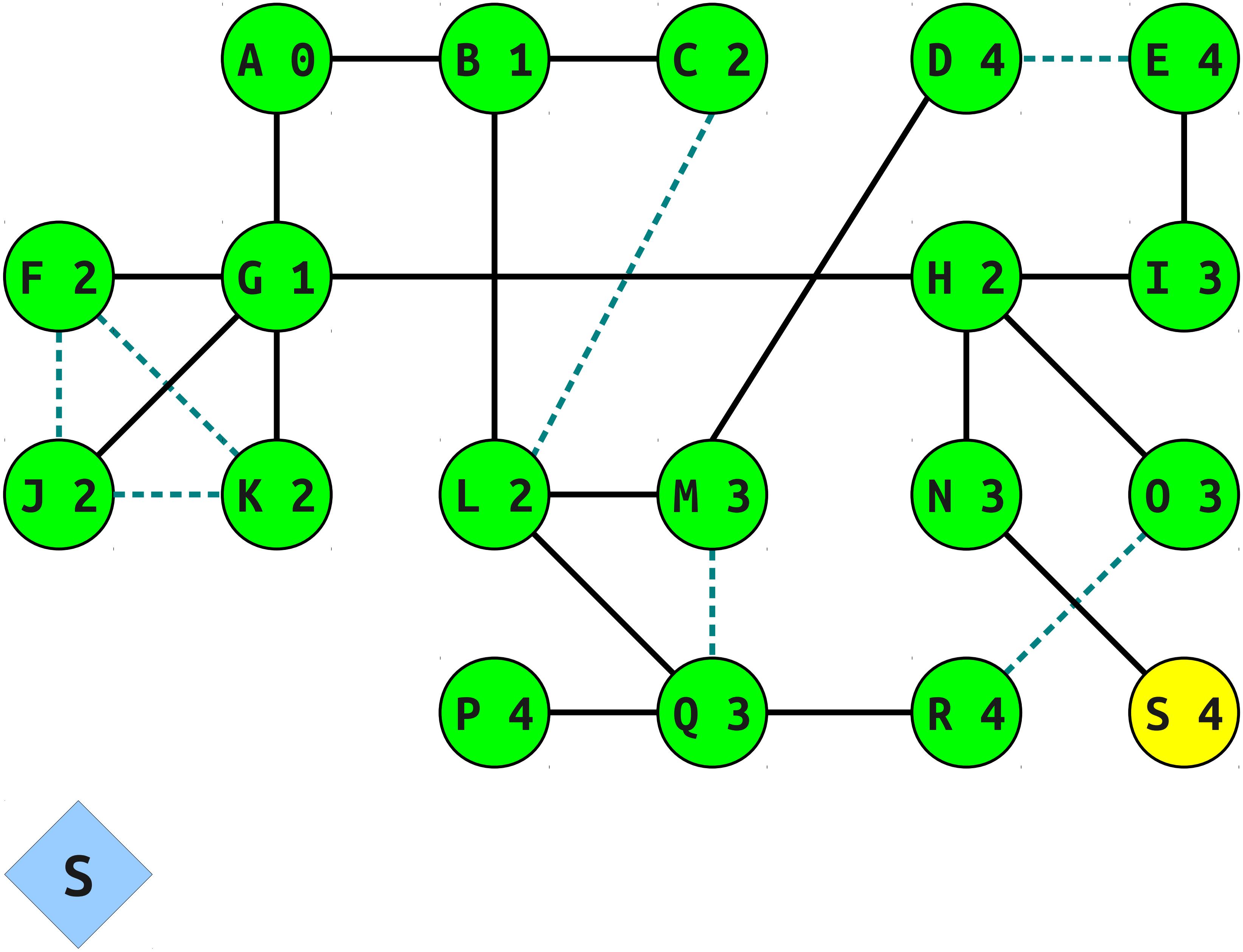

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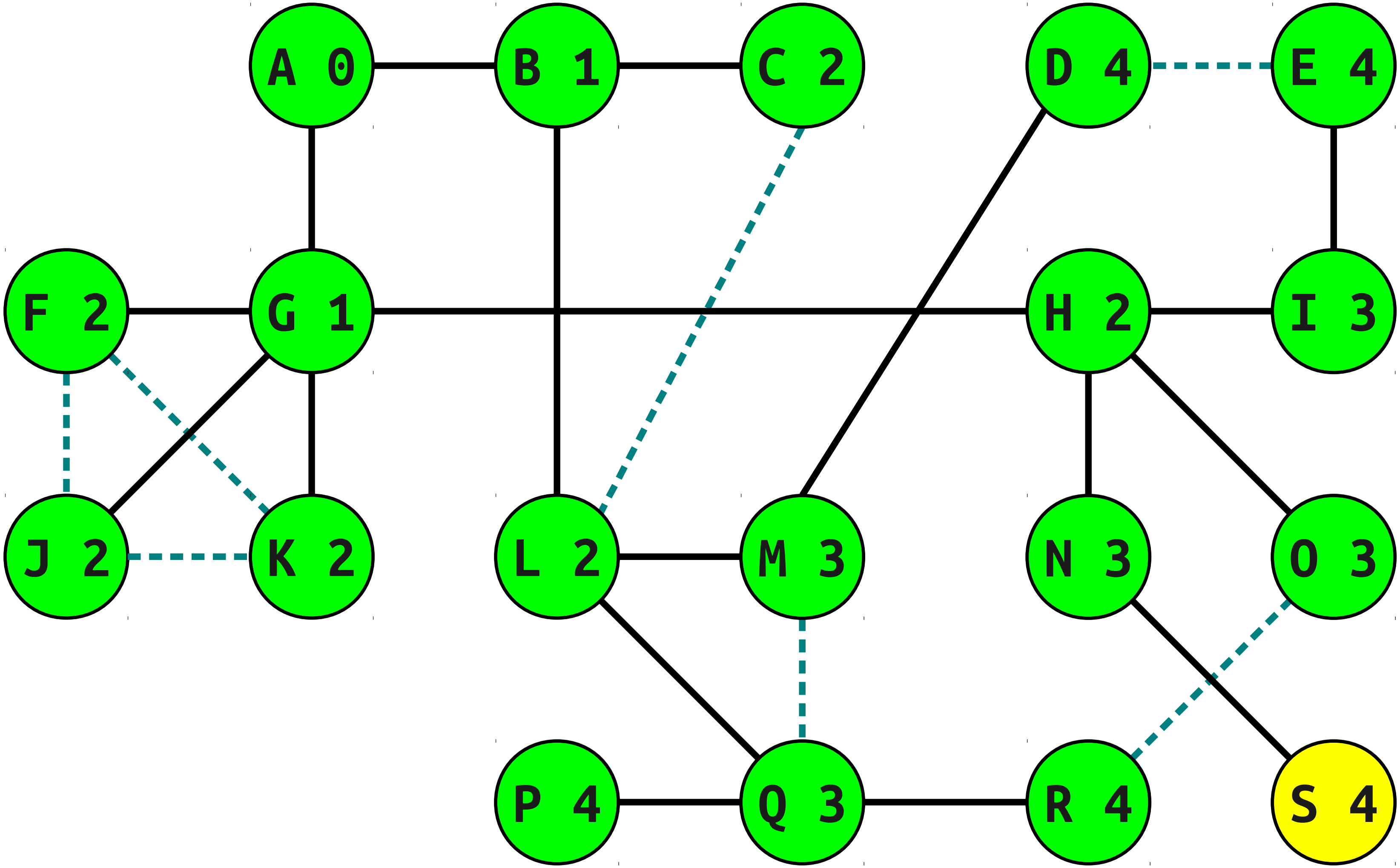

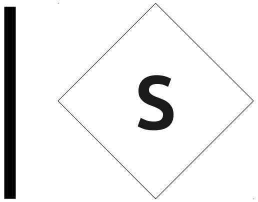

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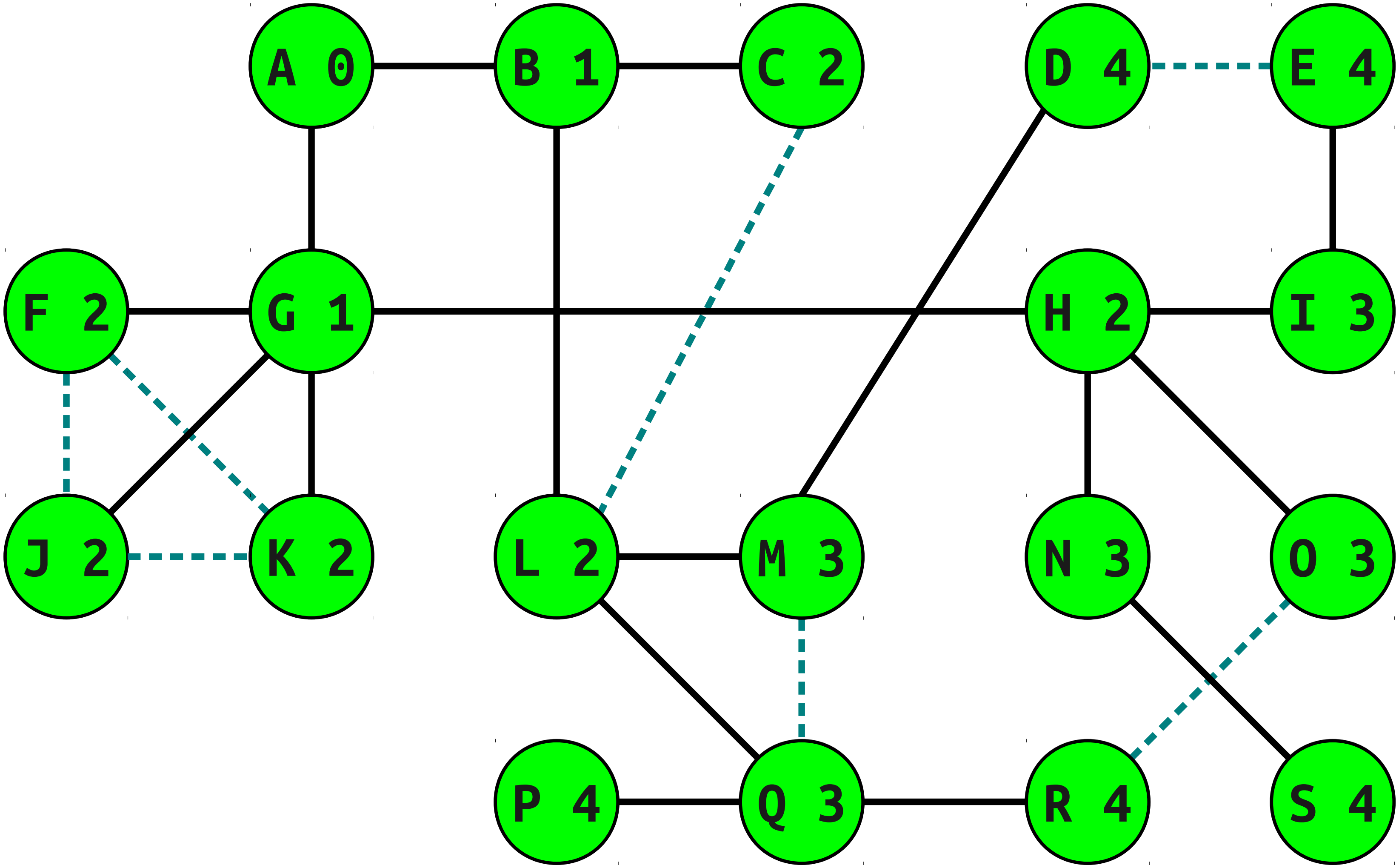

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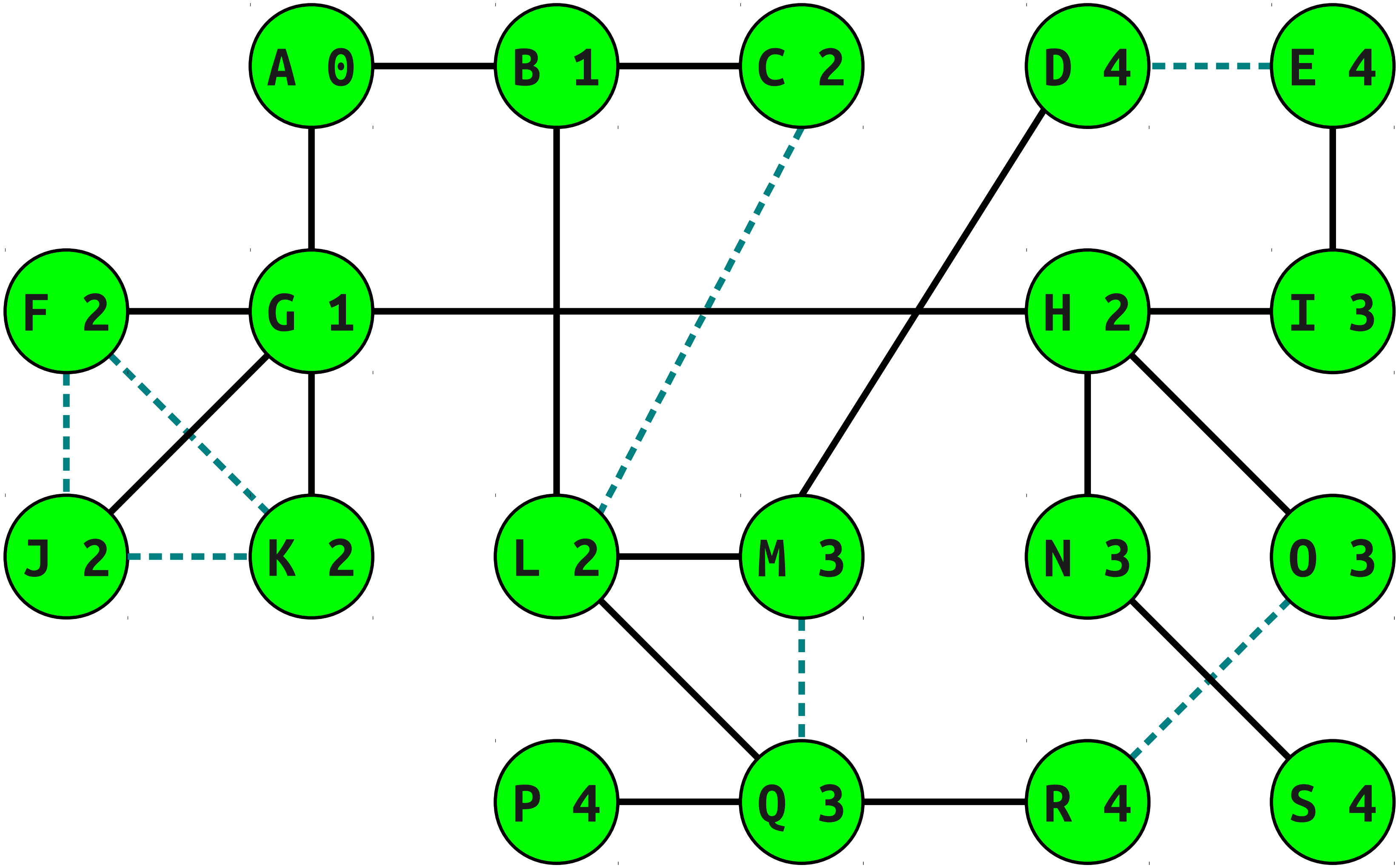

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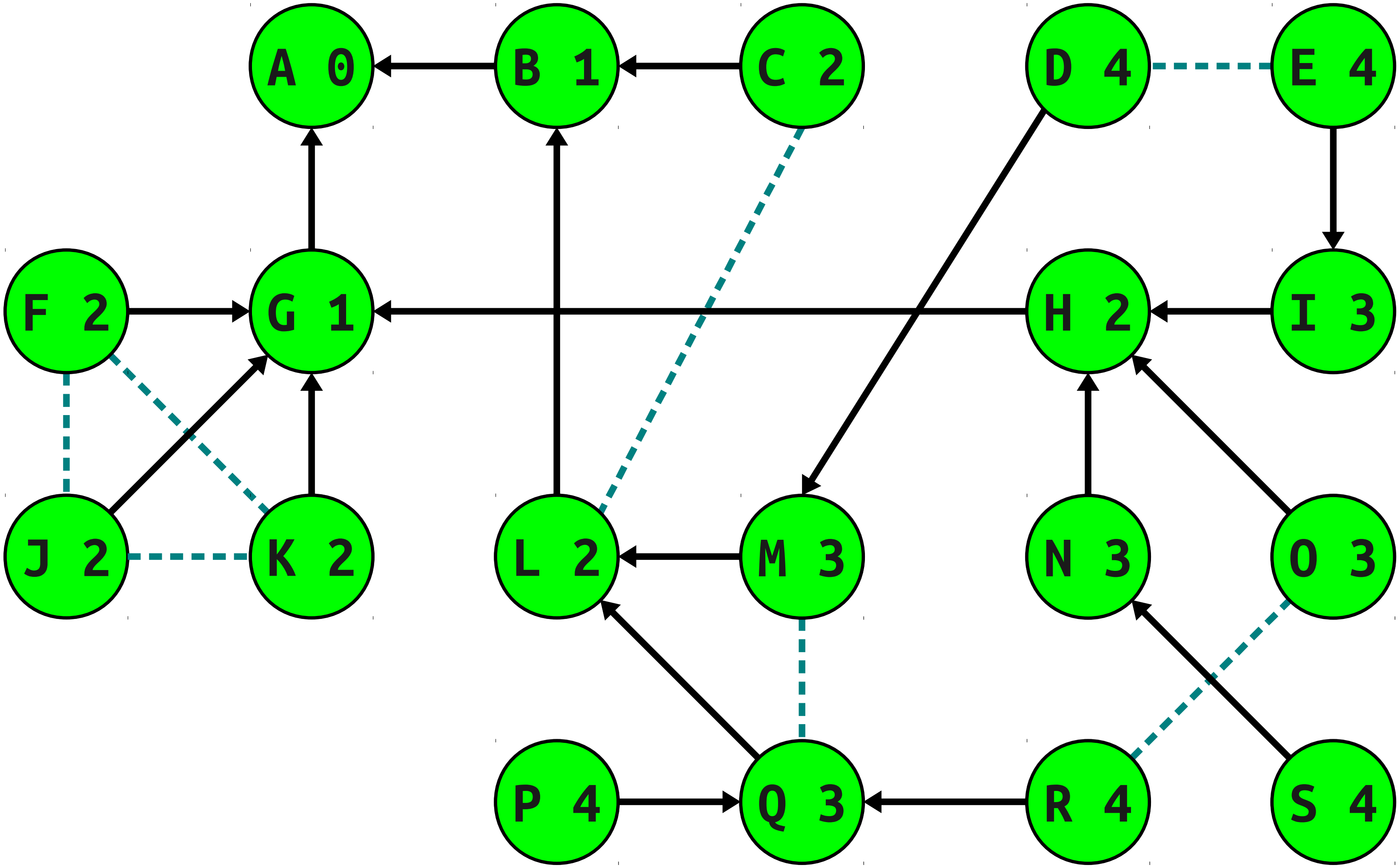

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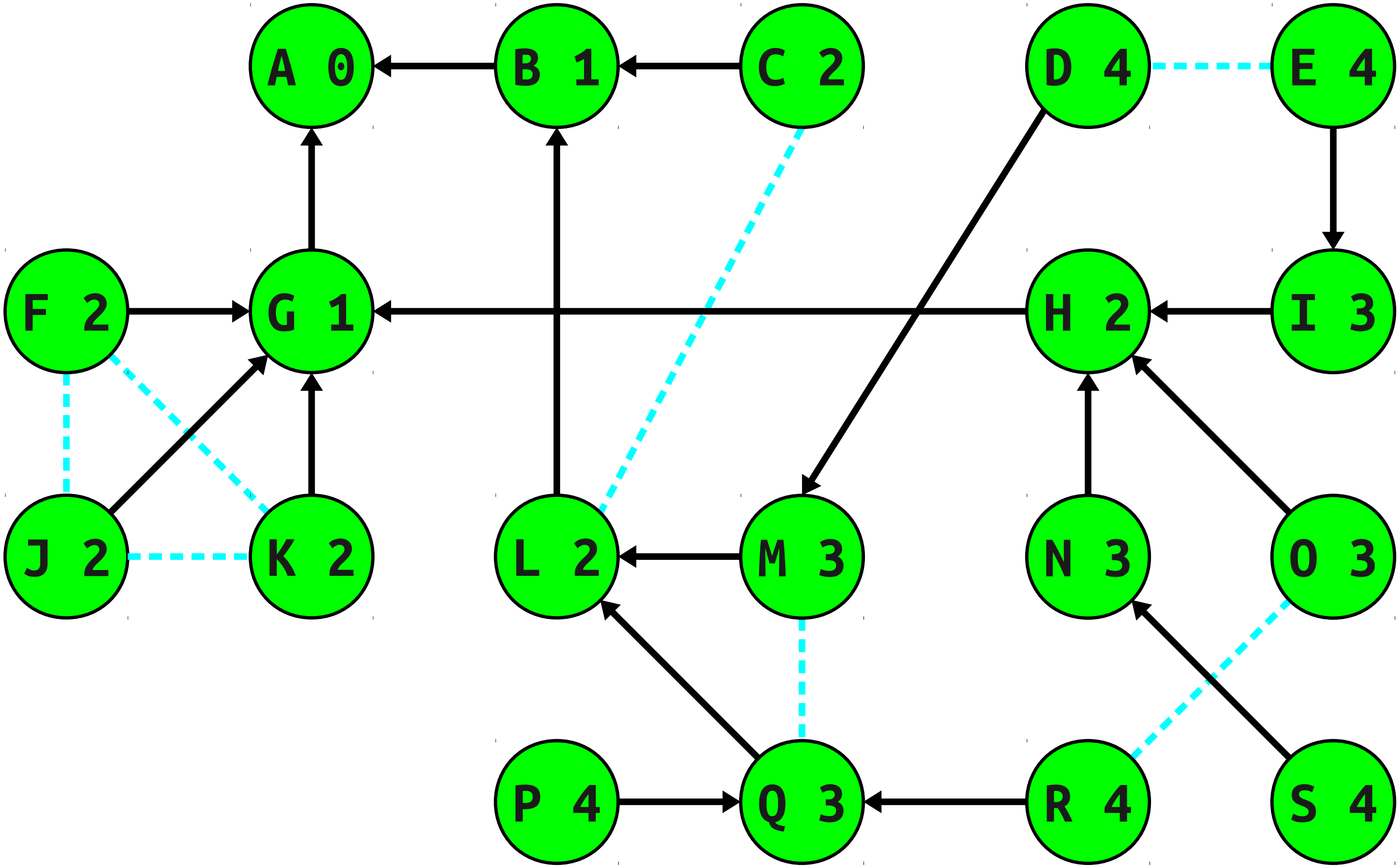

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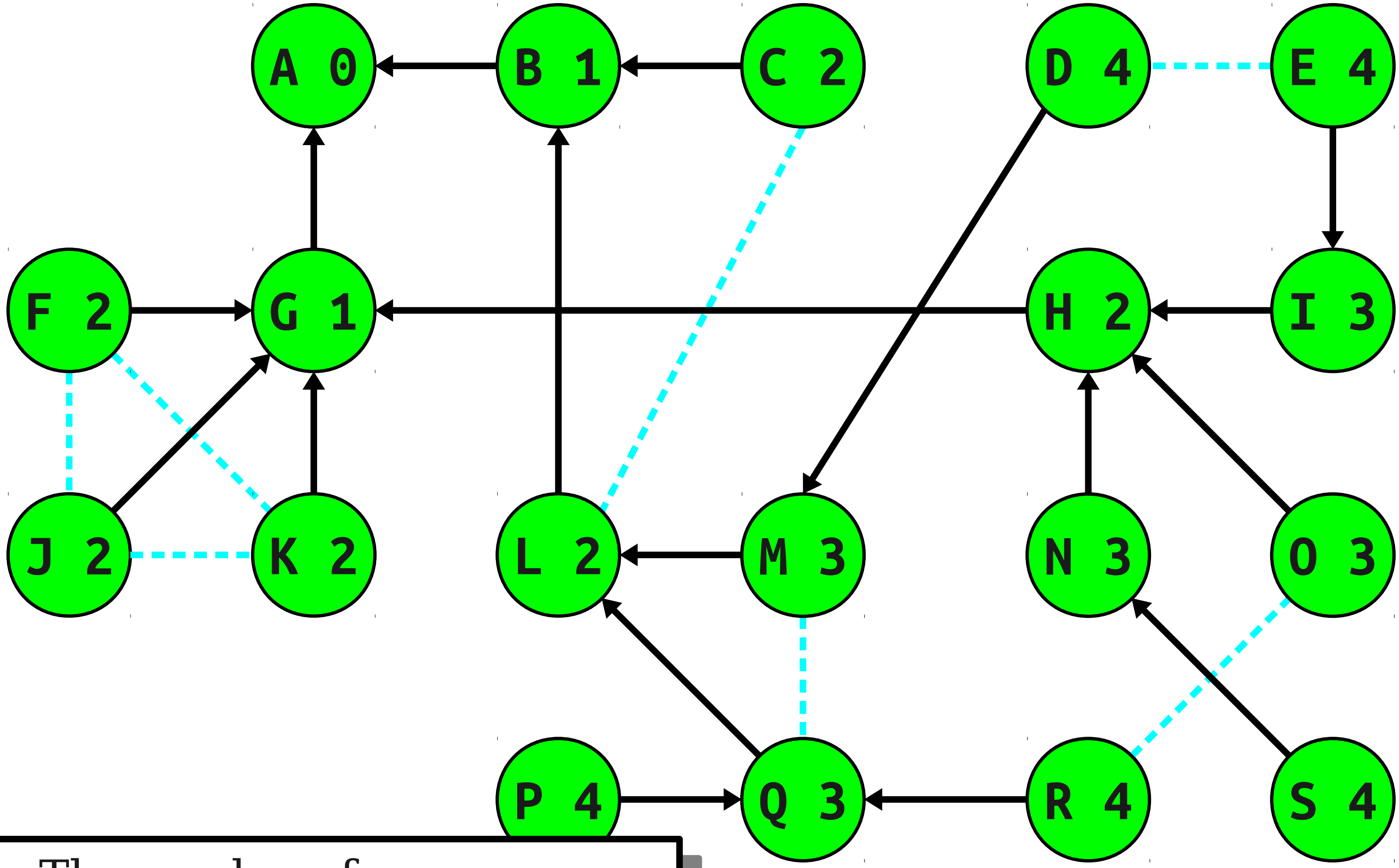


These edges form a **breadth-first search tree**: the path from any *v* to node *A* gives a shortest path from *v* to *A*.

```
procedure breadthFirstSearch(s, G):
   let q be a new queue.
   for each node v in G:
     dist[v] = ∞

   dist[s] = 0
   enqueue(s, q)

   while q is not empty:
      let v = dequeue(q)
      for each neighbor u of v:
         if dist[u] = ∞:
            dist[u] = dist[v] + 1
            enqueue(u, q)
```

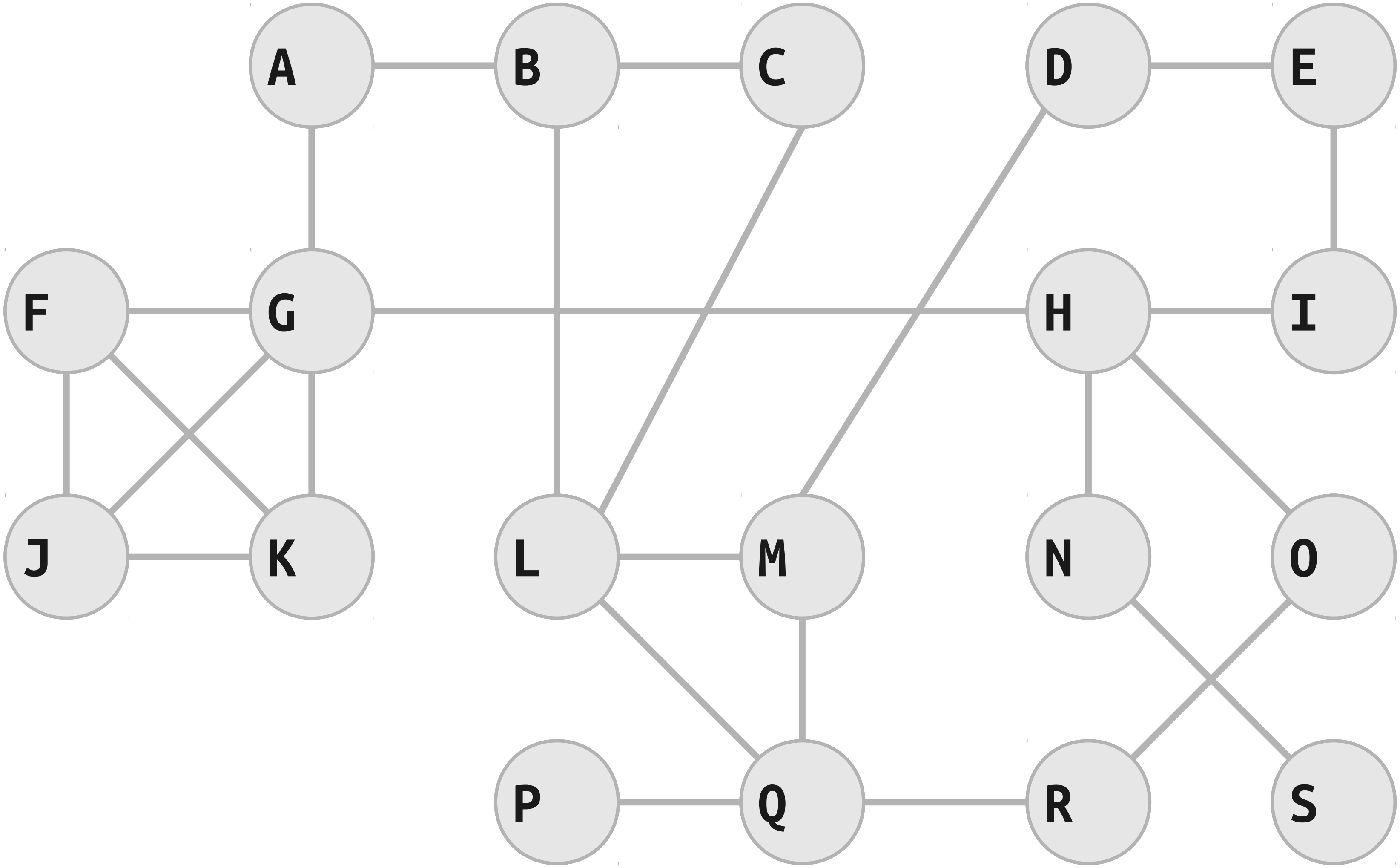
Question 1: How do we prove this always finds the right distances?

Question 2: How *efficiently* does this find the right distances?

Question 1: How do we prove this always finds the right distances?

Question 2: How *efficiently* does this find the right distances?

Breadth-First Search



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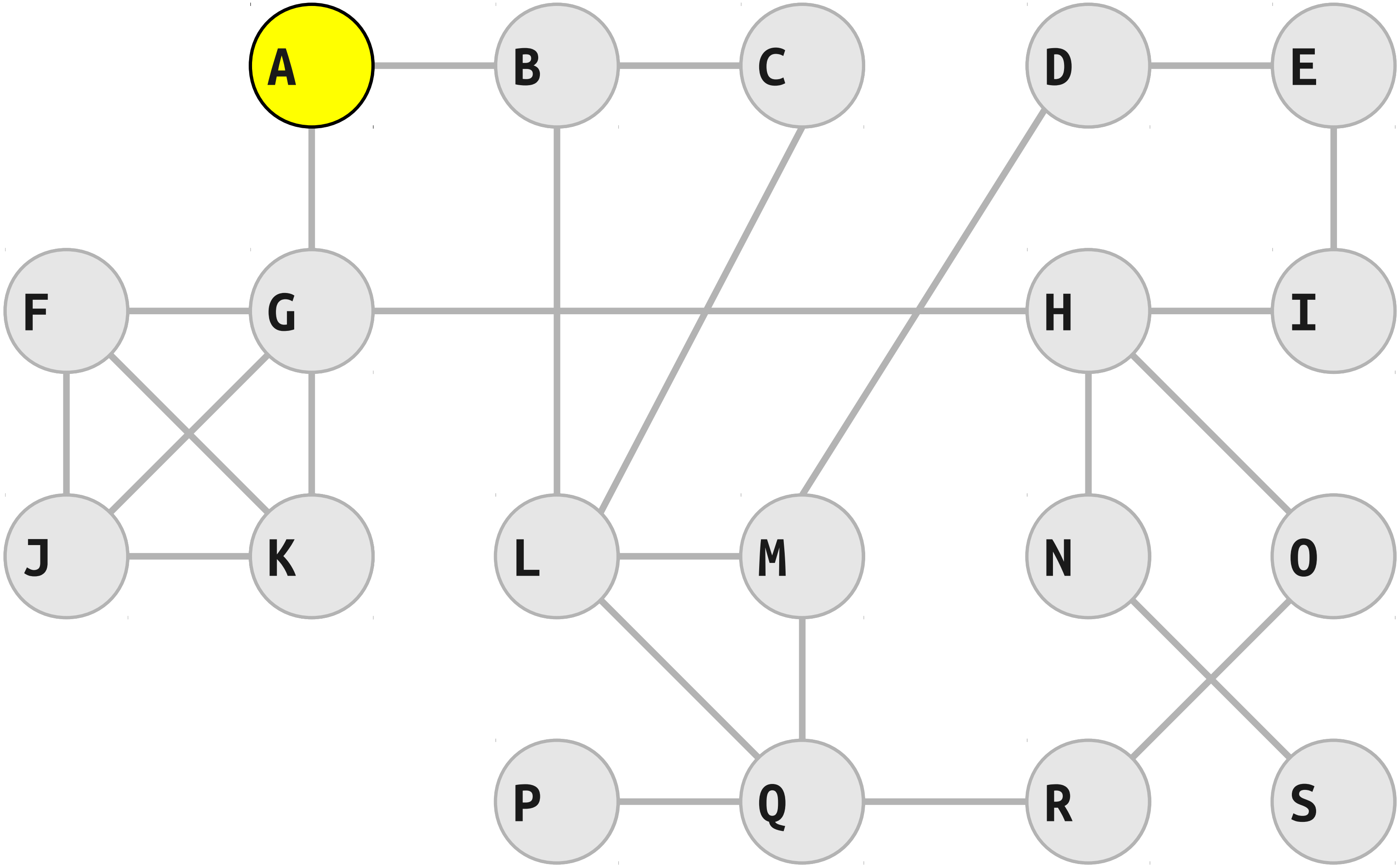

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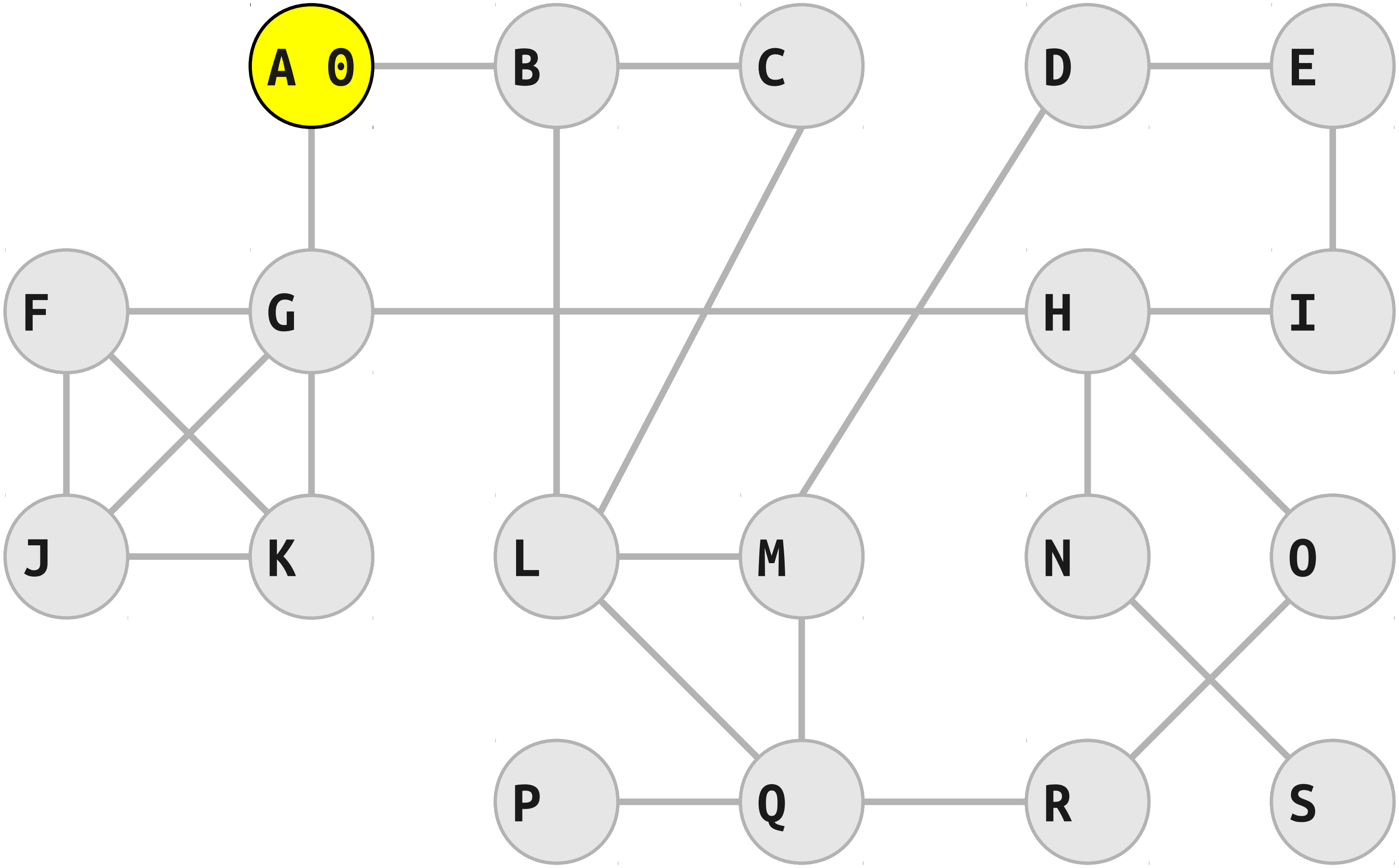

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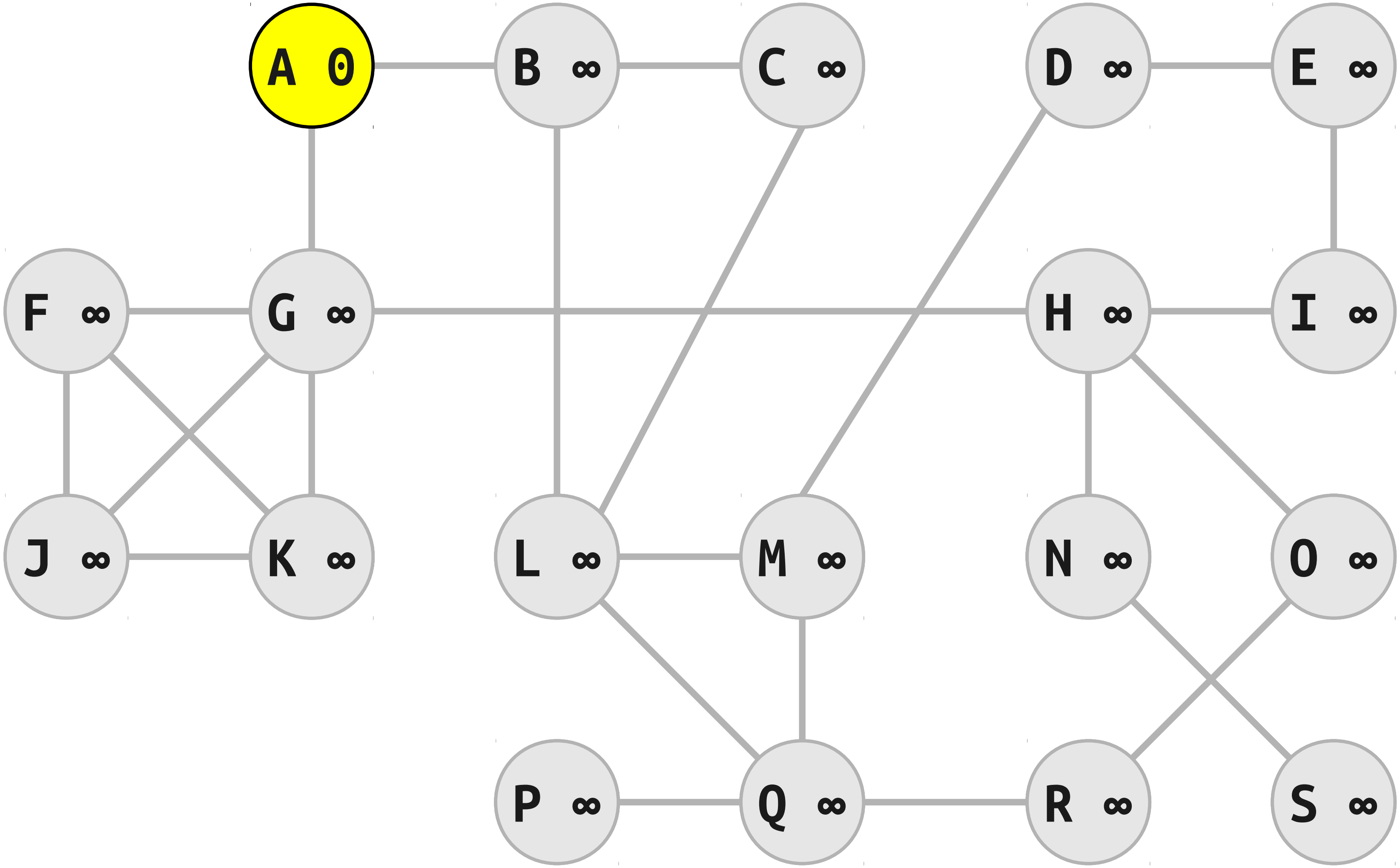

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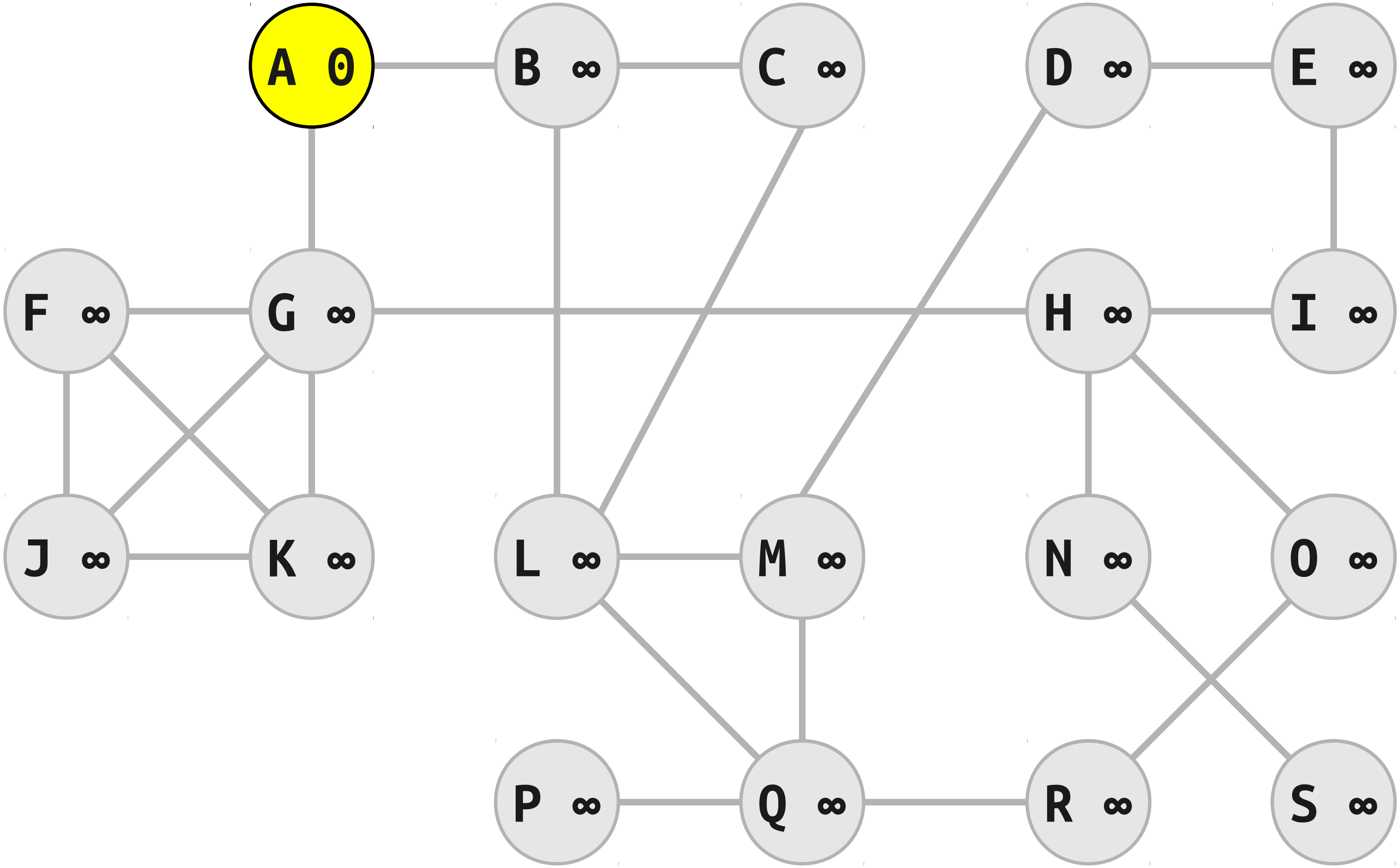

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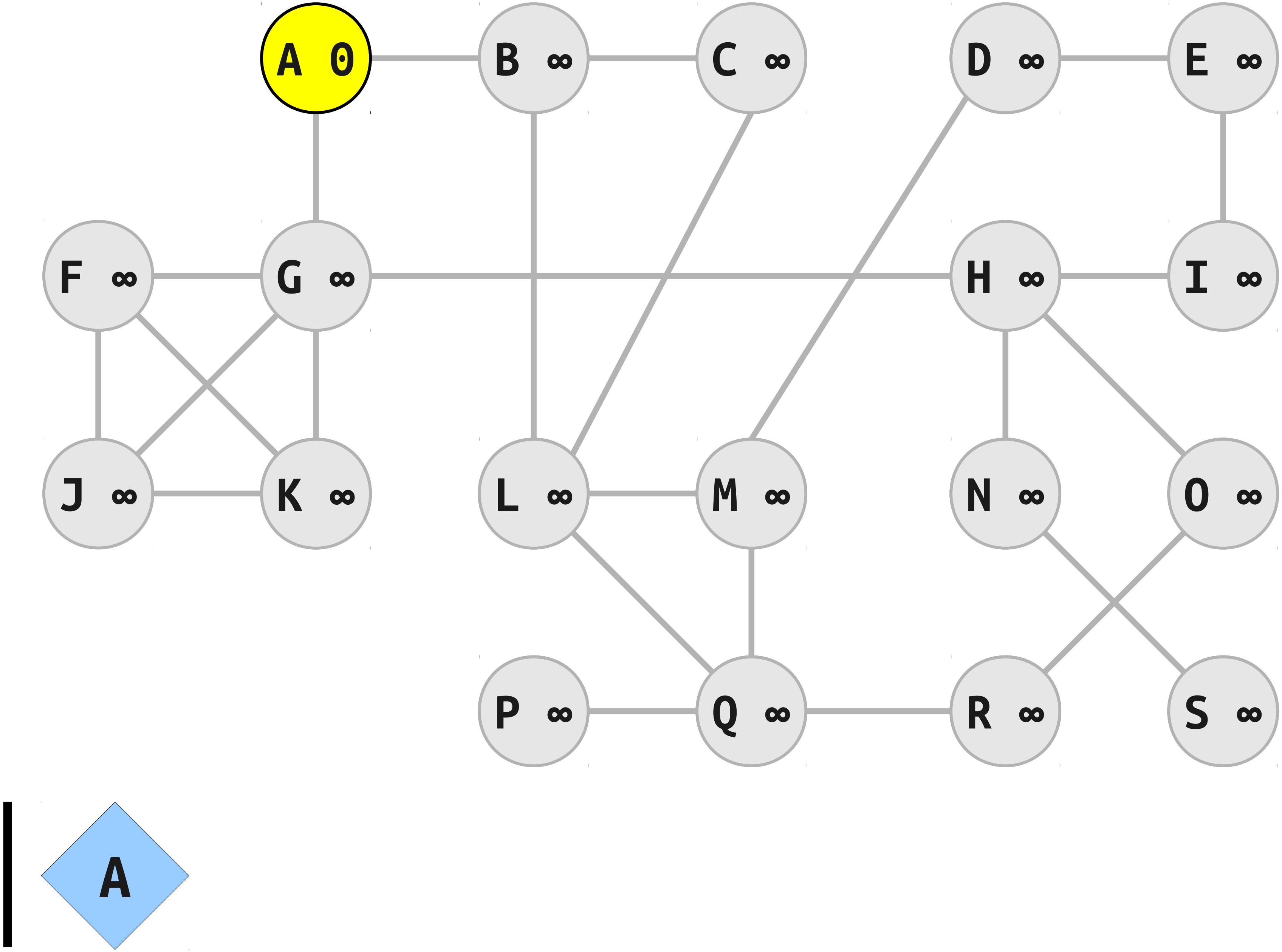

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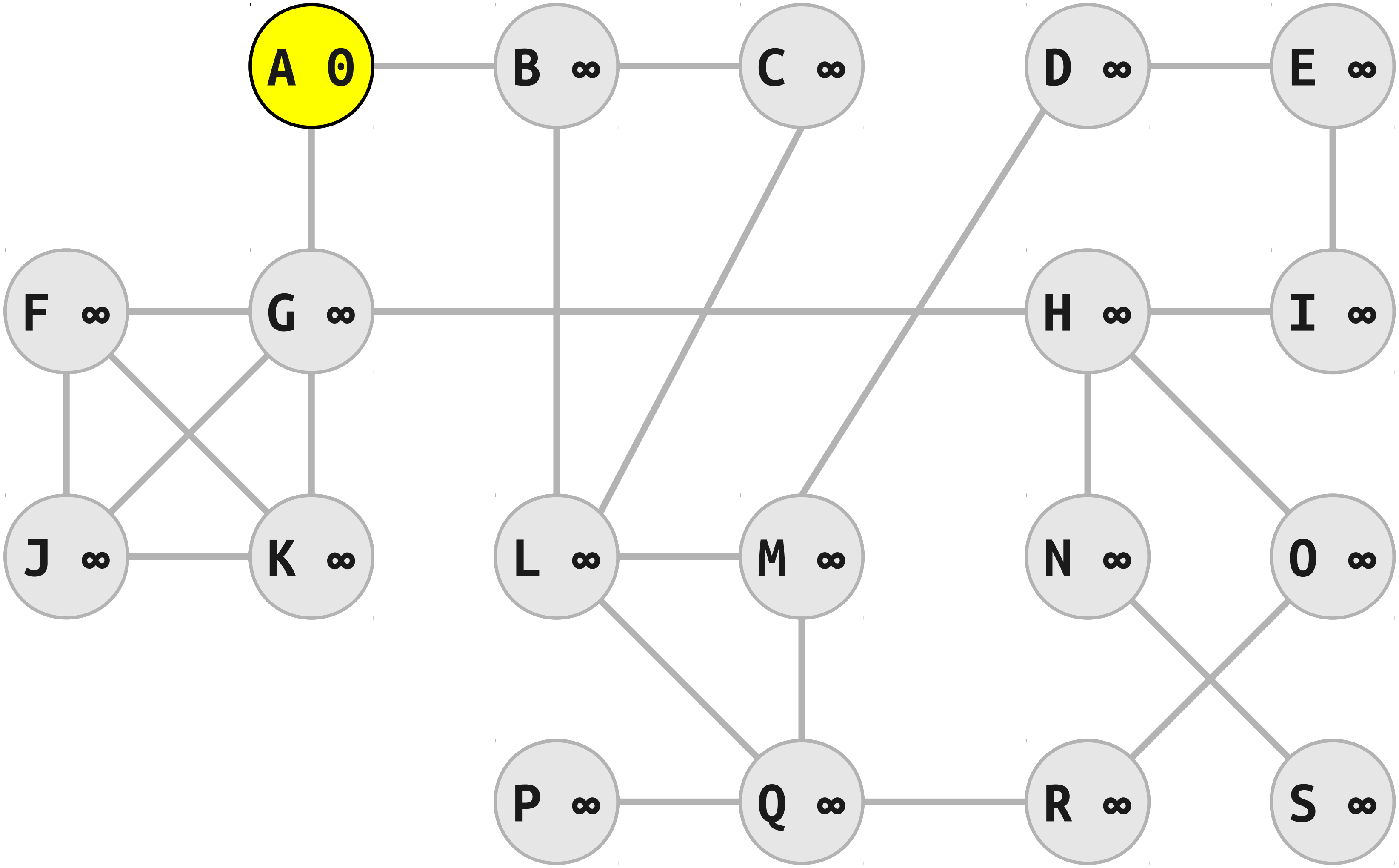
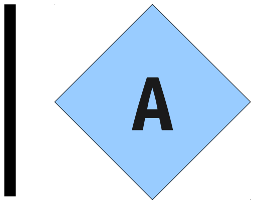
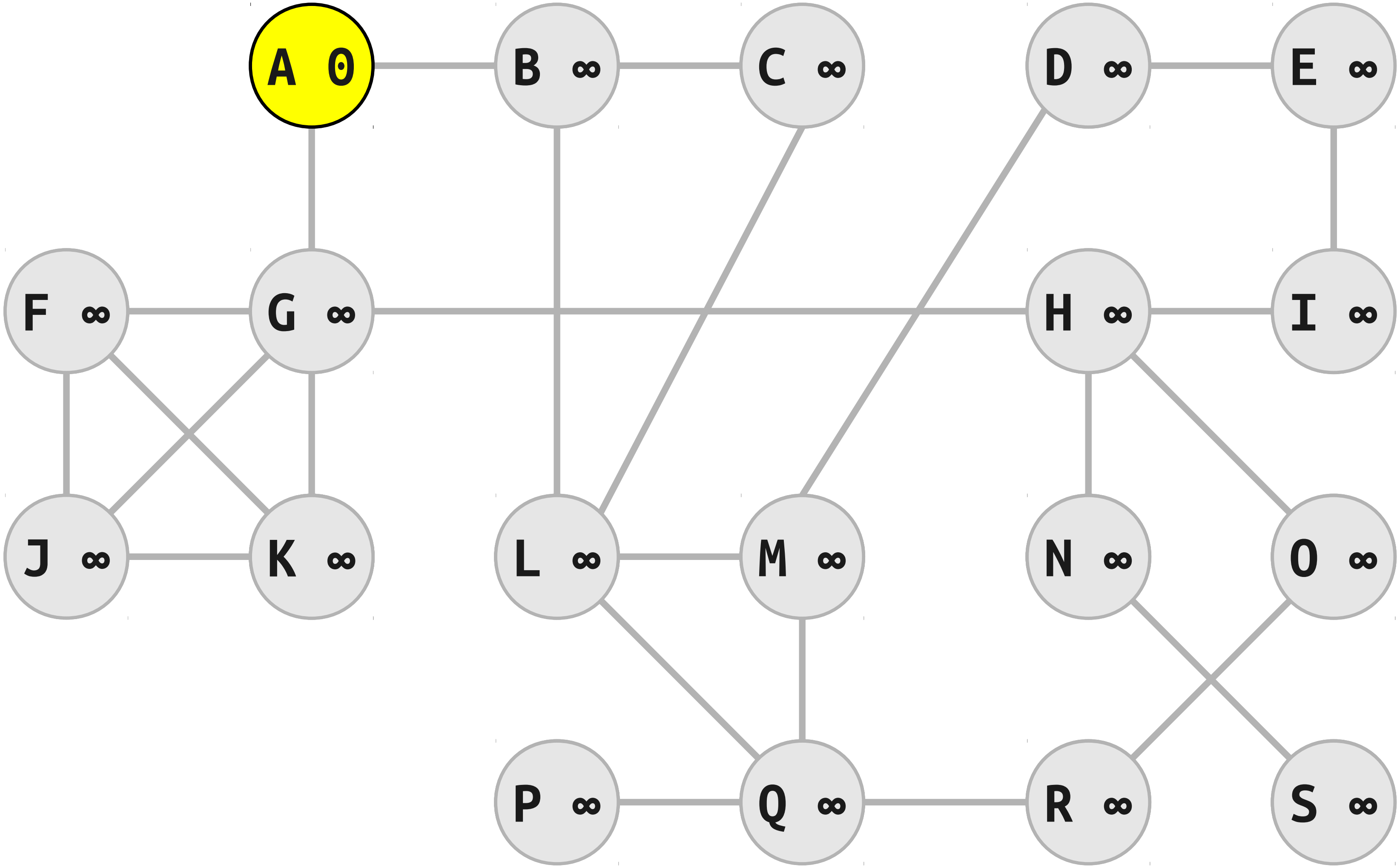
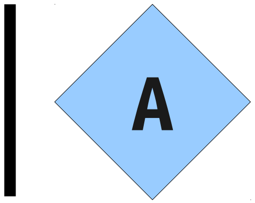




All nodes in the queue are at distance **0** from A.

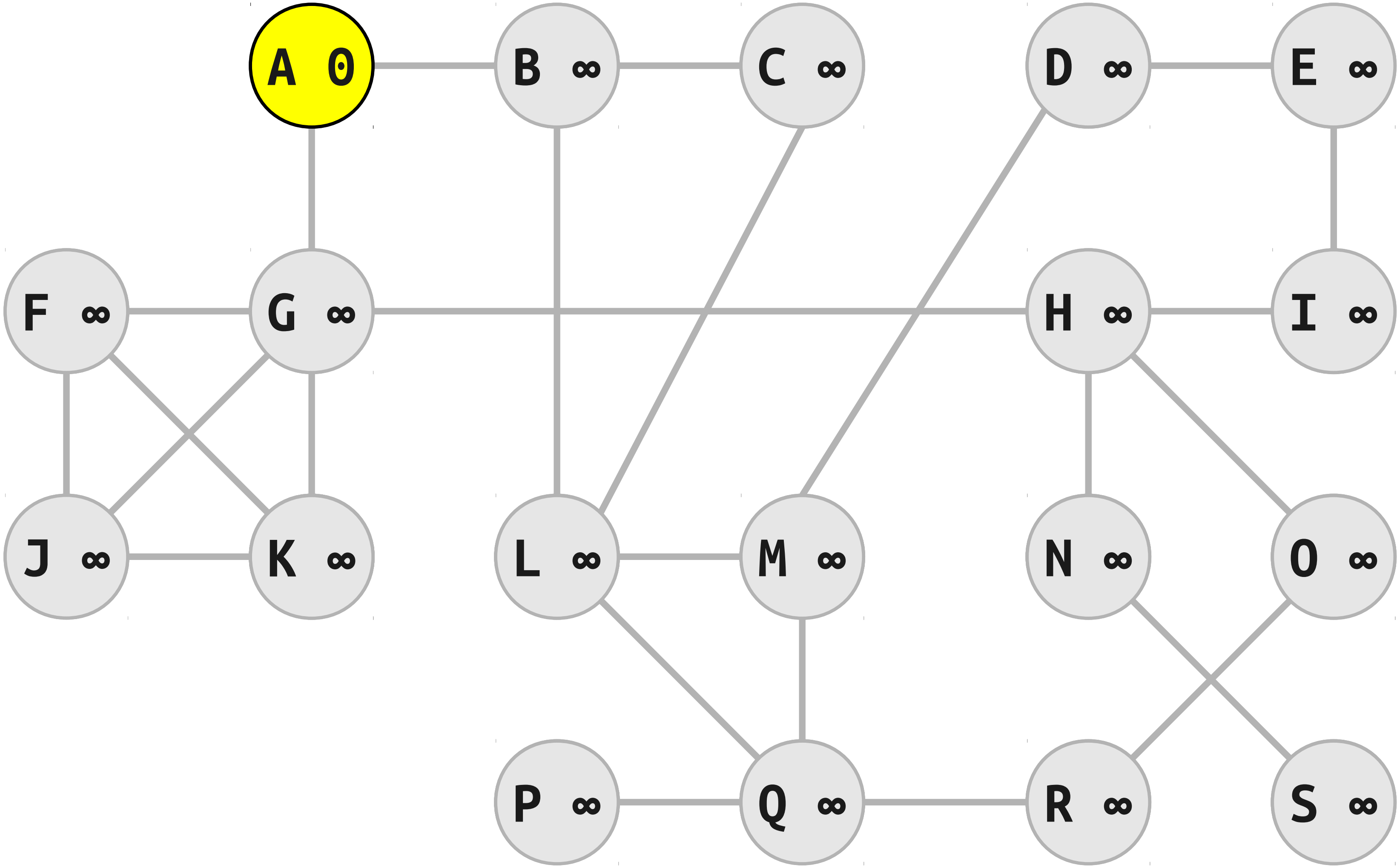
Breadth-First Search

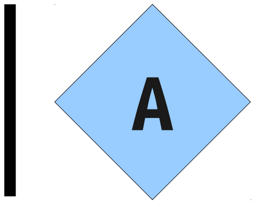




All nodes at distance **0** from *A* are in the queue.

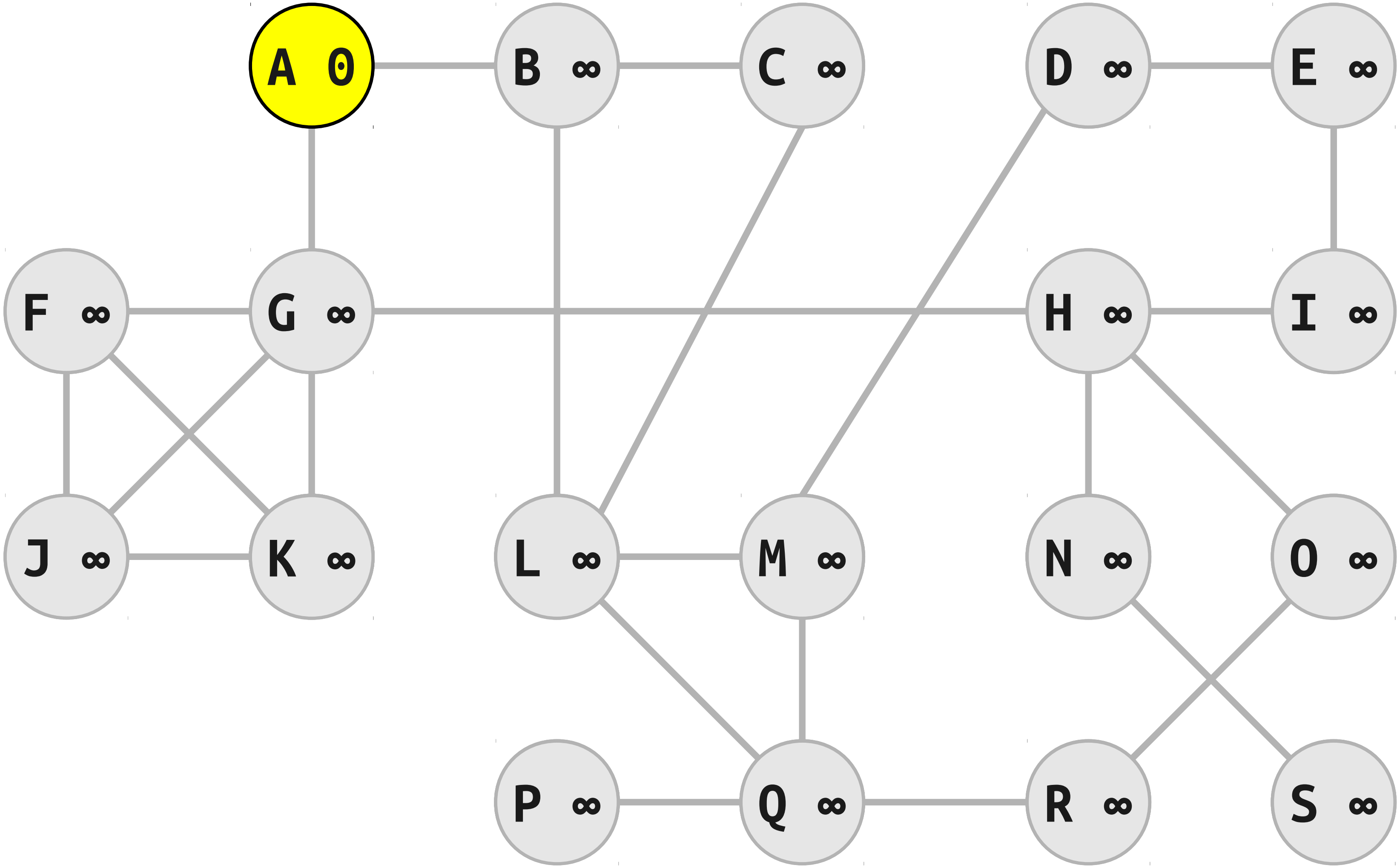
Breadth-First Search

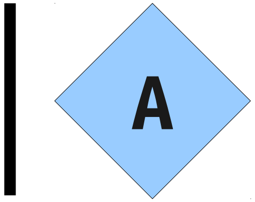




All nodes at distance ≤ **0** from *A* have the right distance set.

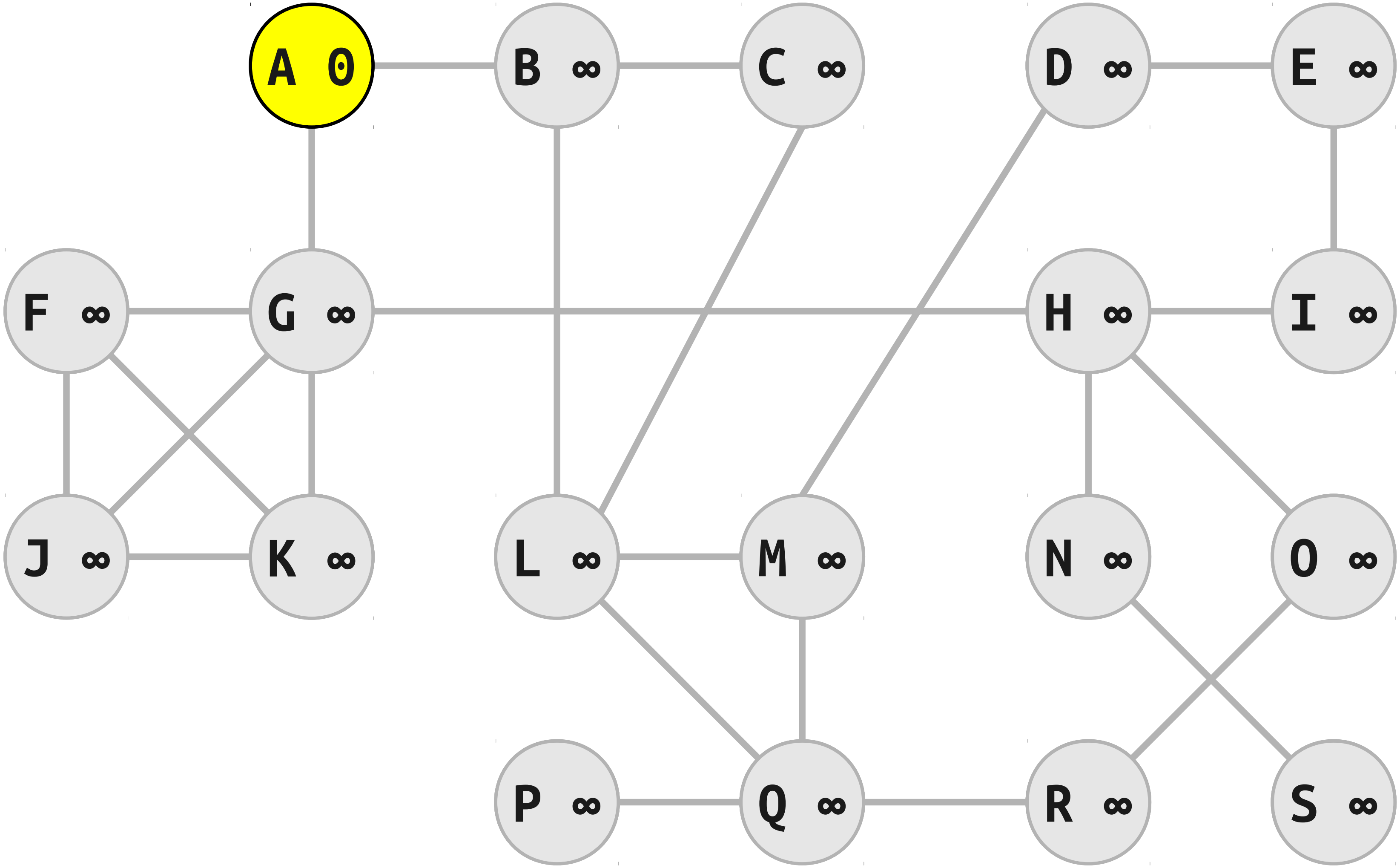
Breadth-First Search

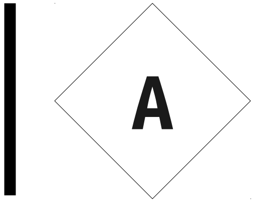




All nodes at distance > **0** from *A* have distance set to ∞

Breadth-First Search





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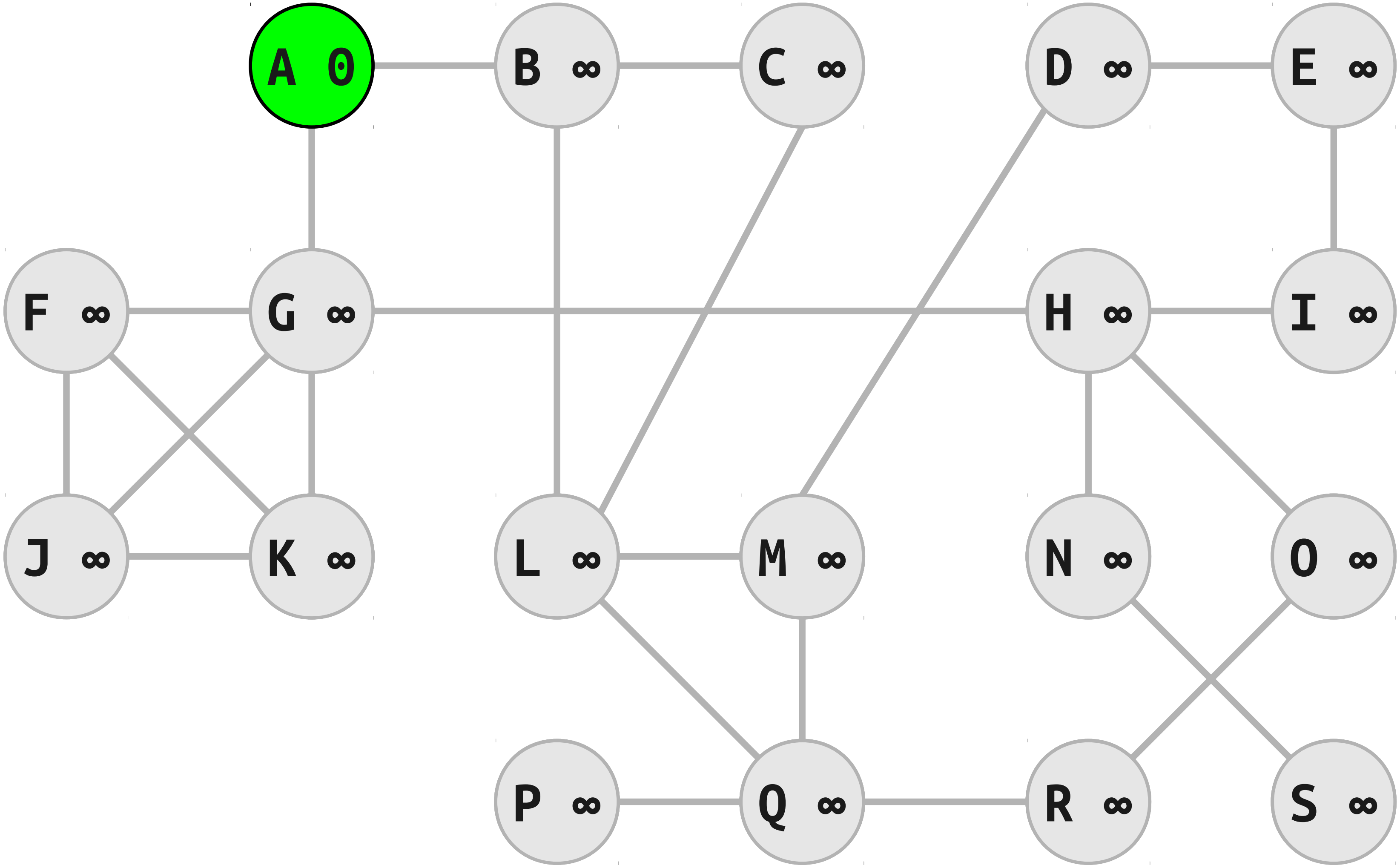

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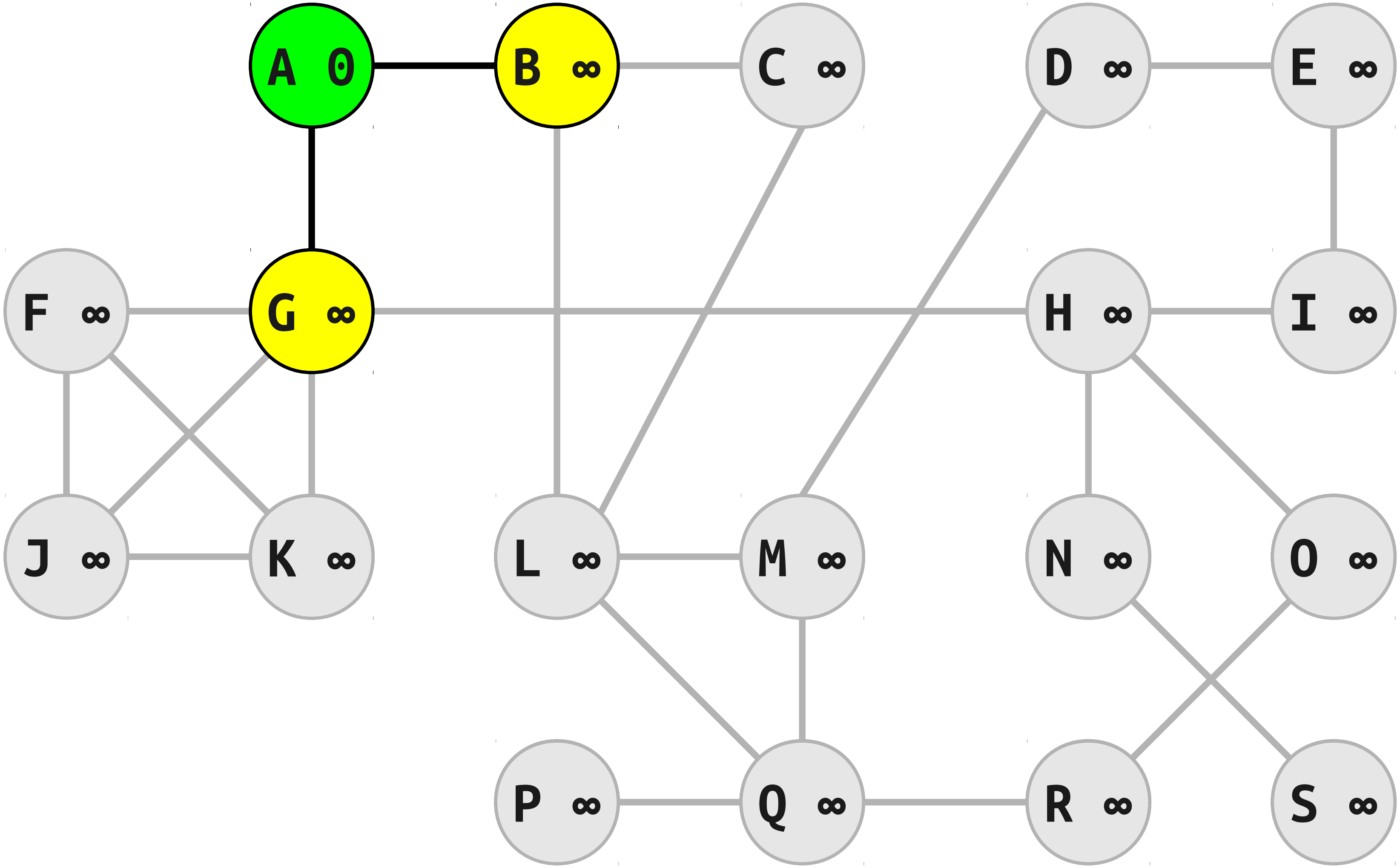

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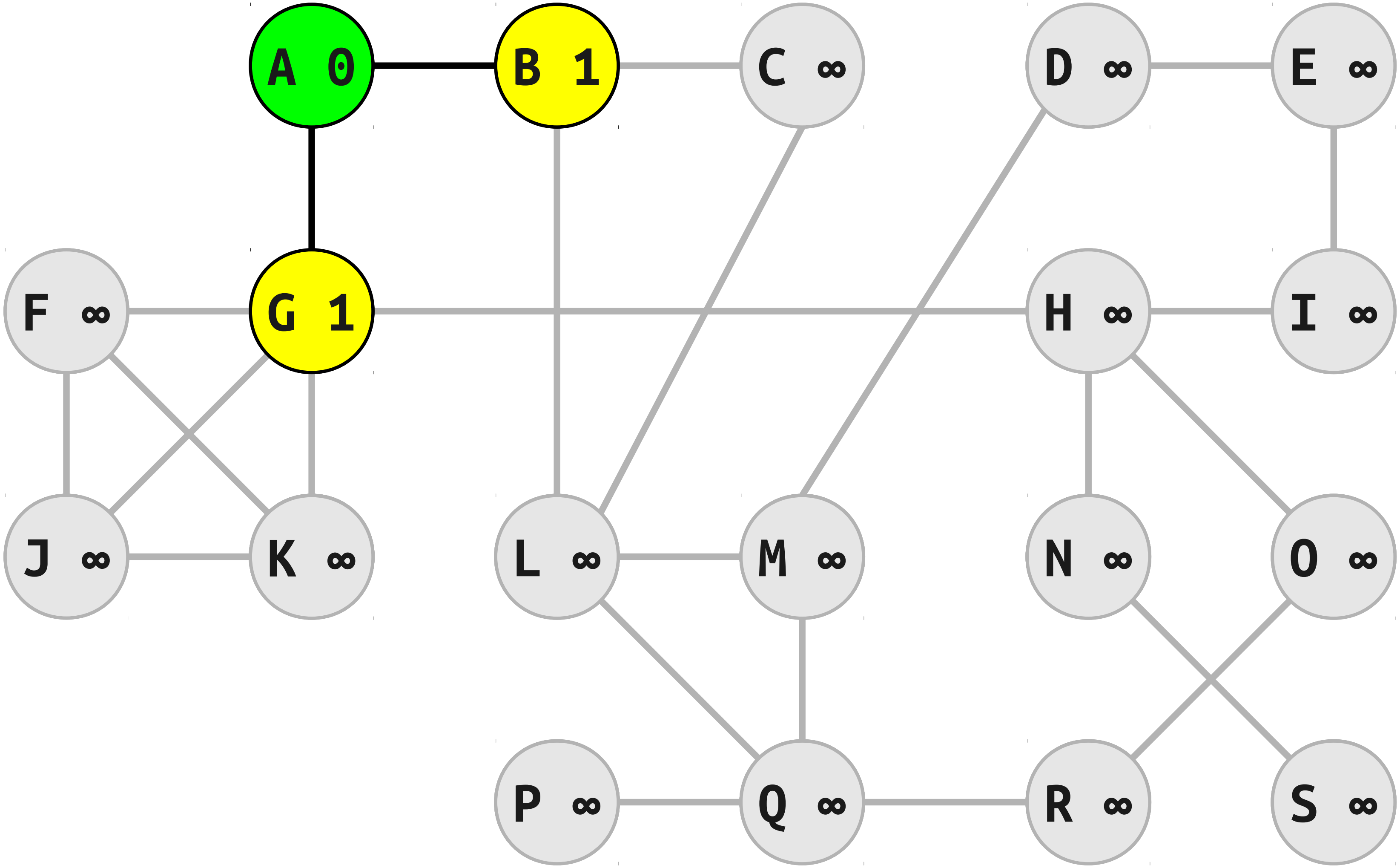

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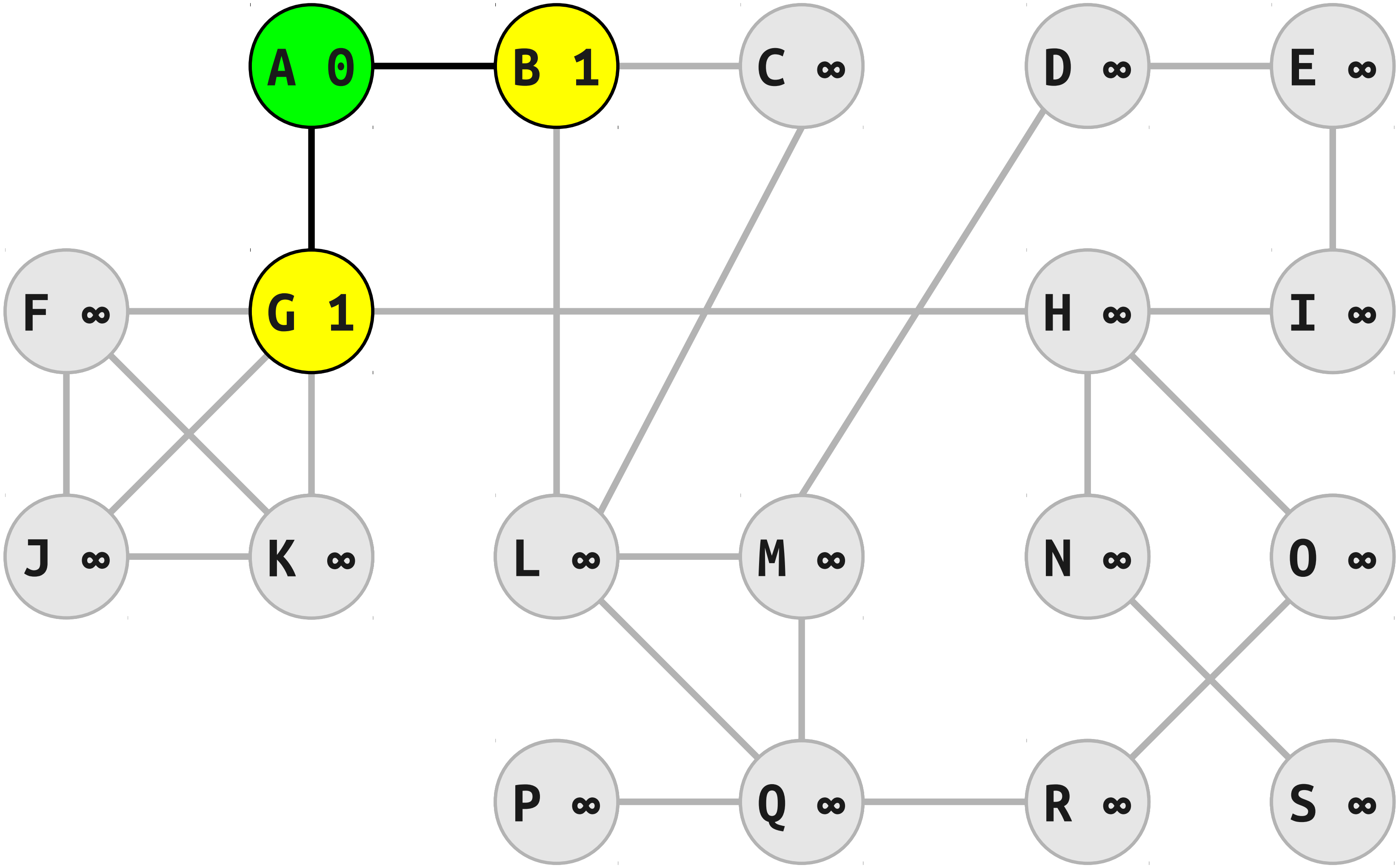

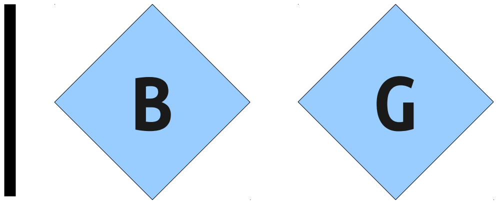

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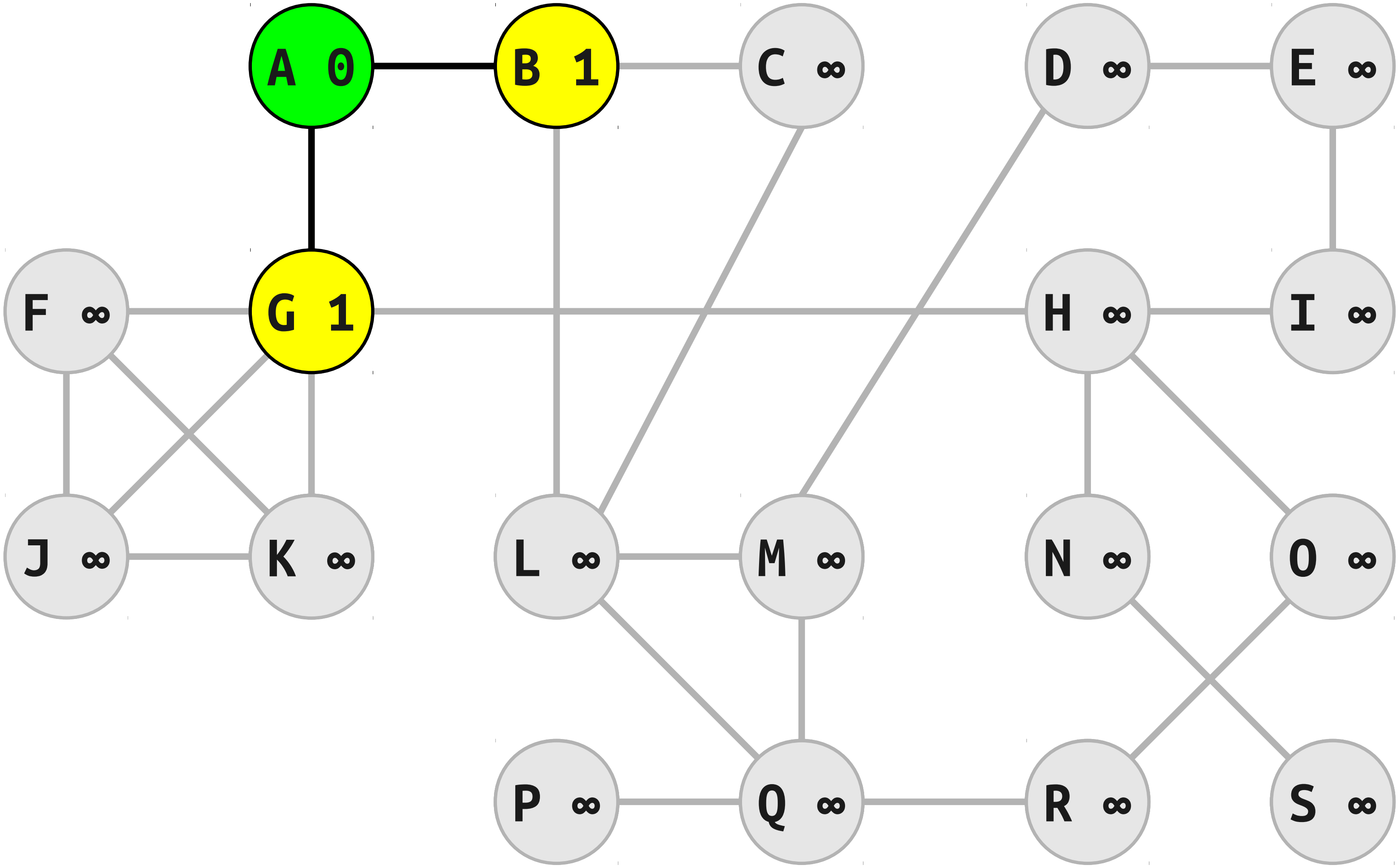
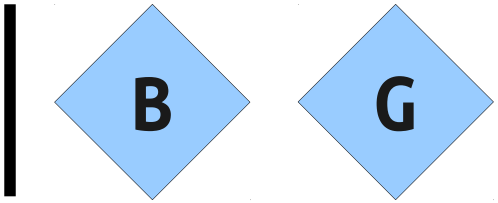
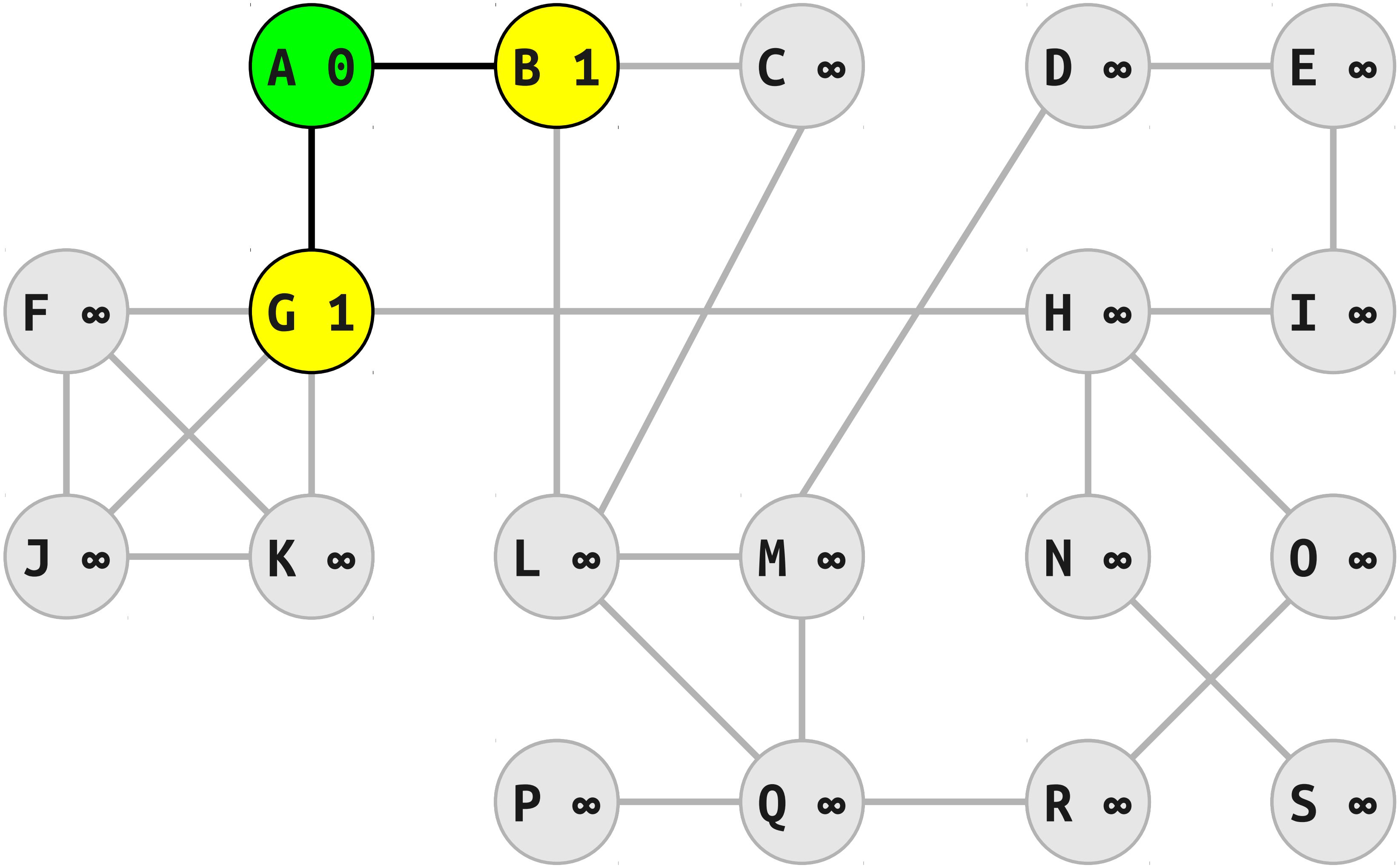
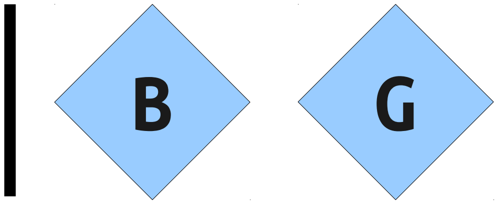




All nodes in the queue are at distance **1** from *A*.

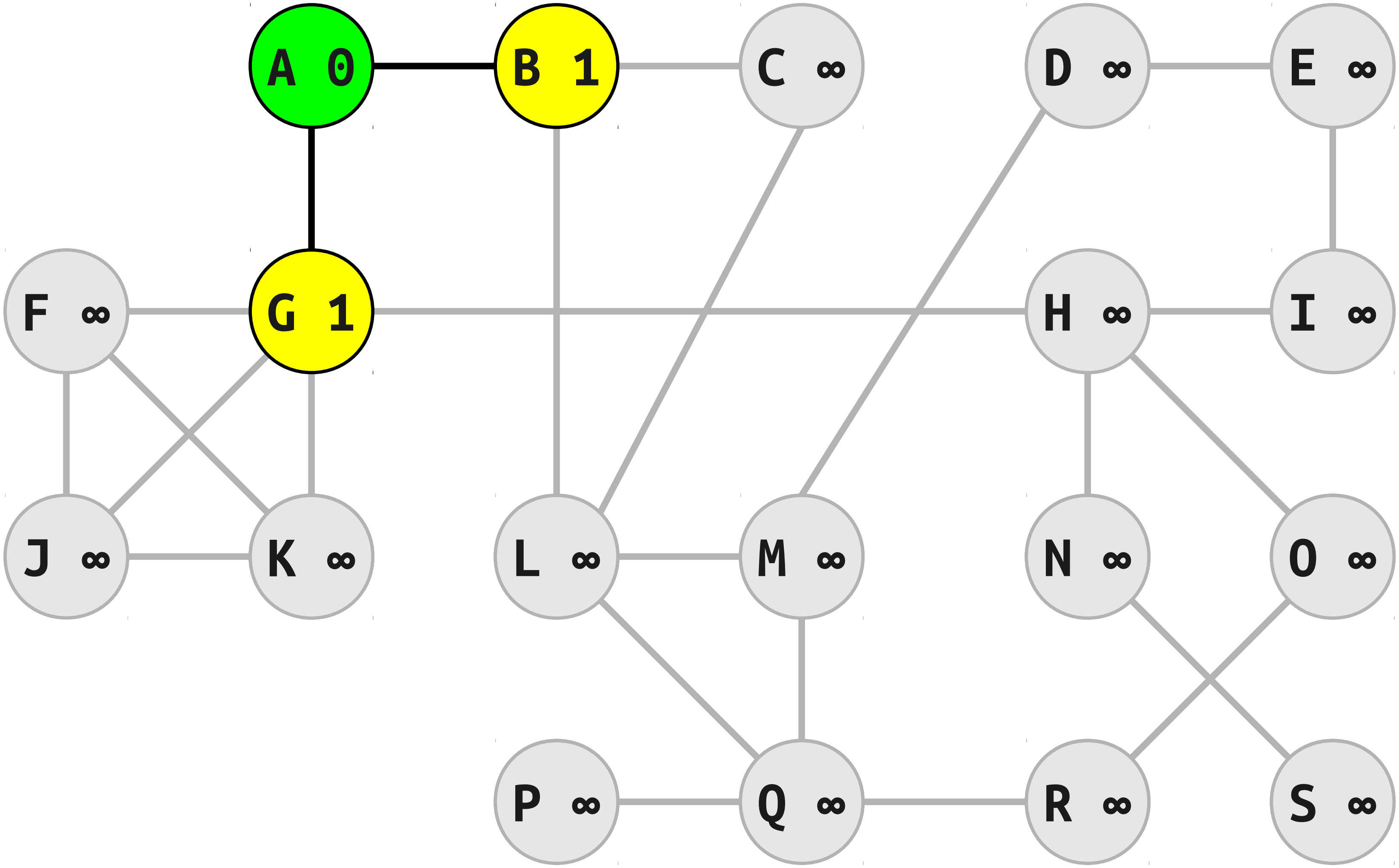
Breadth-First Search

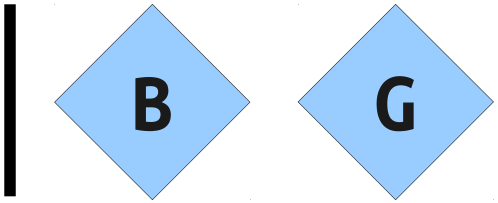




All nodes at distance 1 from *A* are in the queue.

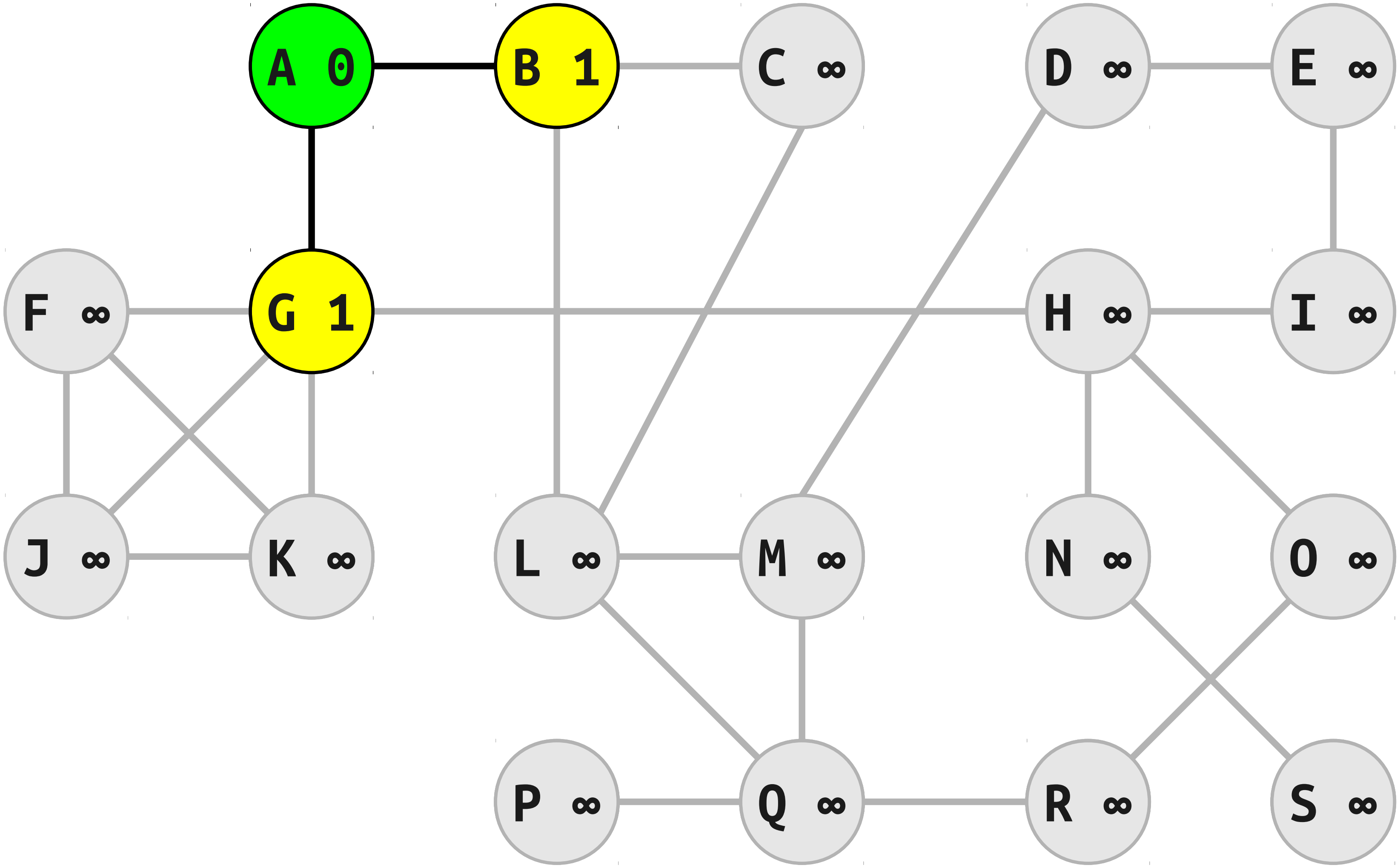
Breadth-First Search

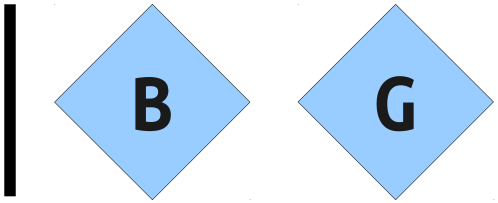




All nodes at distance ≤ 1 from *A* have the right distance set.

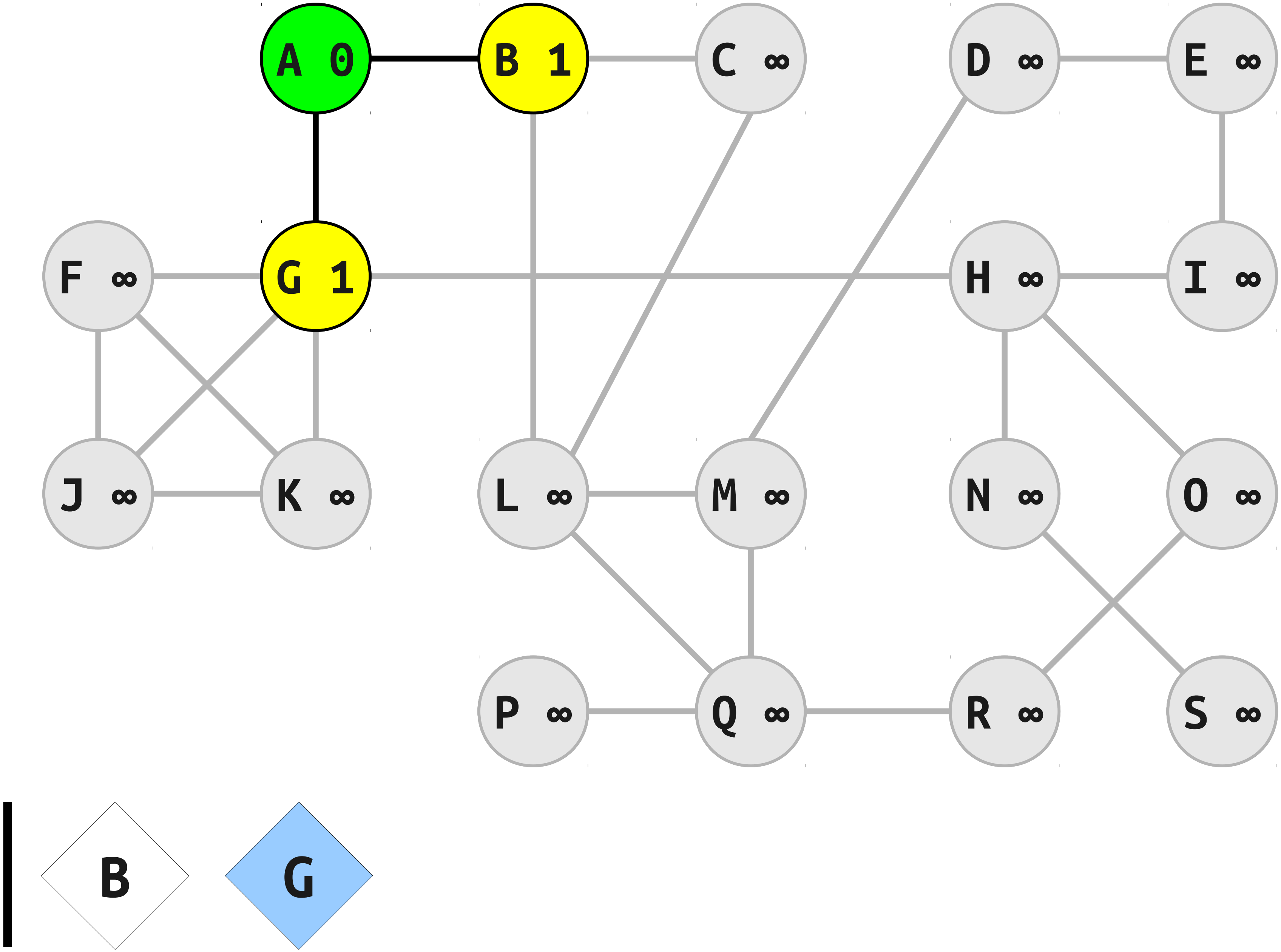
Breadth-First Search



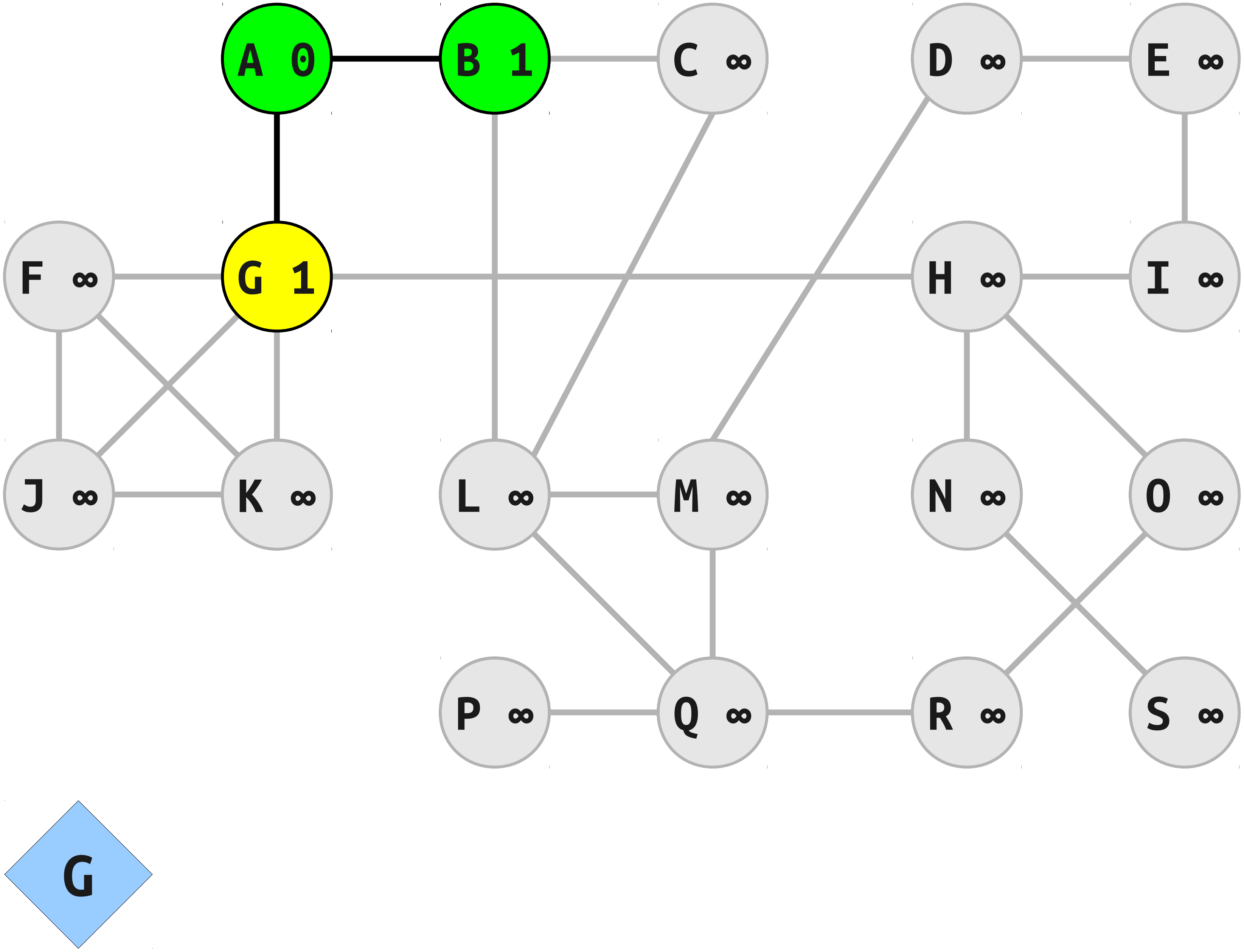


All nodes at distance > 1 from *A* have distance set to ∞

Breadth-First Search



Breadth-First Search



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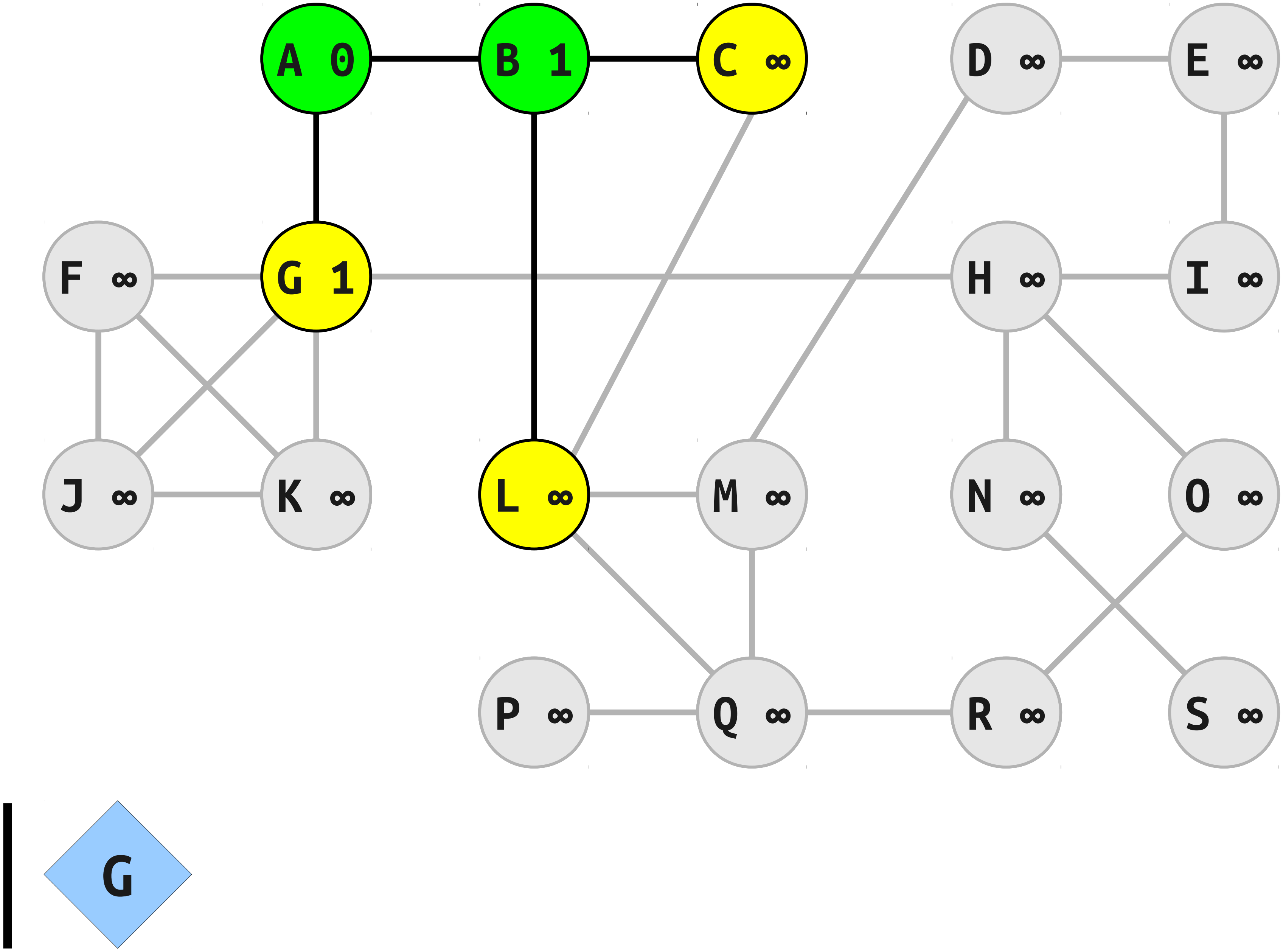

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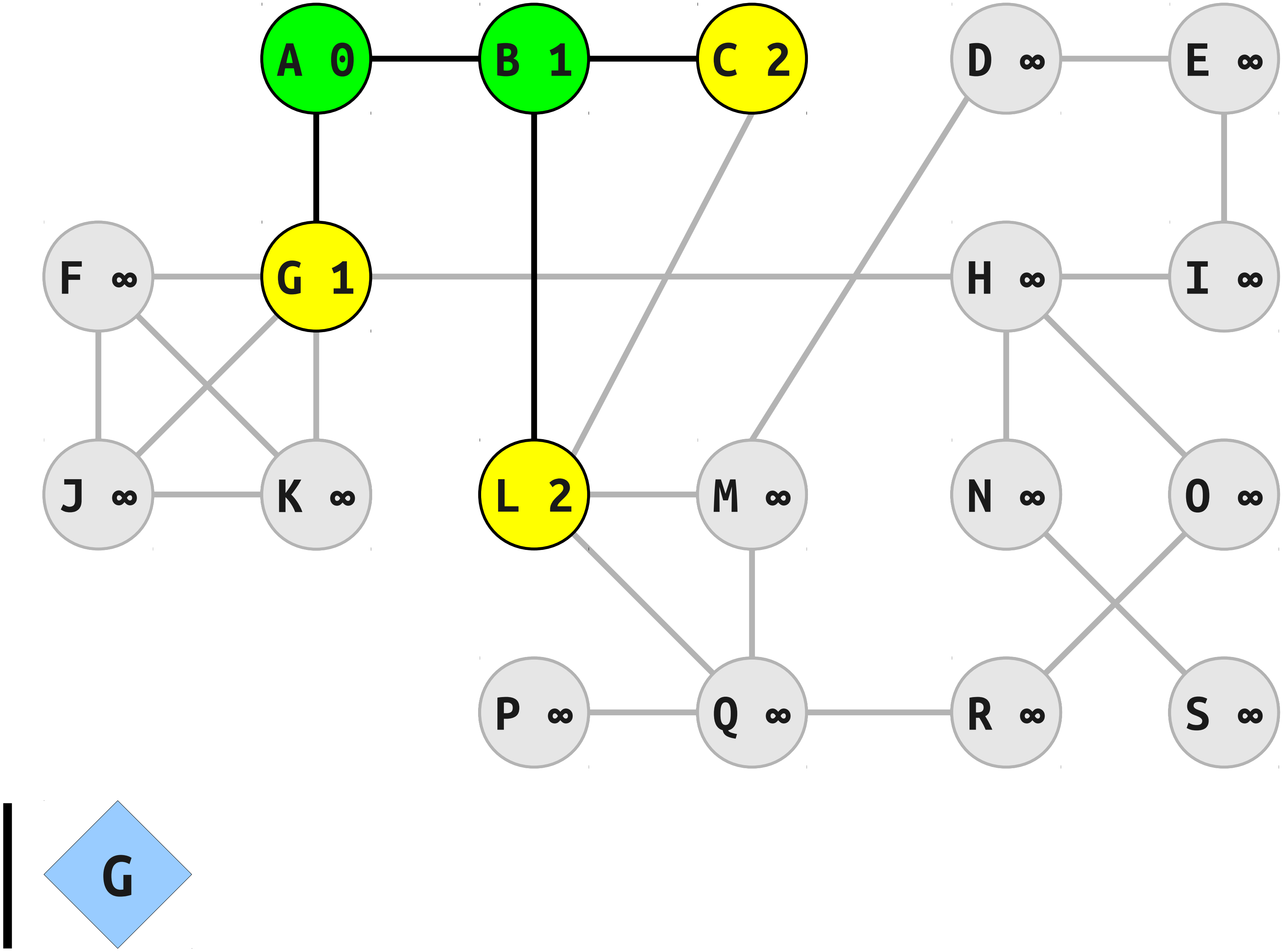

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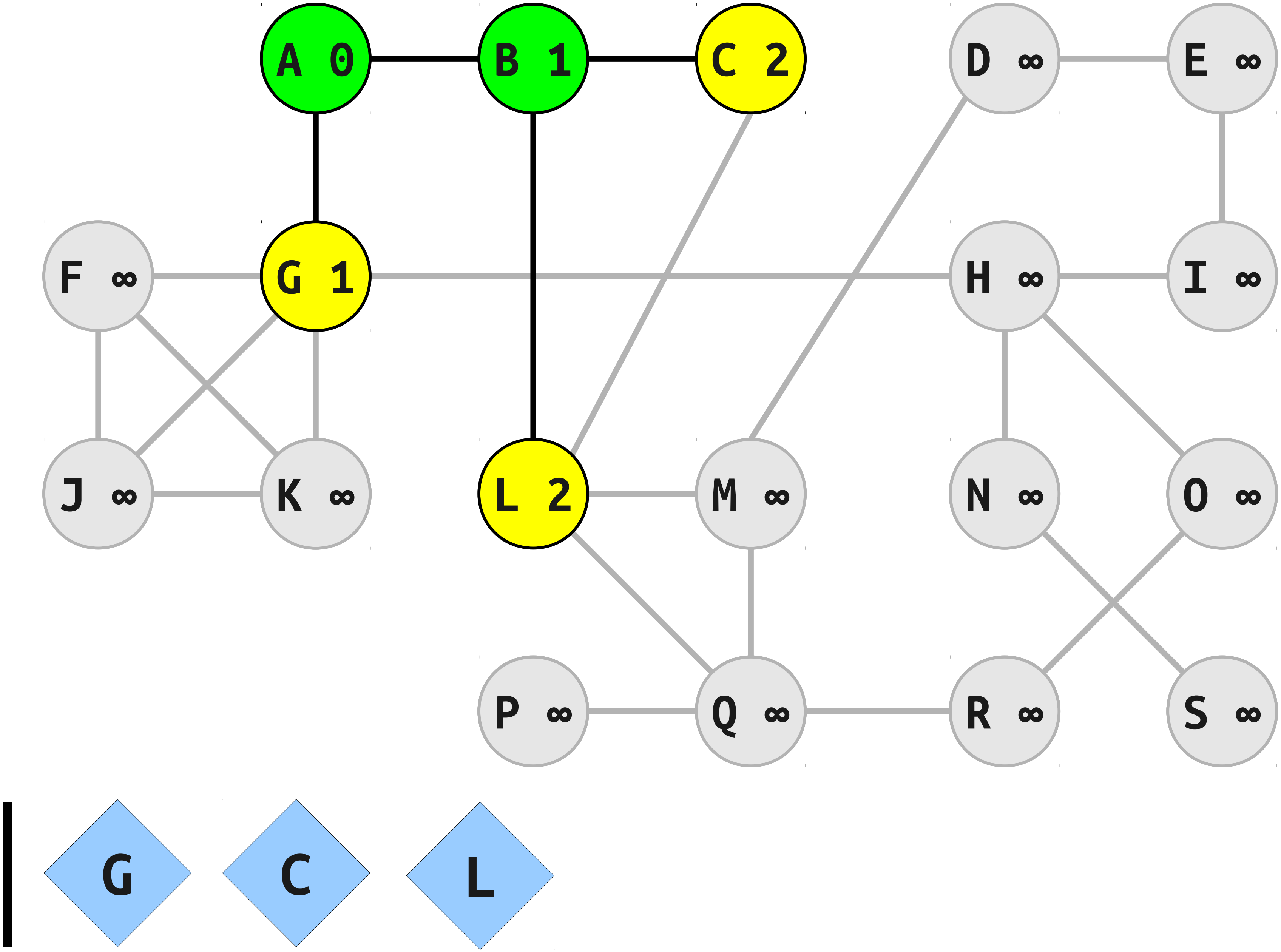

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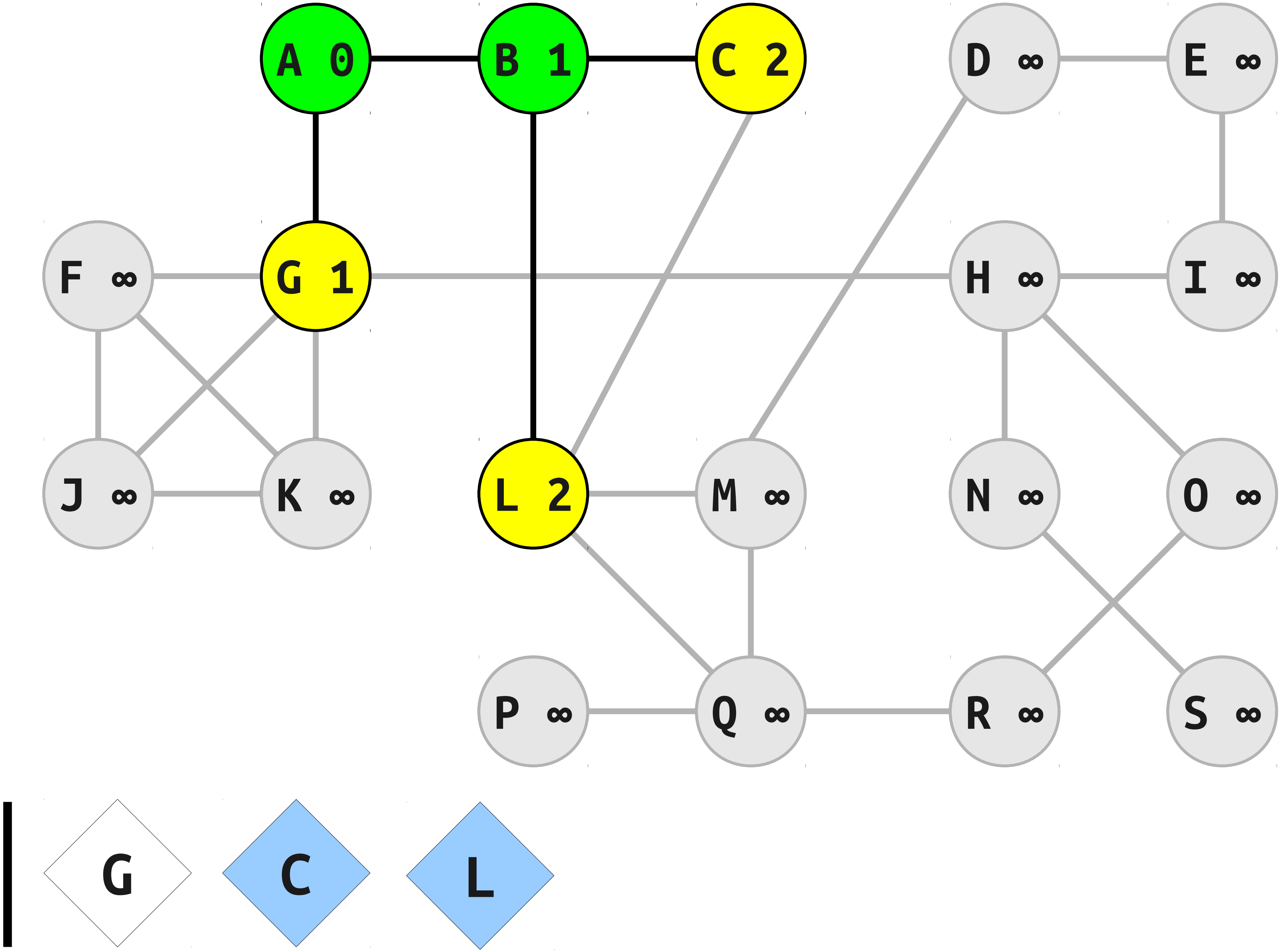

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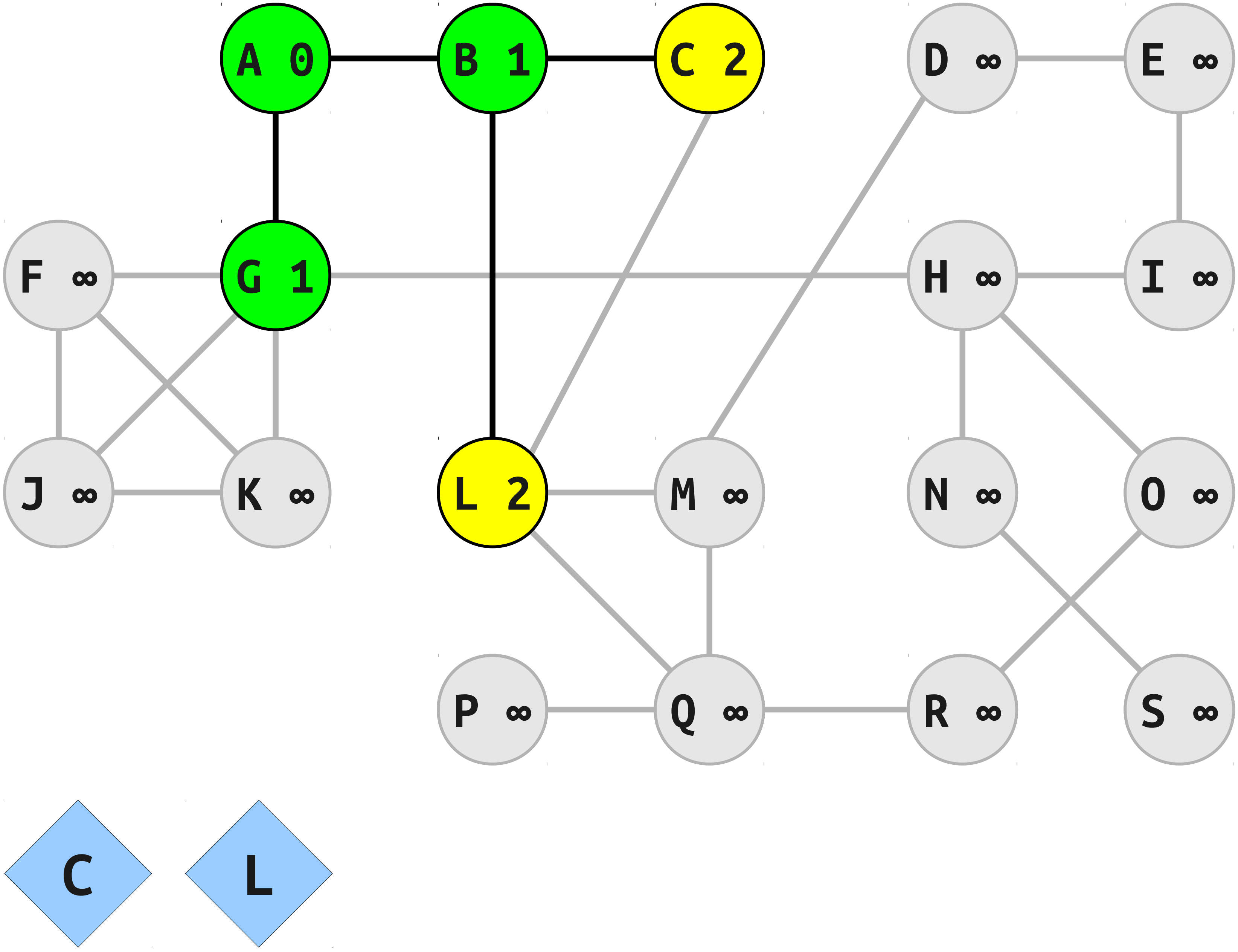

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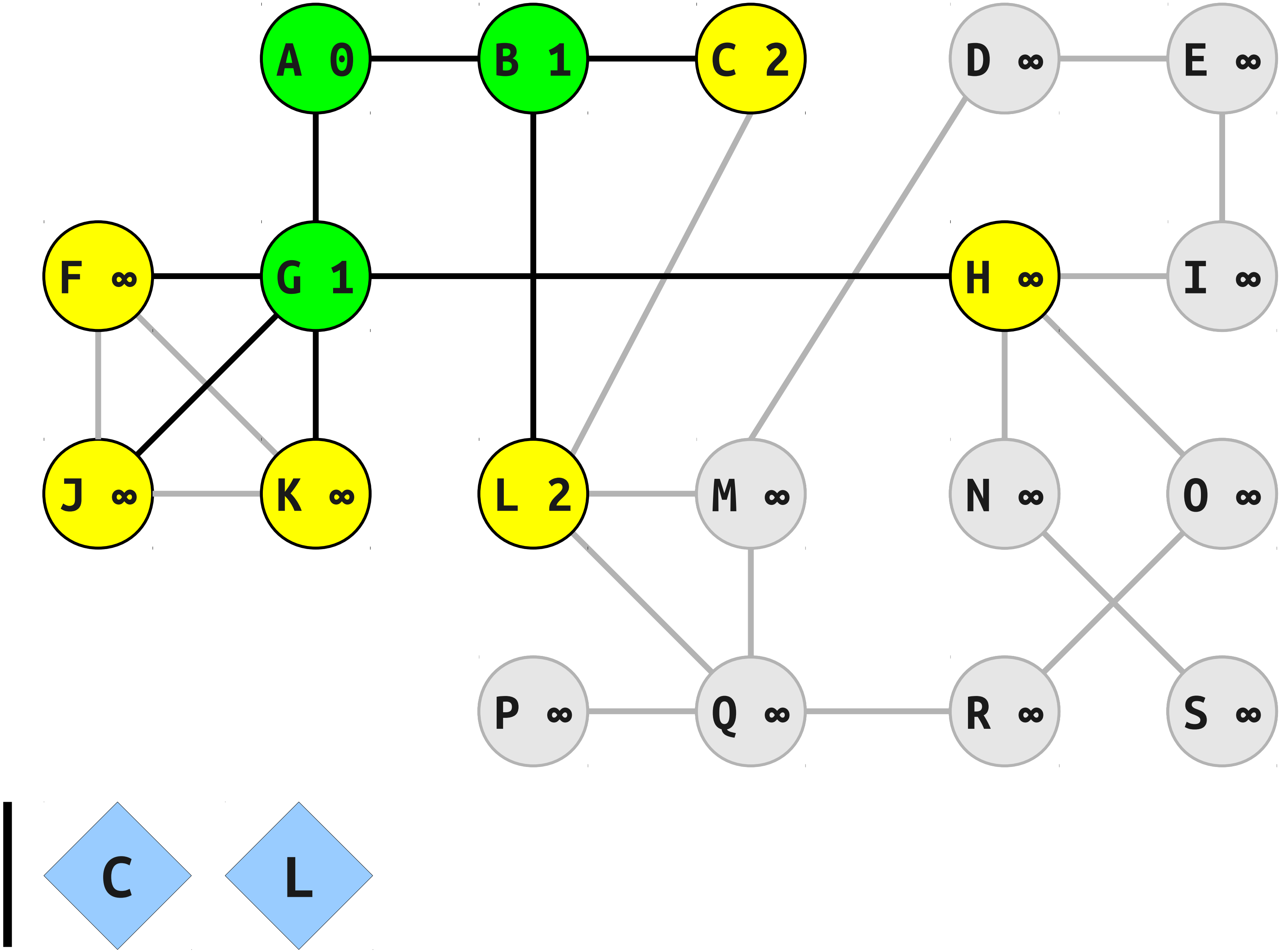

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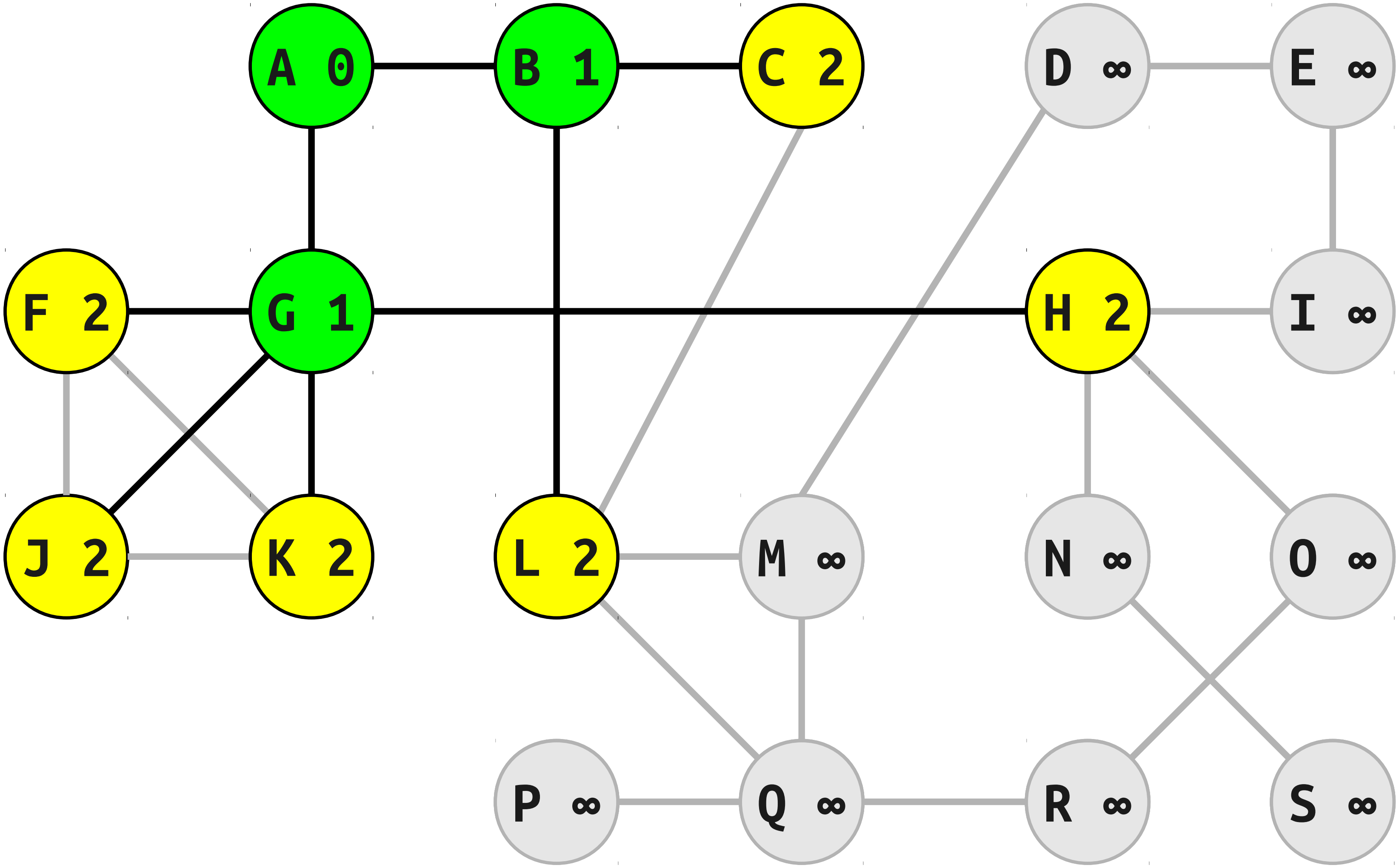

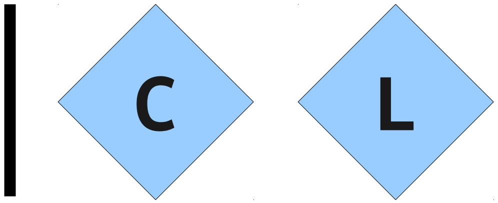

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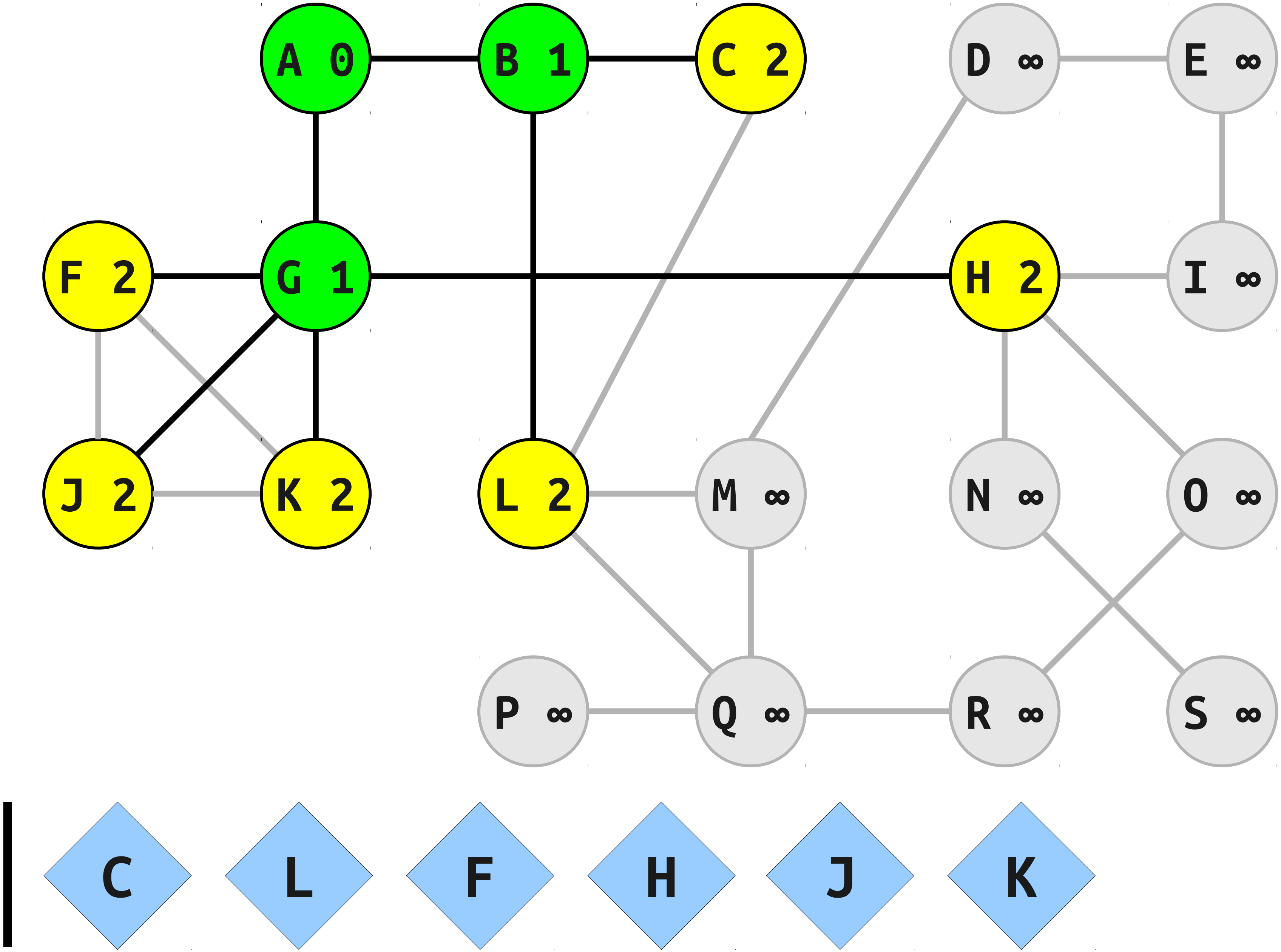

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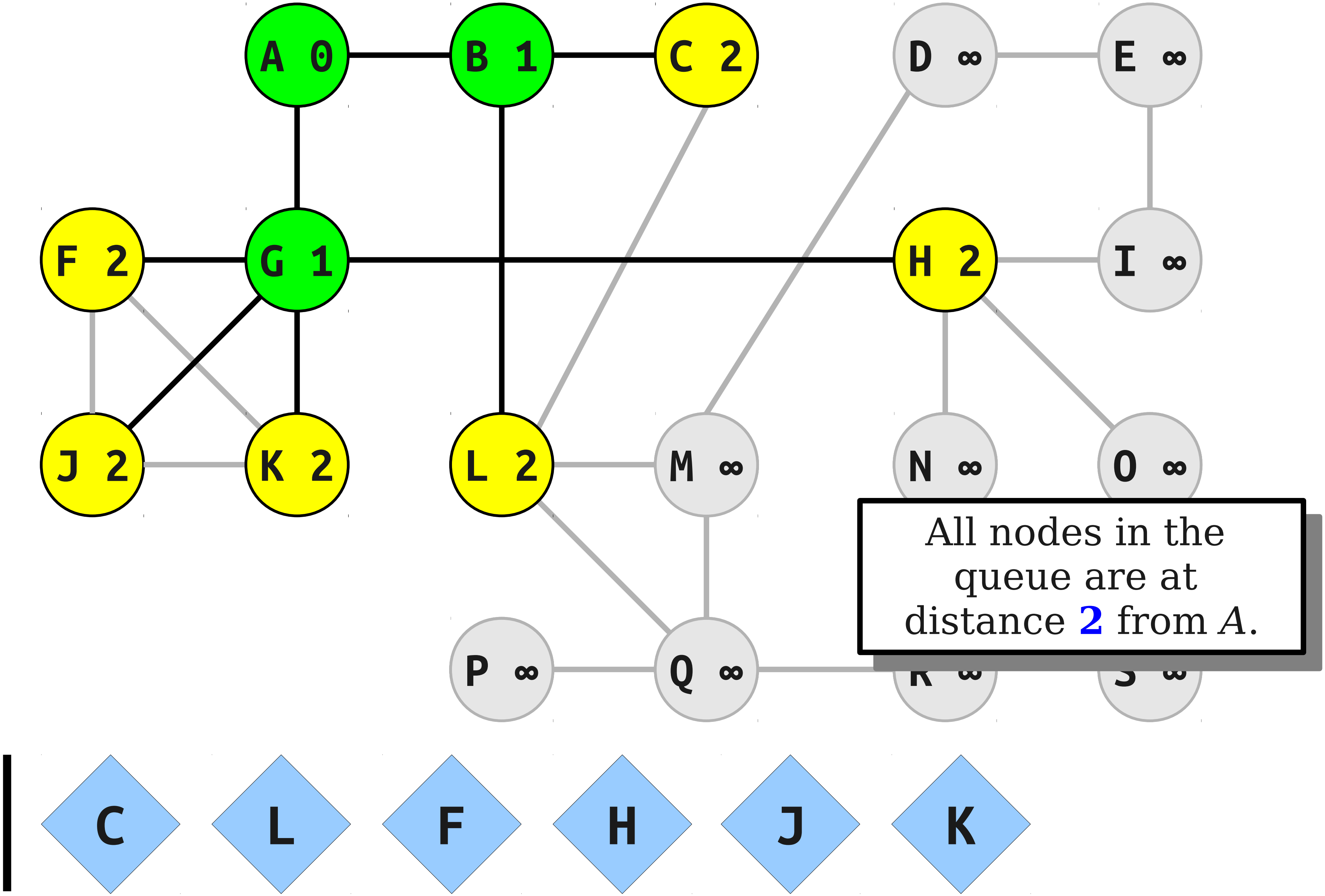
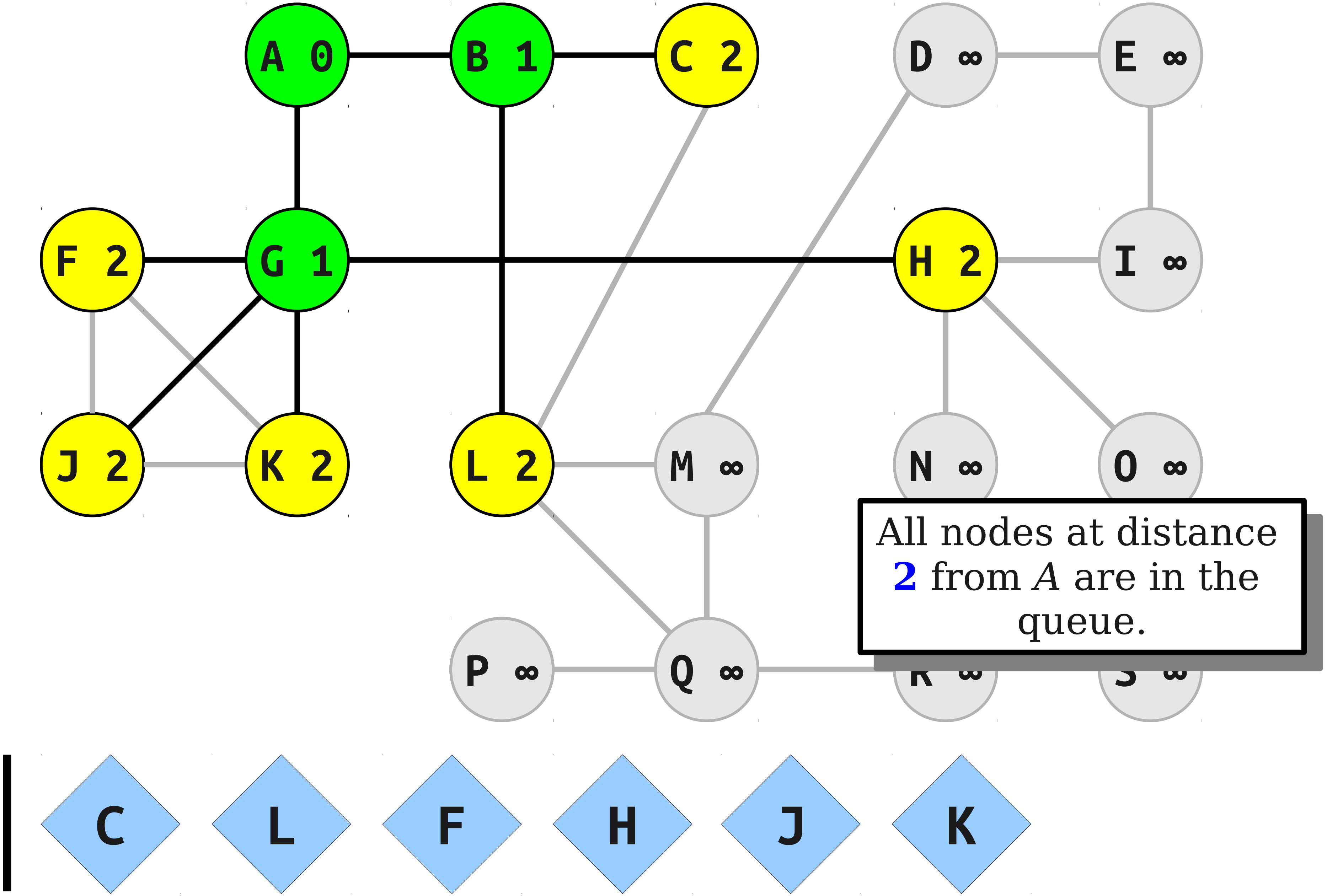


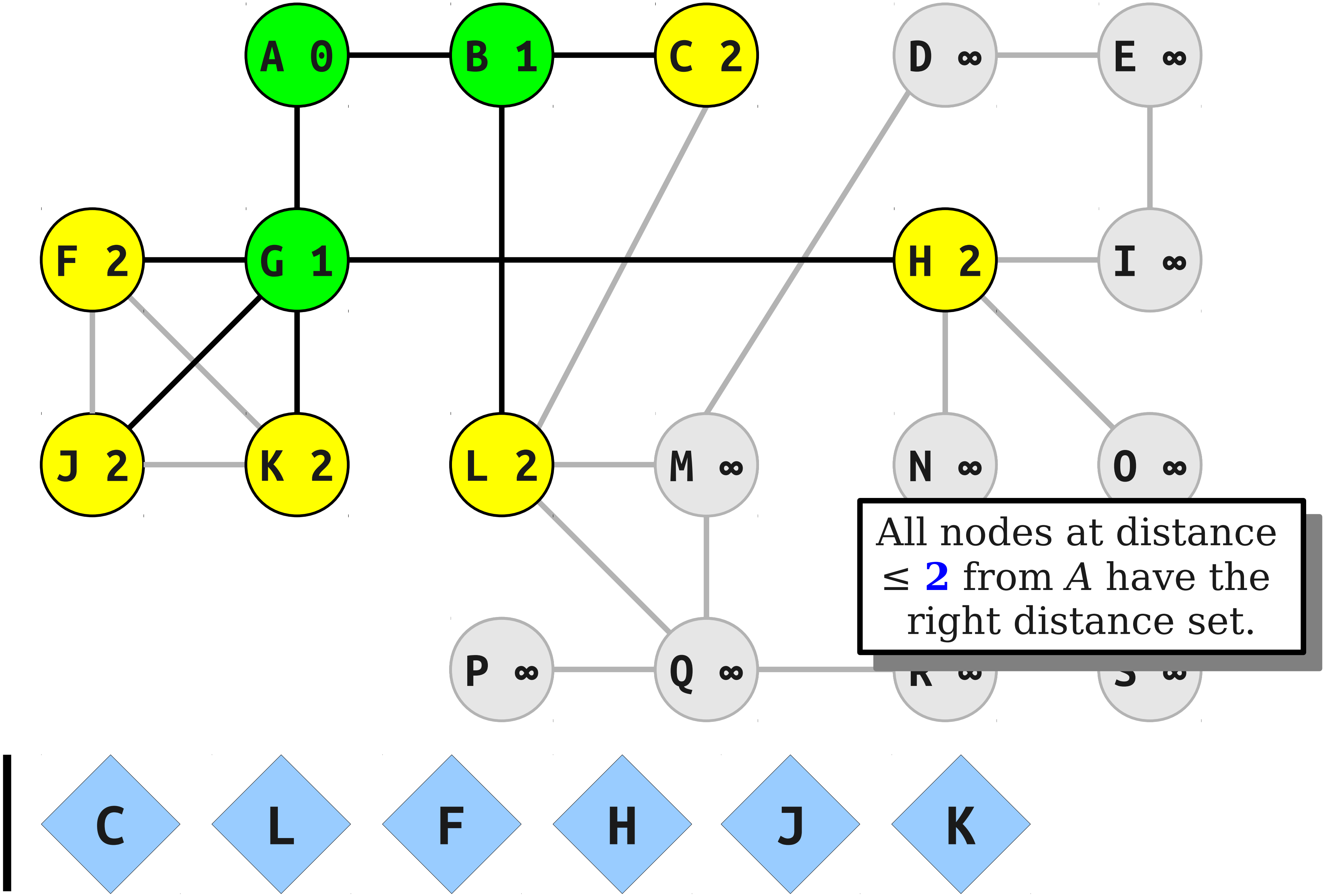
All nodes in the queue are at distance **2** from *A*.

Breadth-First Search



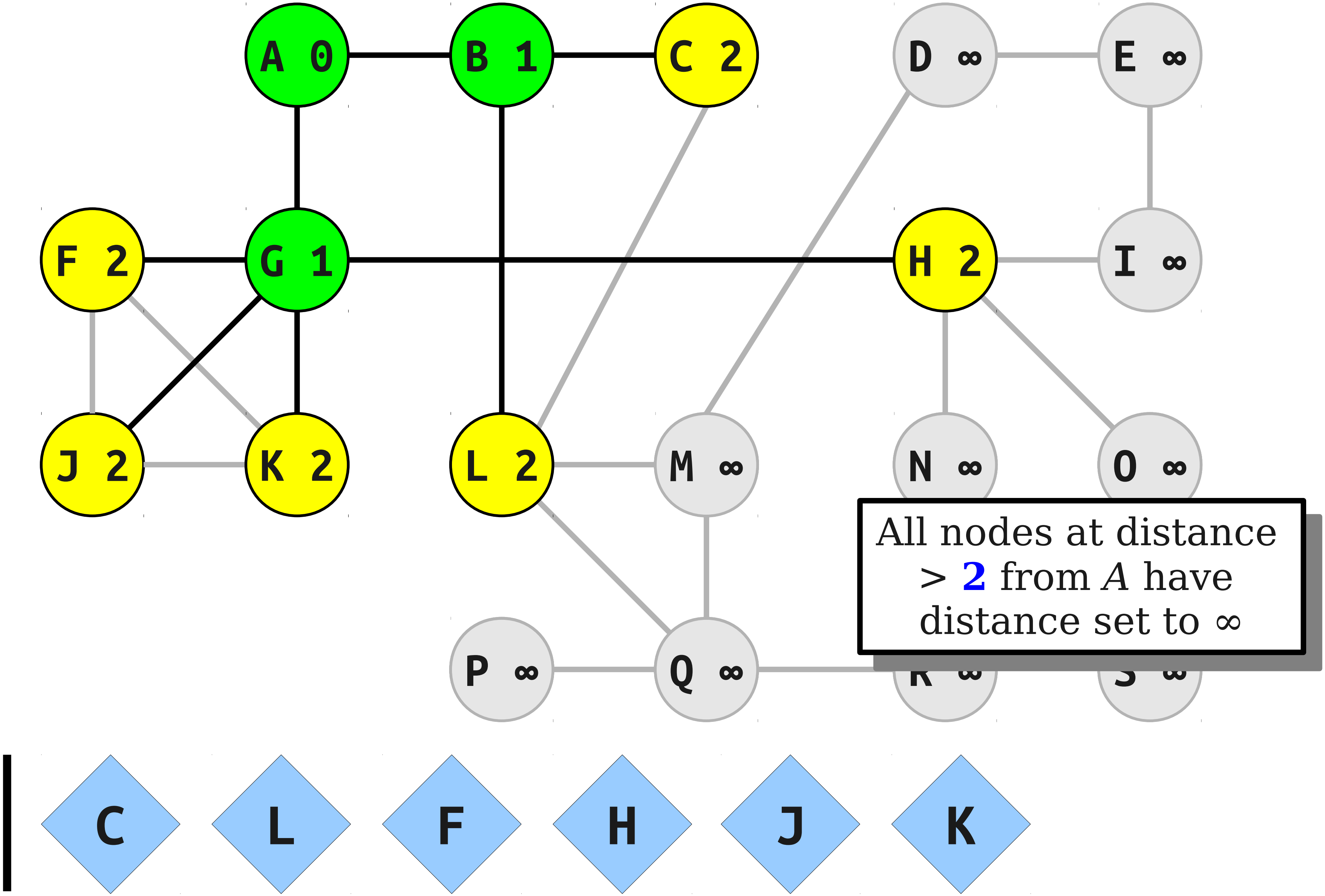
All nodes at distance 2 from *A* are in the queue.

Breadth-First Search



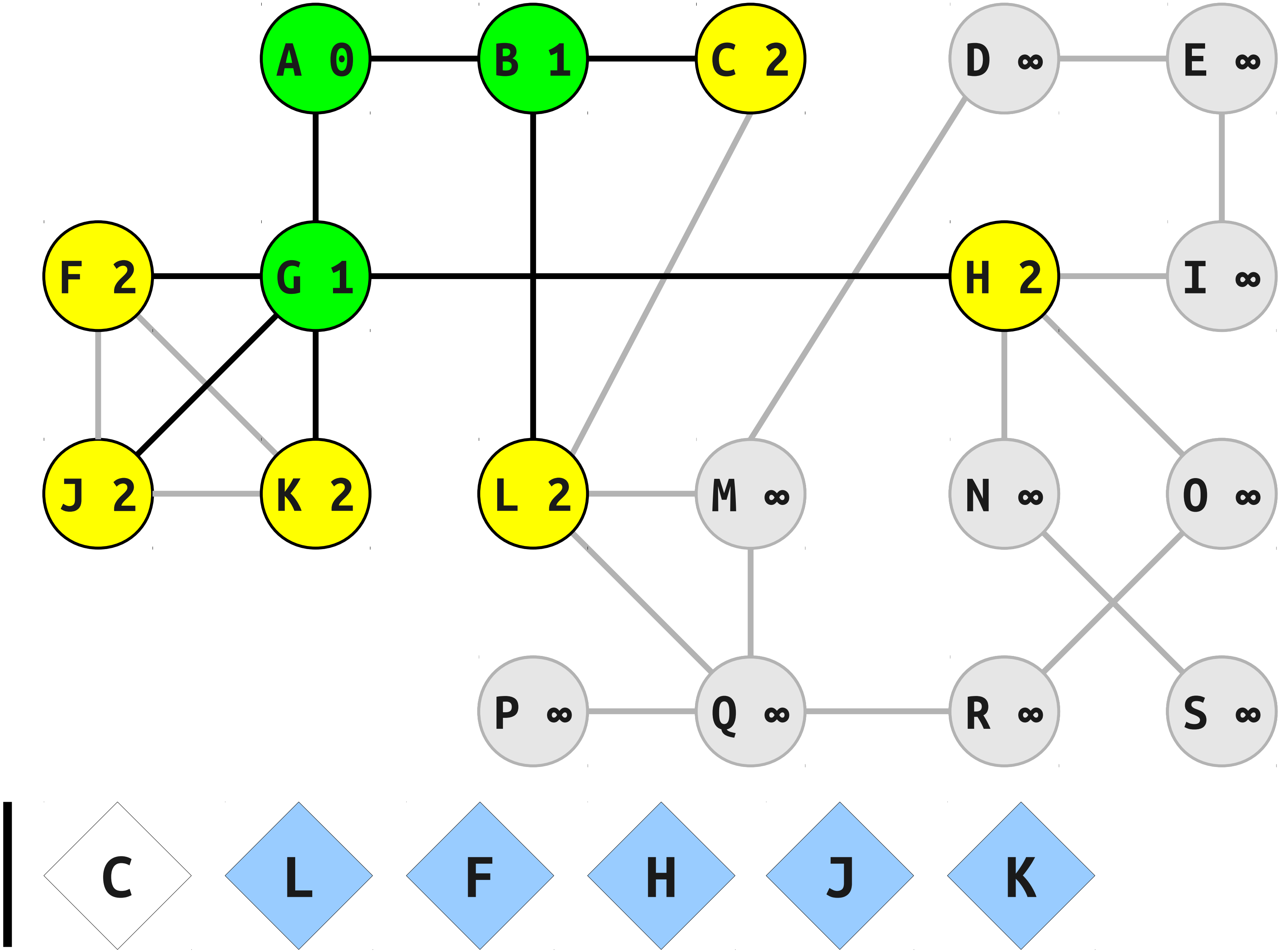
All nodes at distance ≤ **2** from *A* have the right distance set.

Breadth-First Search



All nodes at distance > **2** from *A* have distance set to ∞

Breadth-First Search



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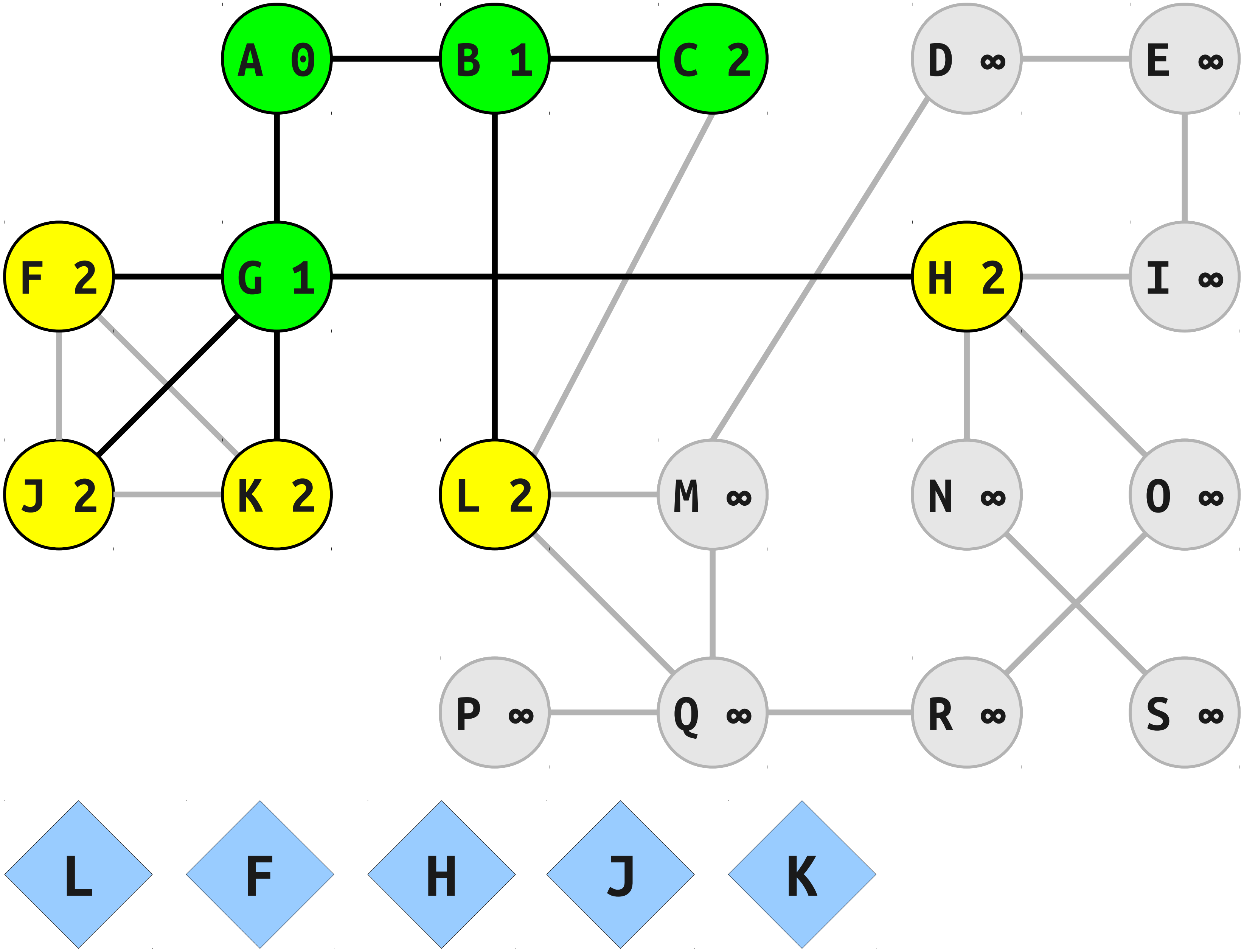

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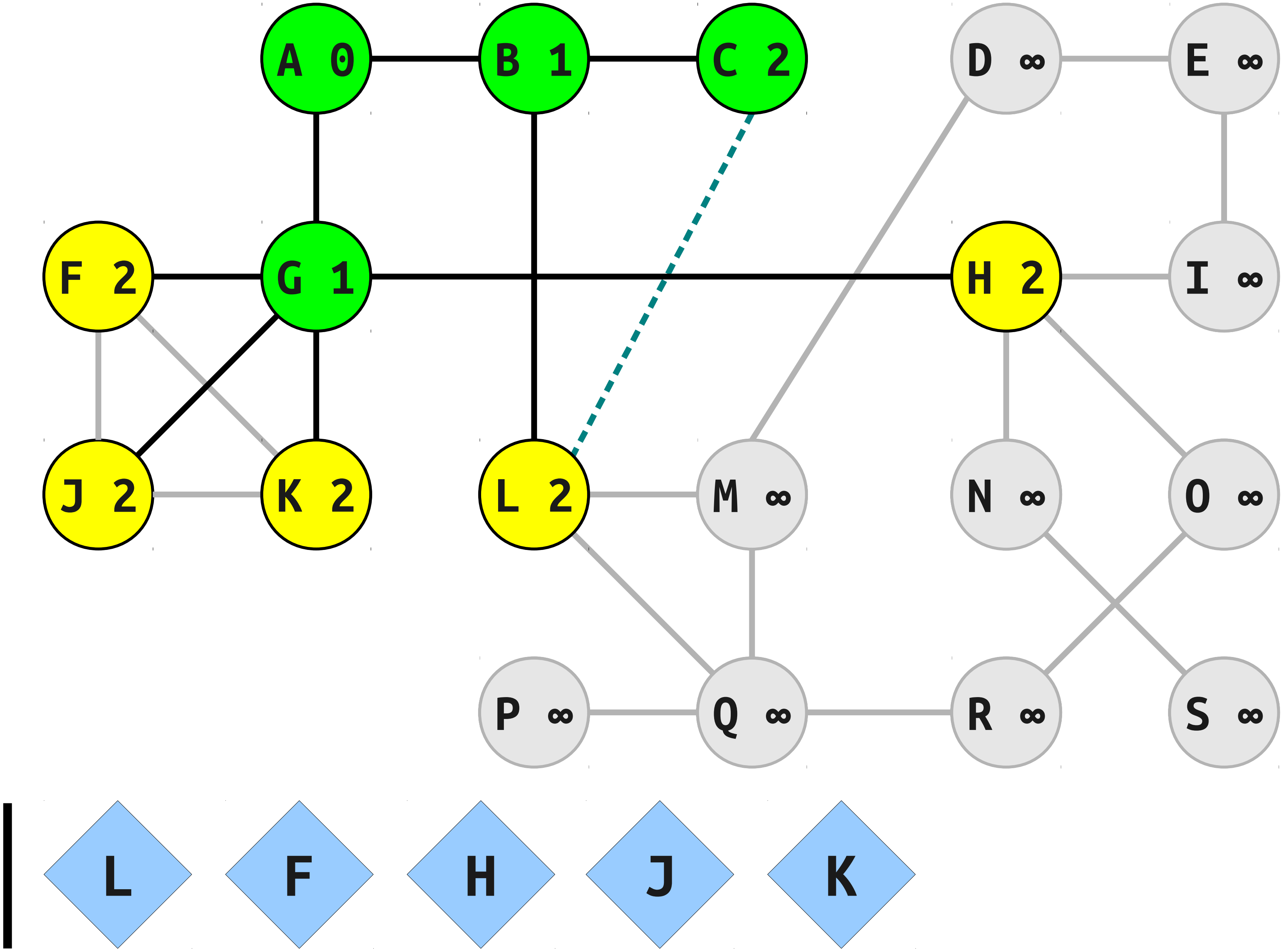

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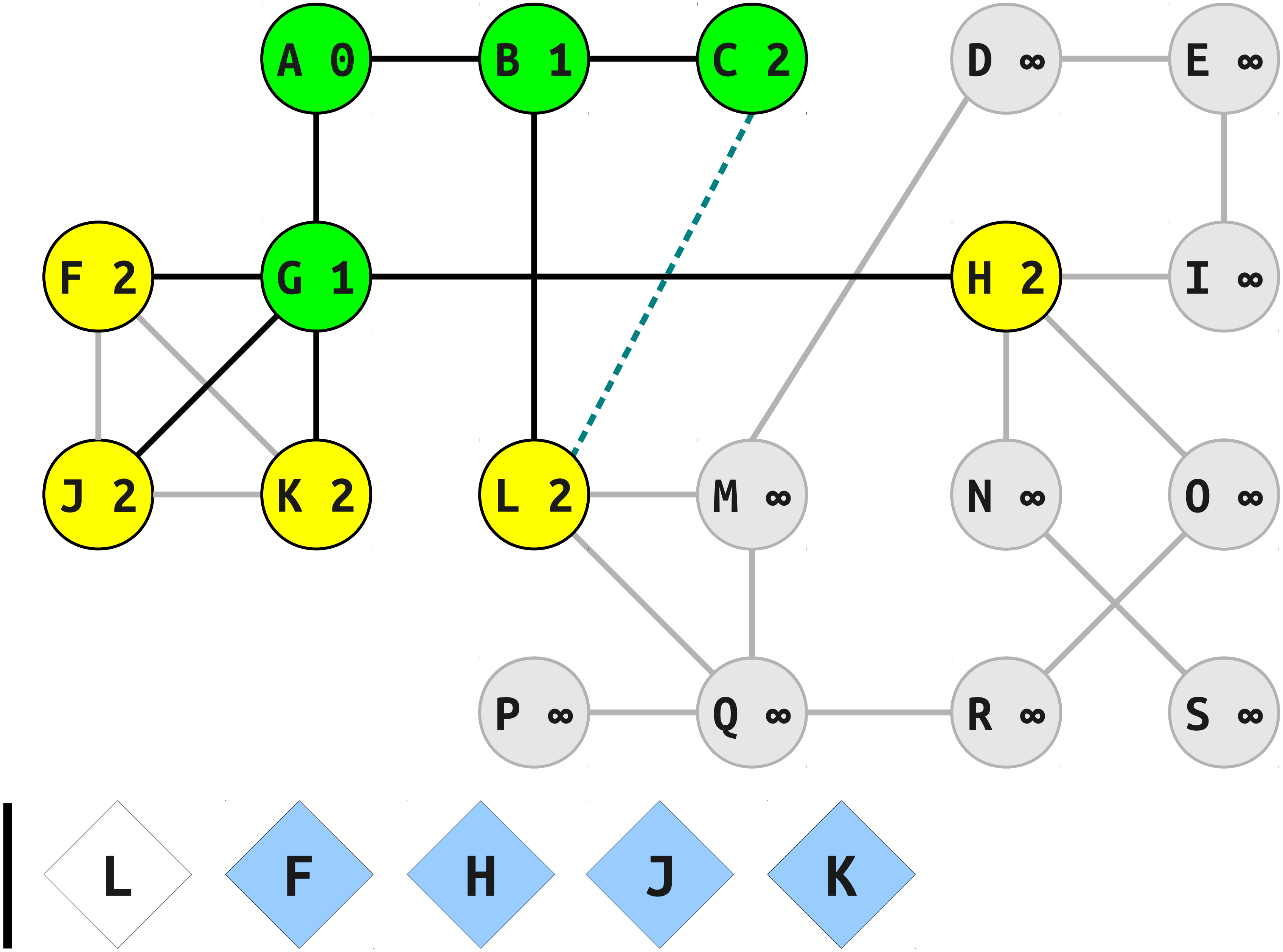

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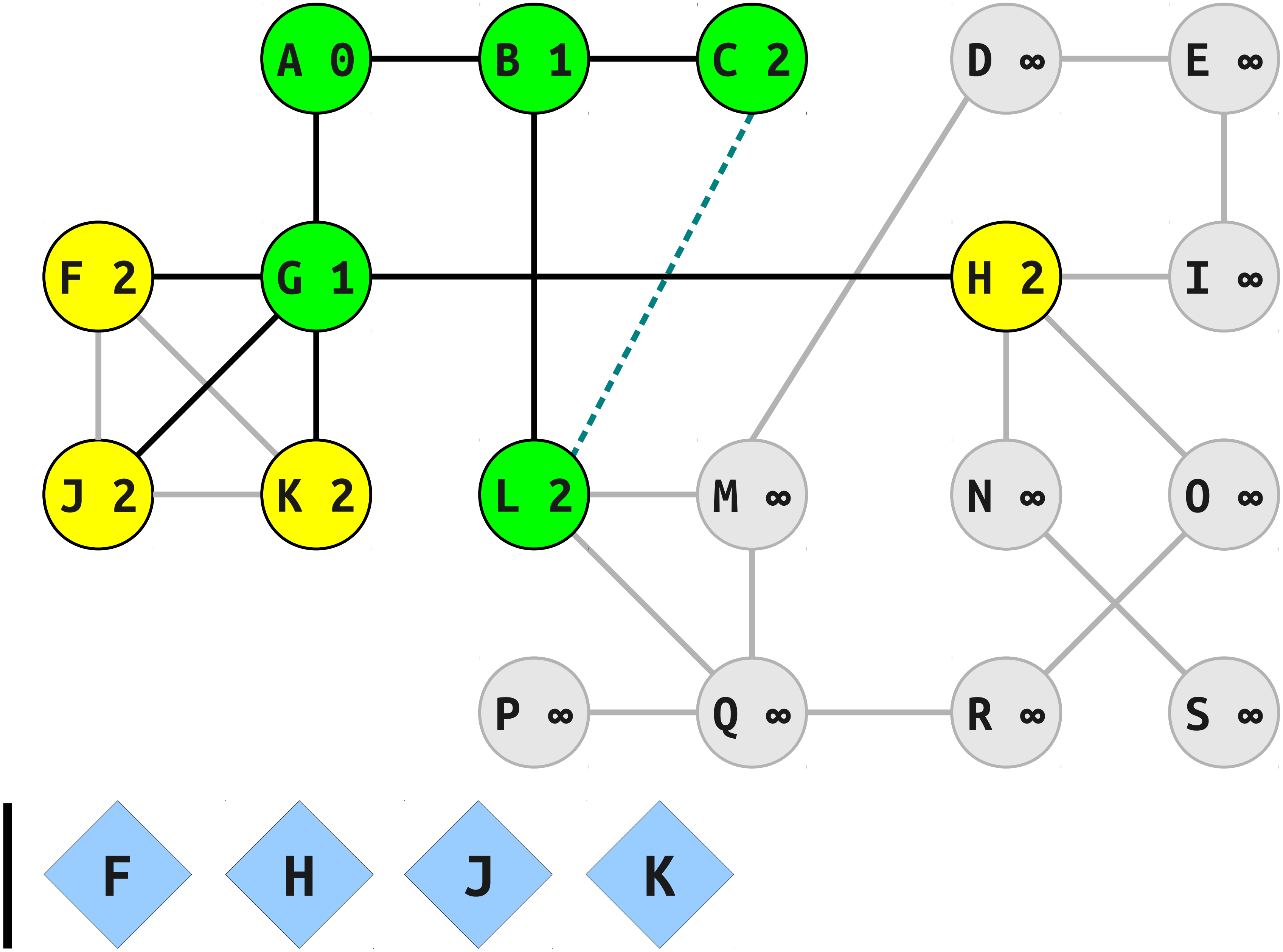

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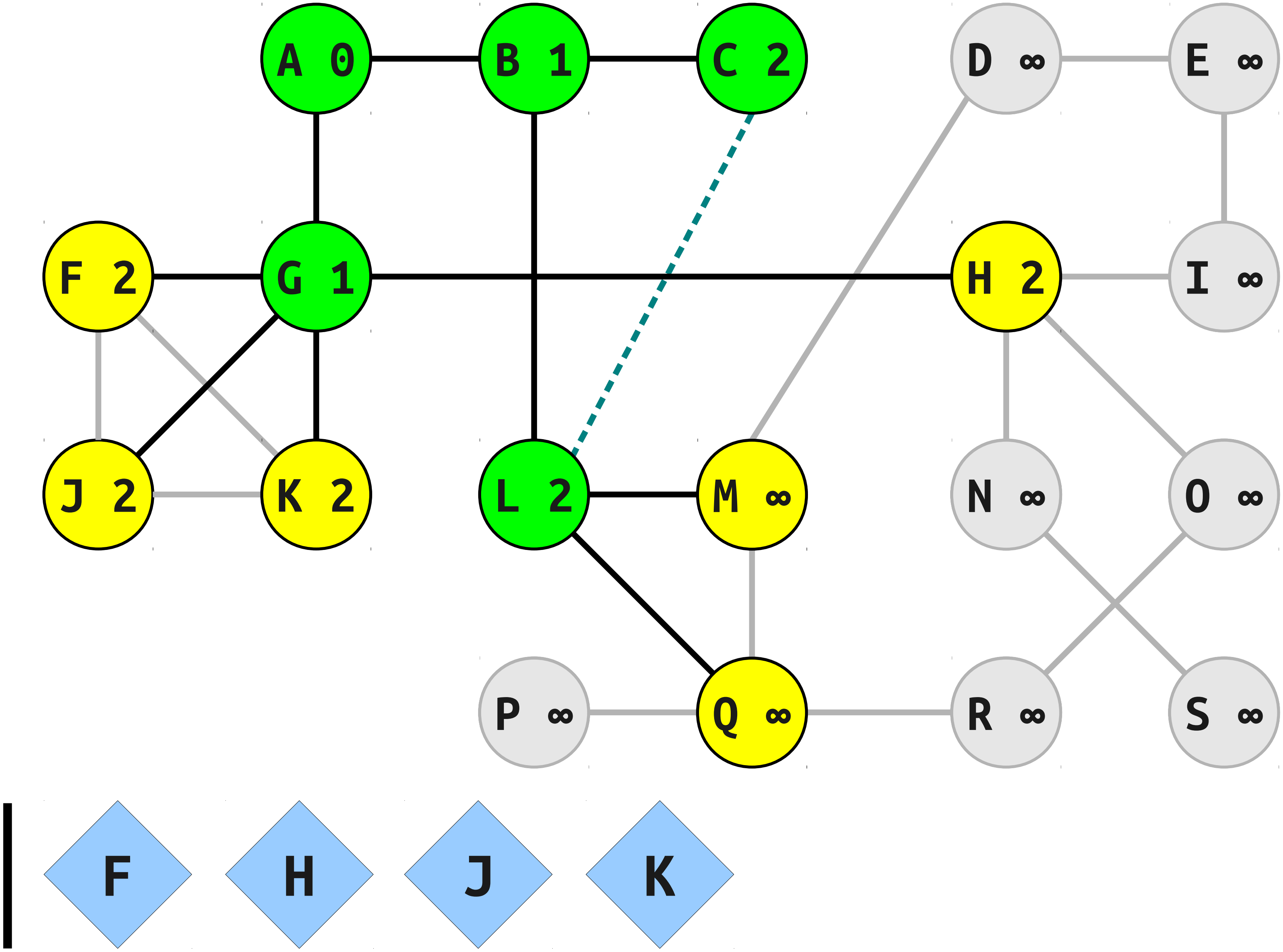

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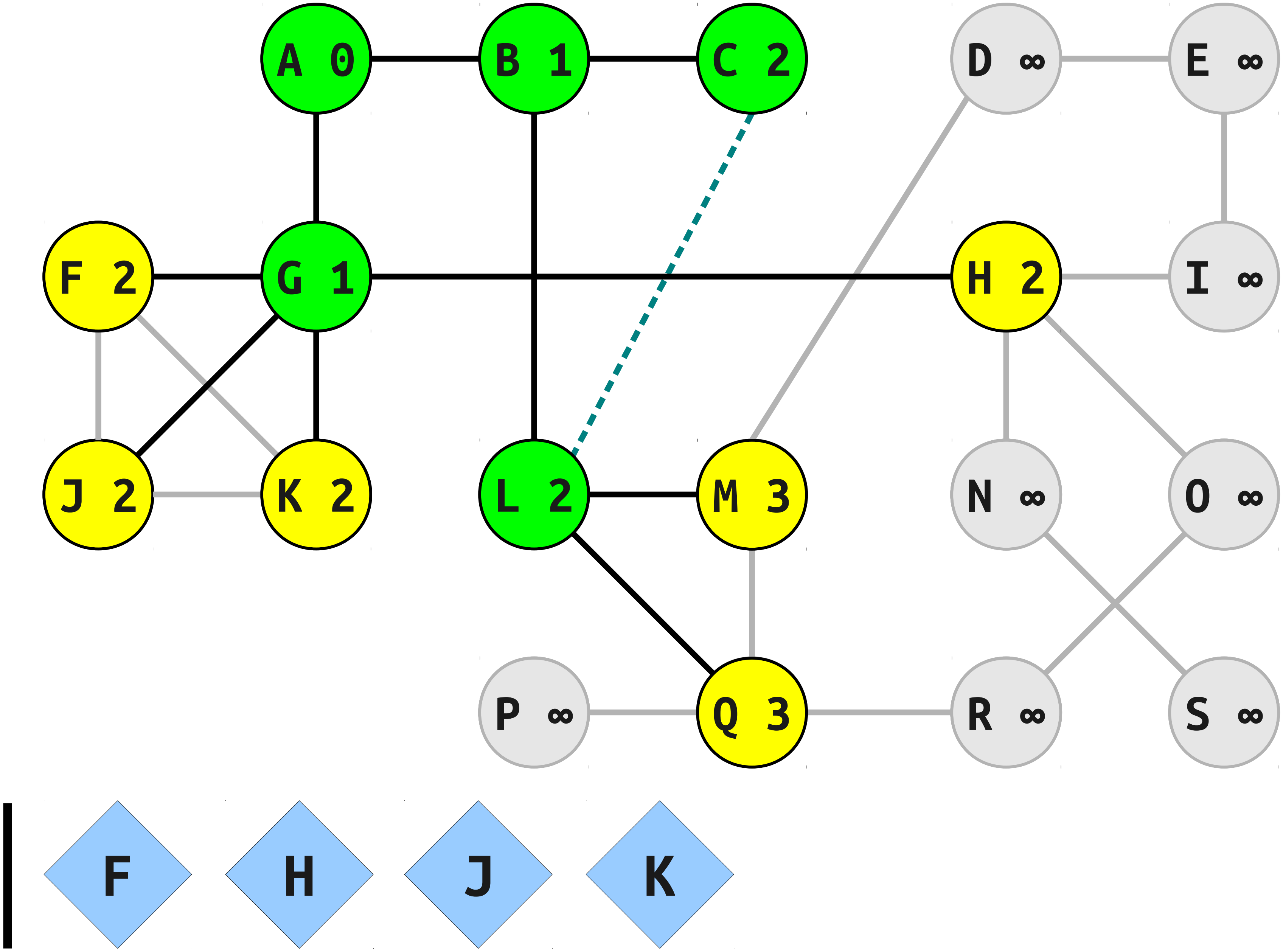

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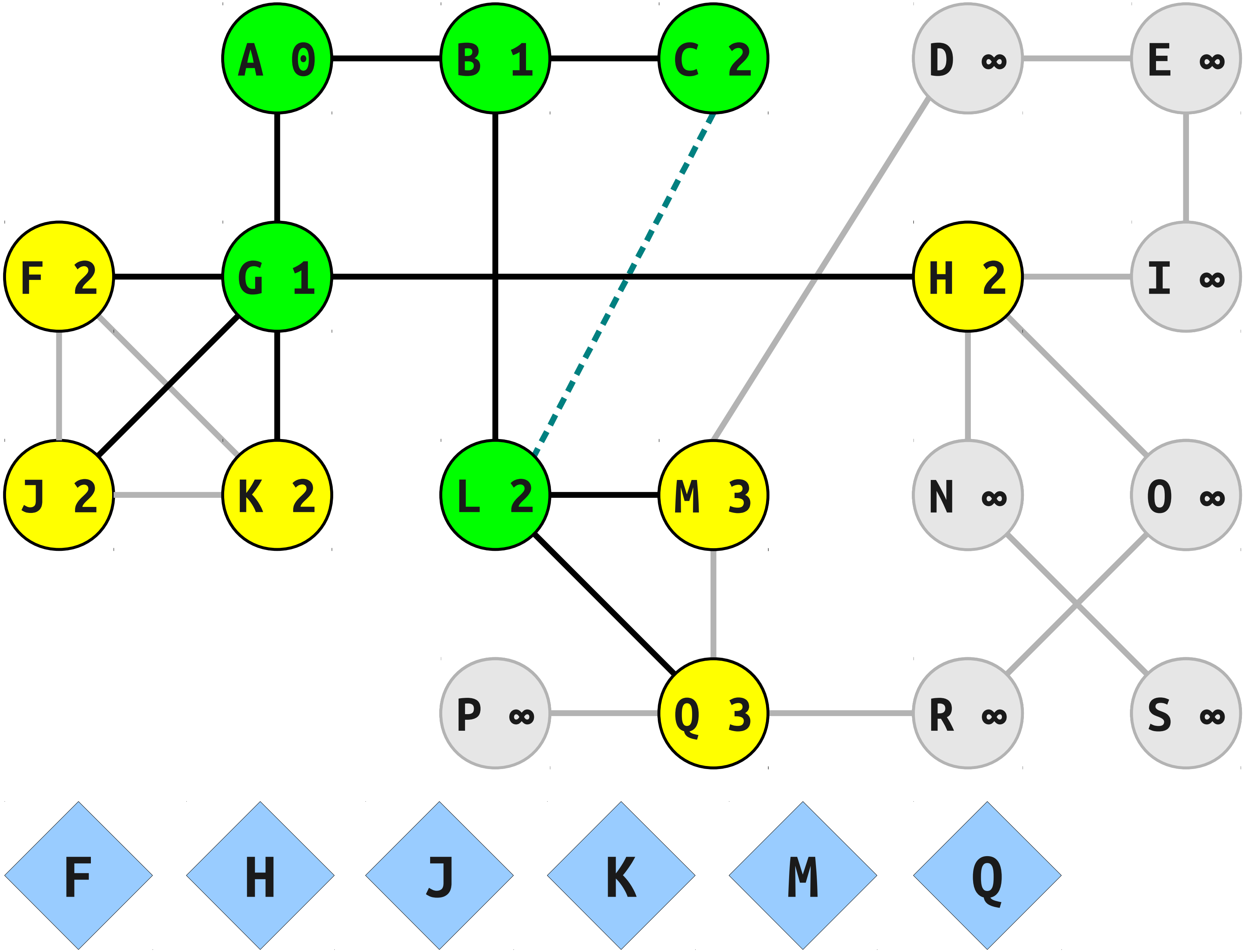

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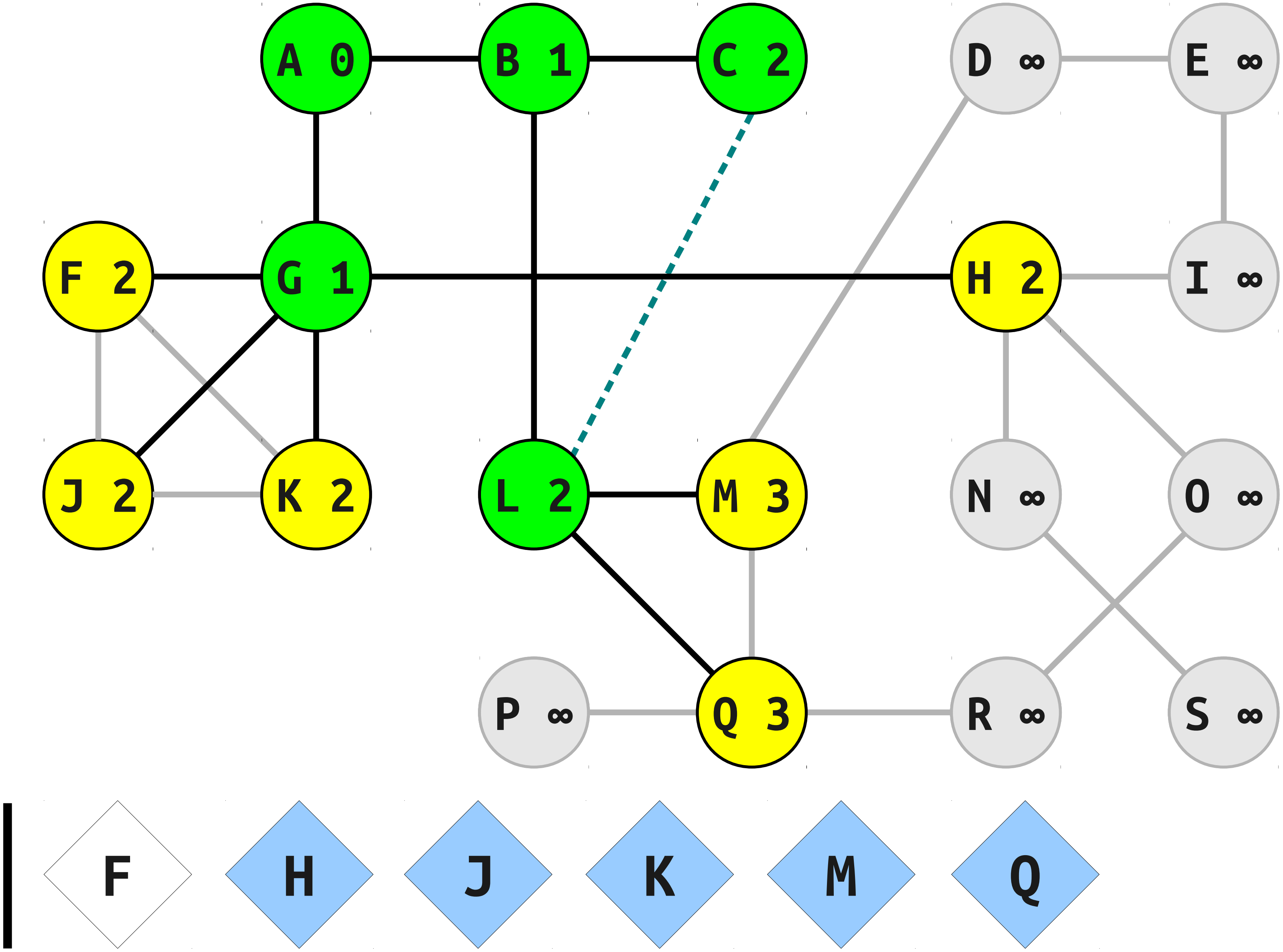

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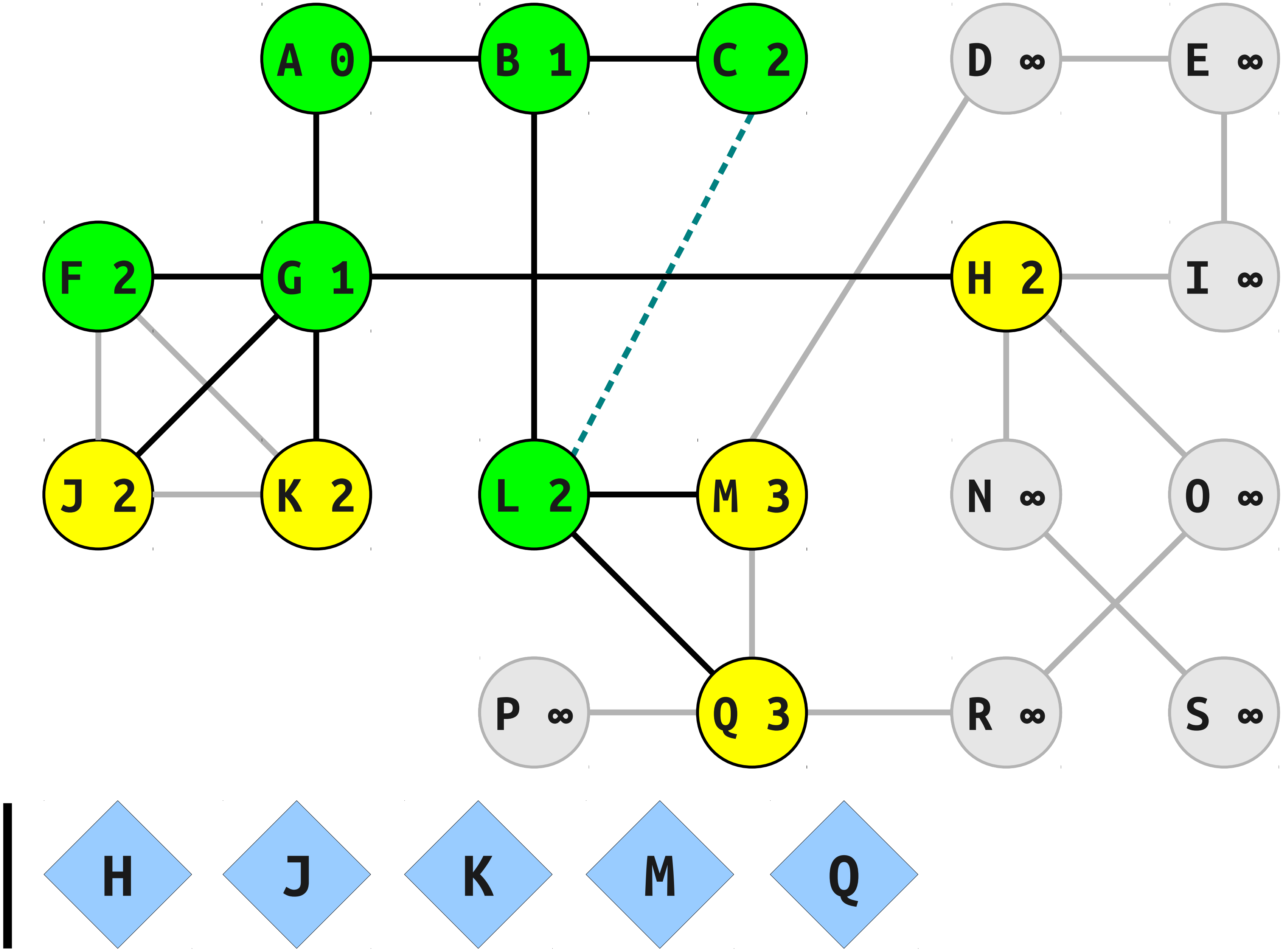

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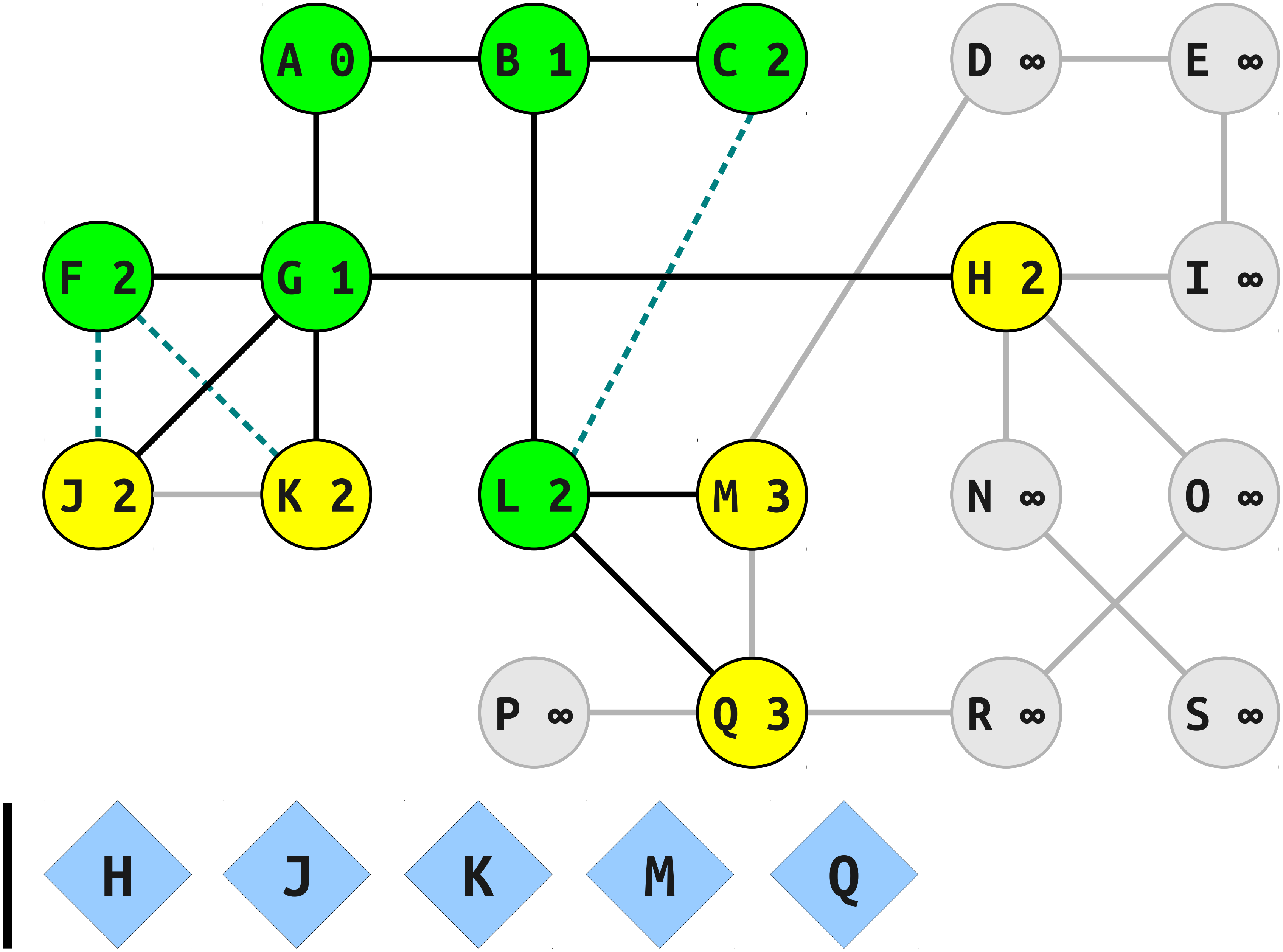

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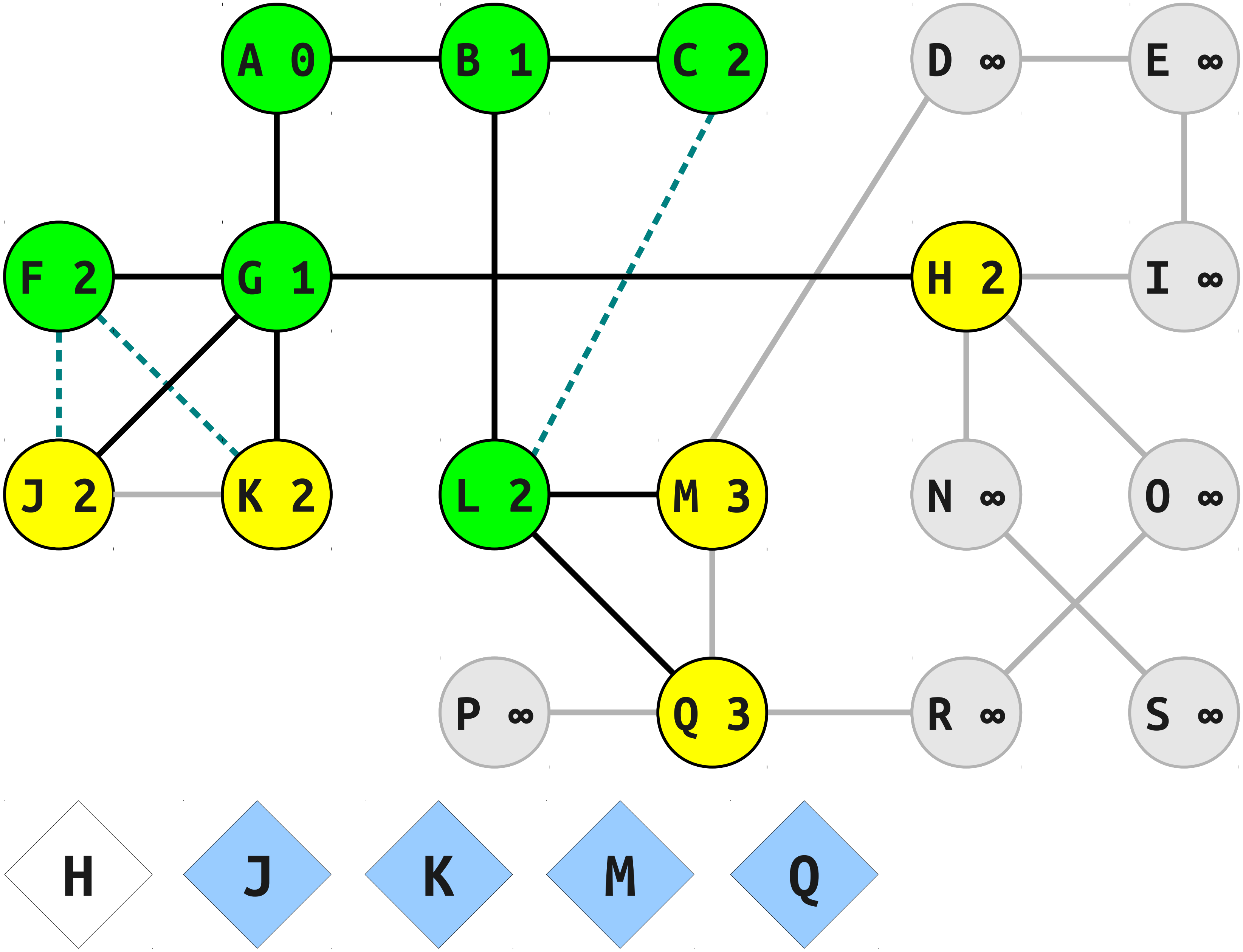

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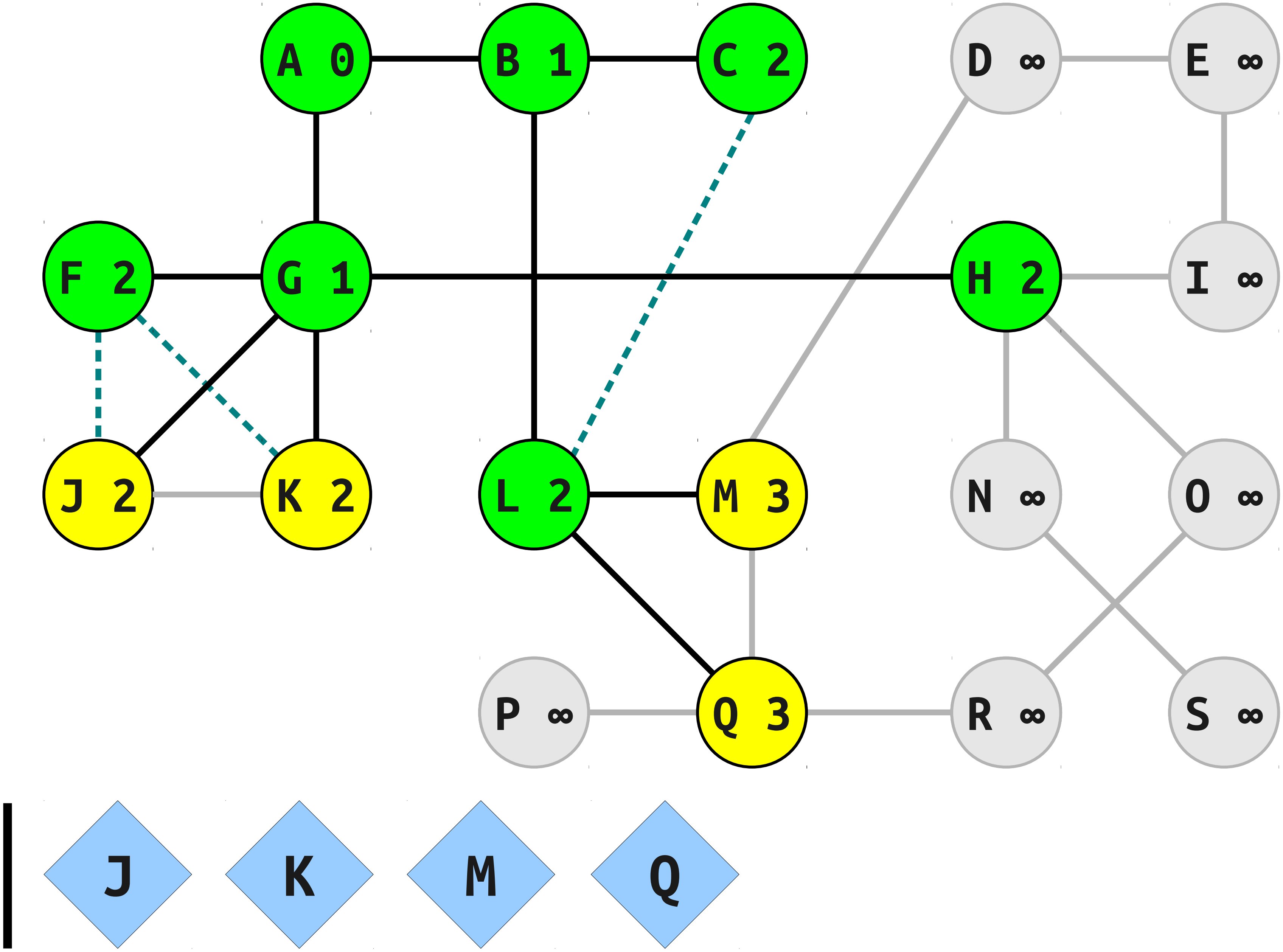

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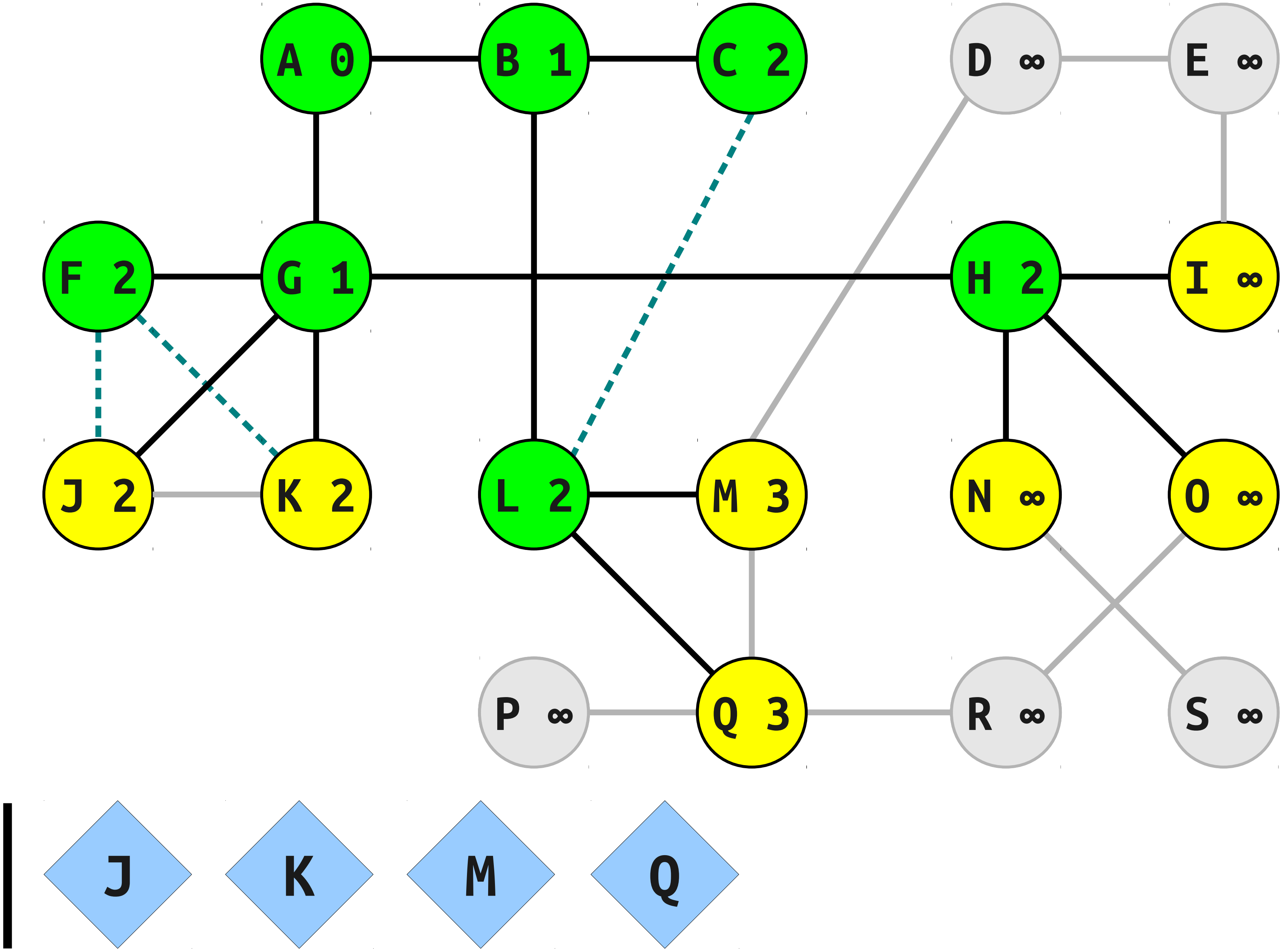

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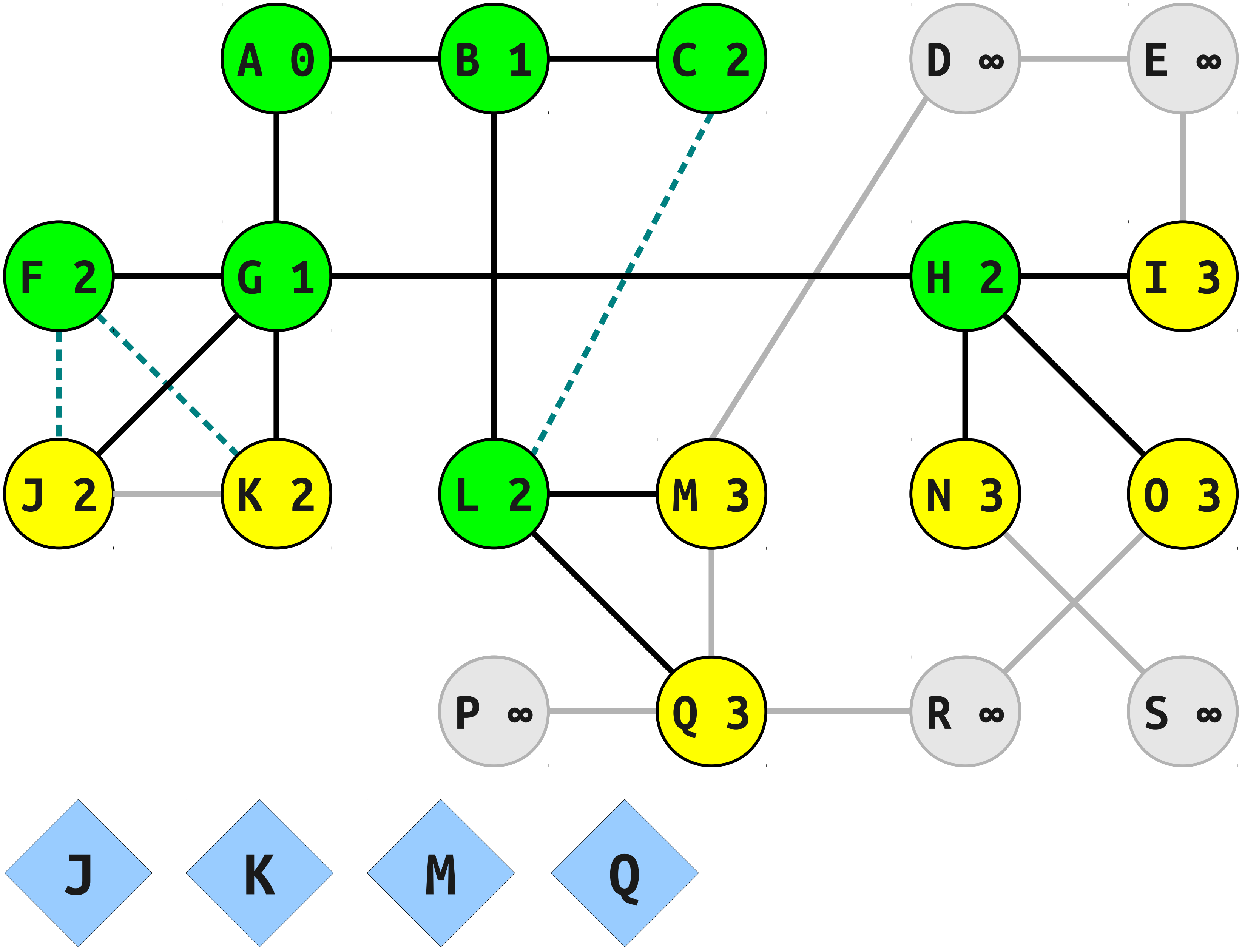

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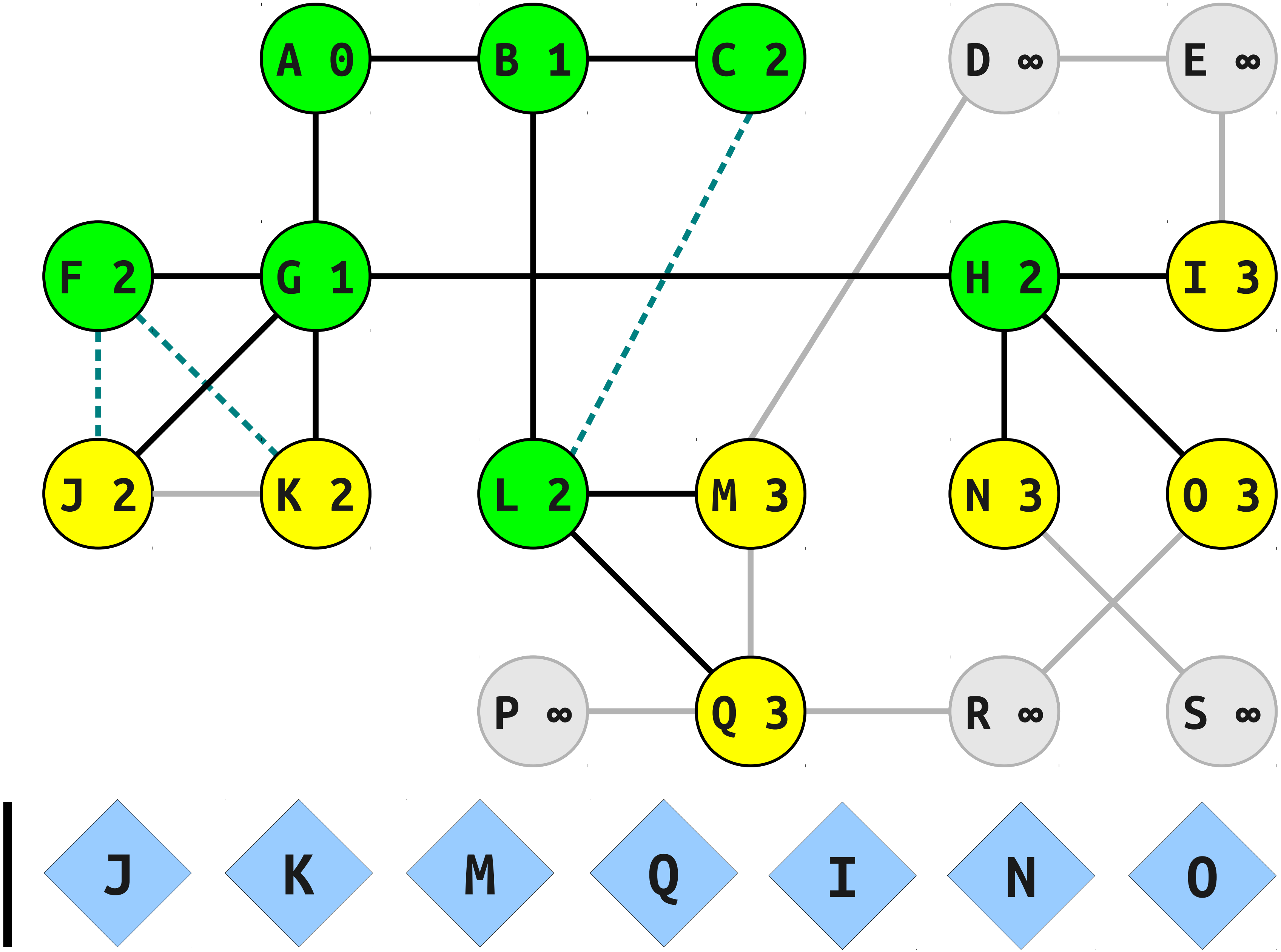
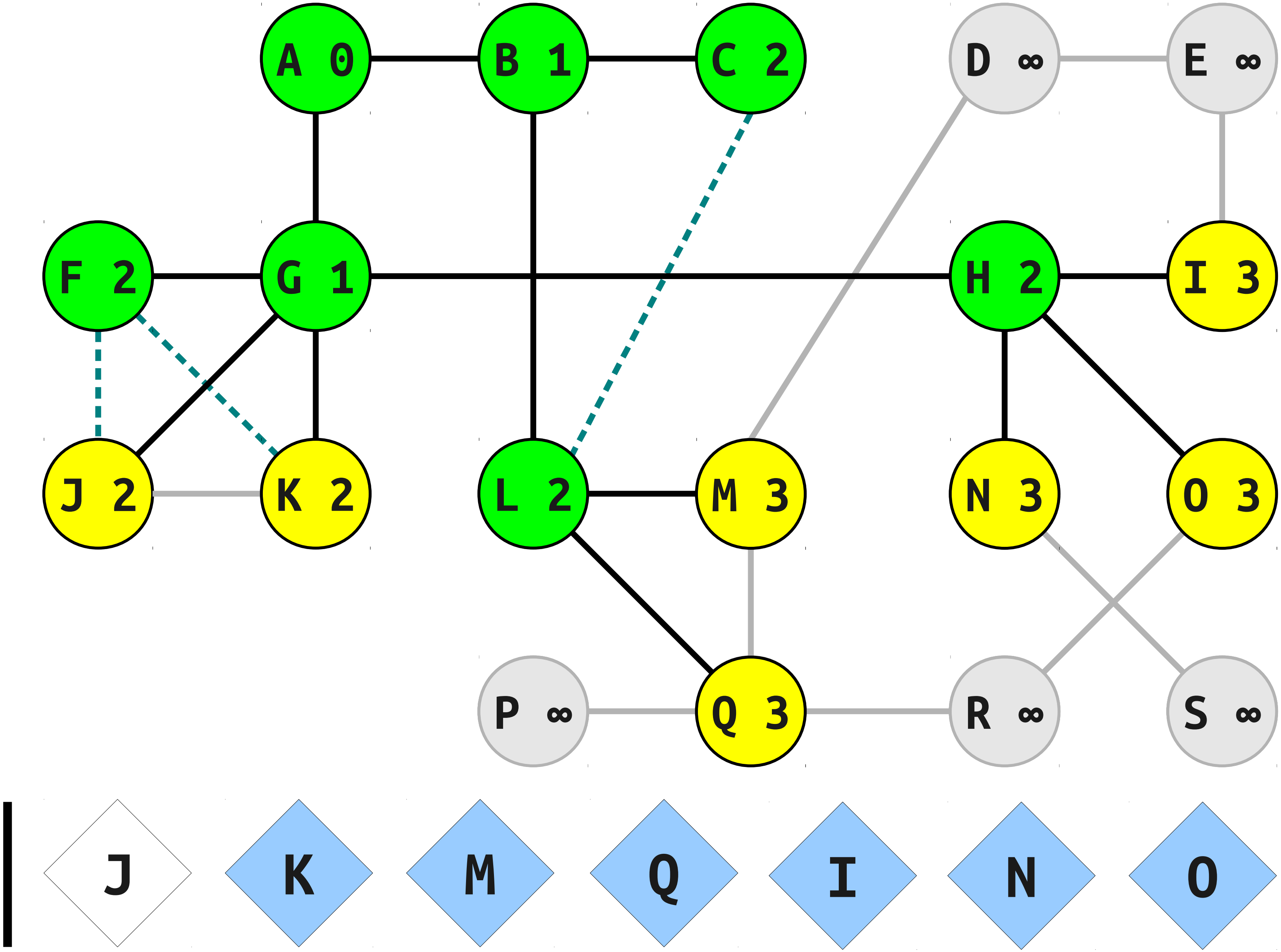


Breadth-First Search



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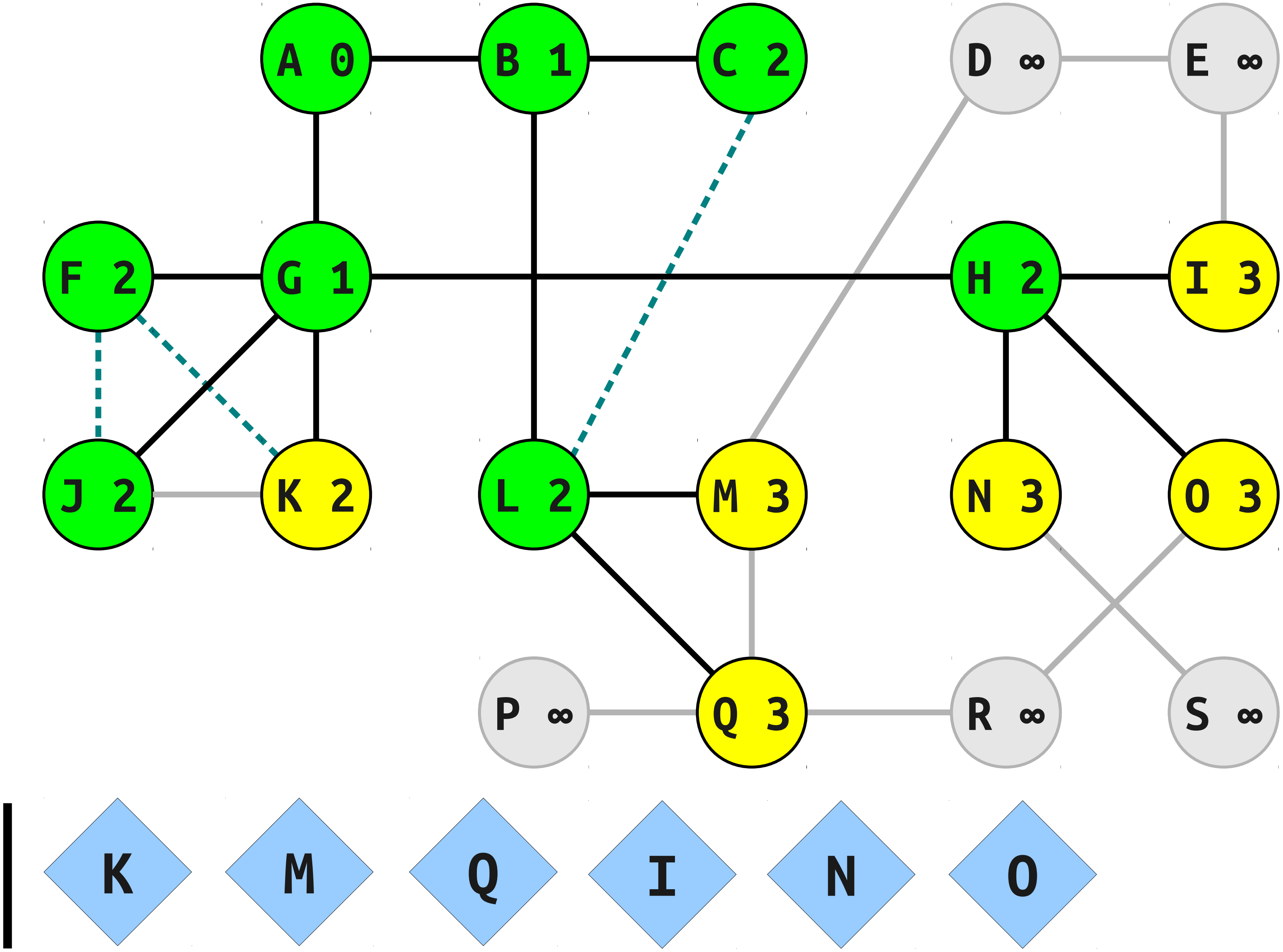
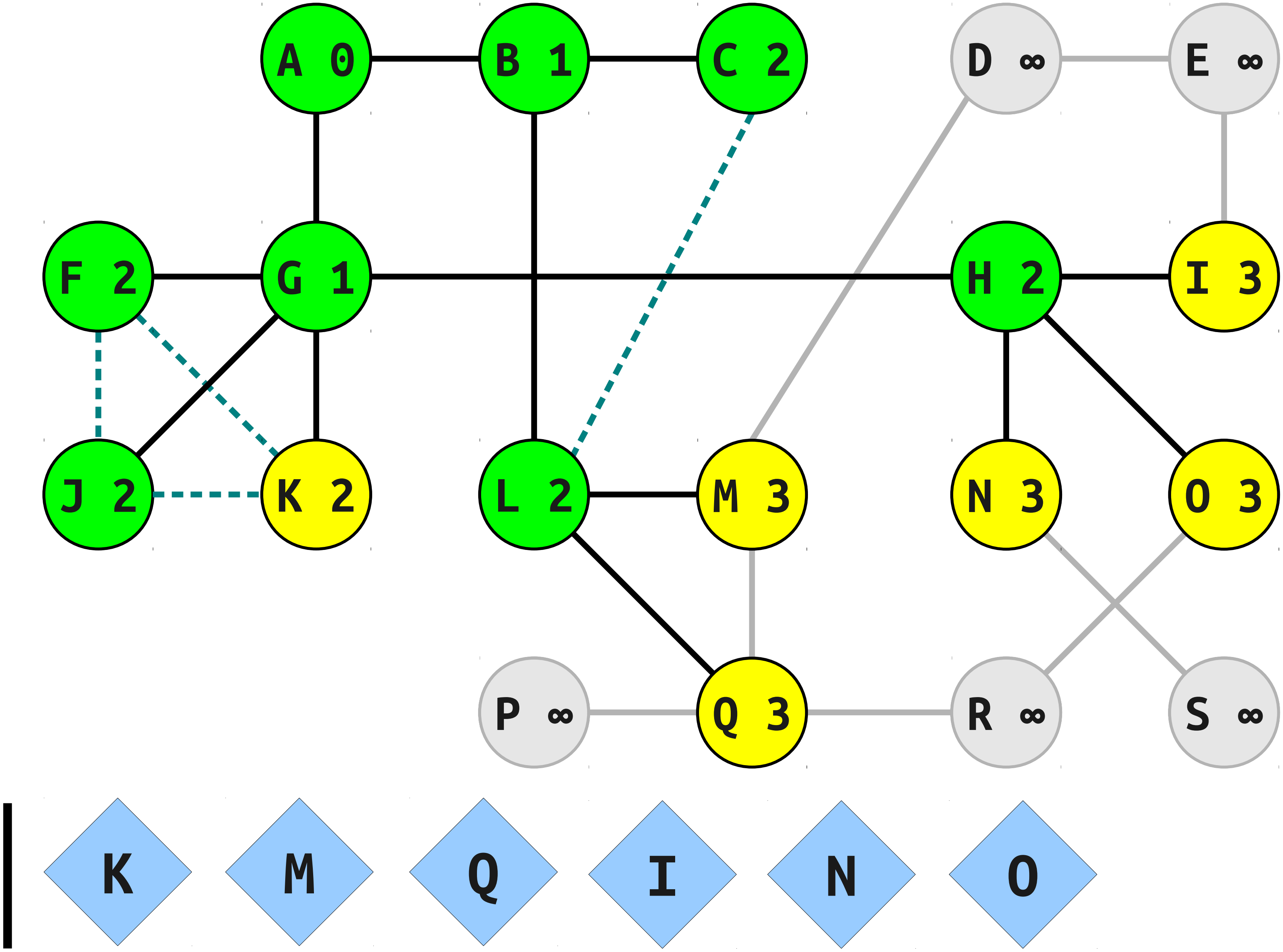


Breadth-First Search



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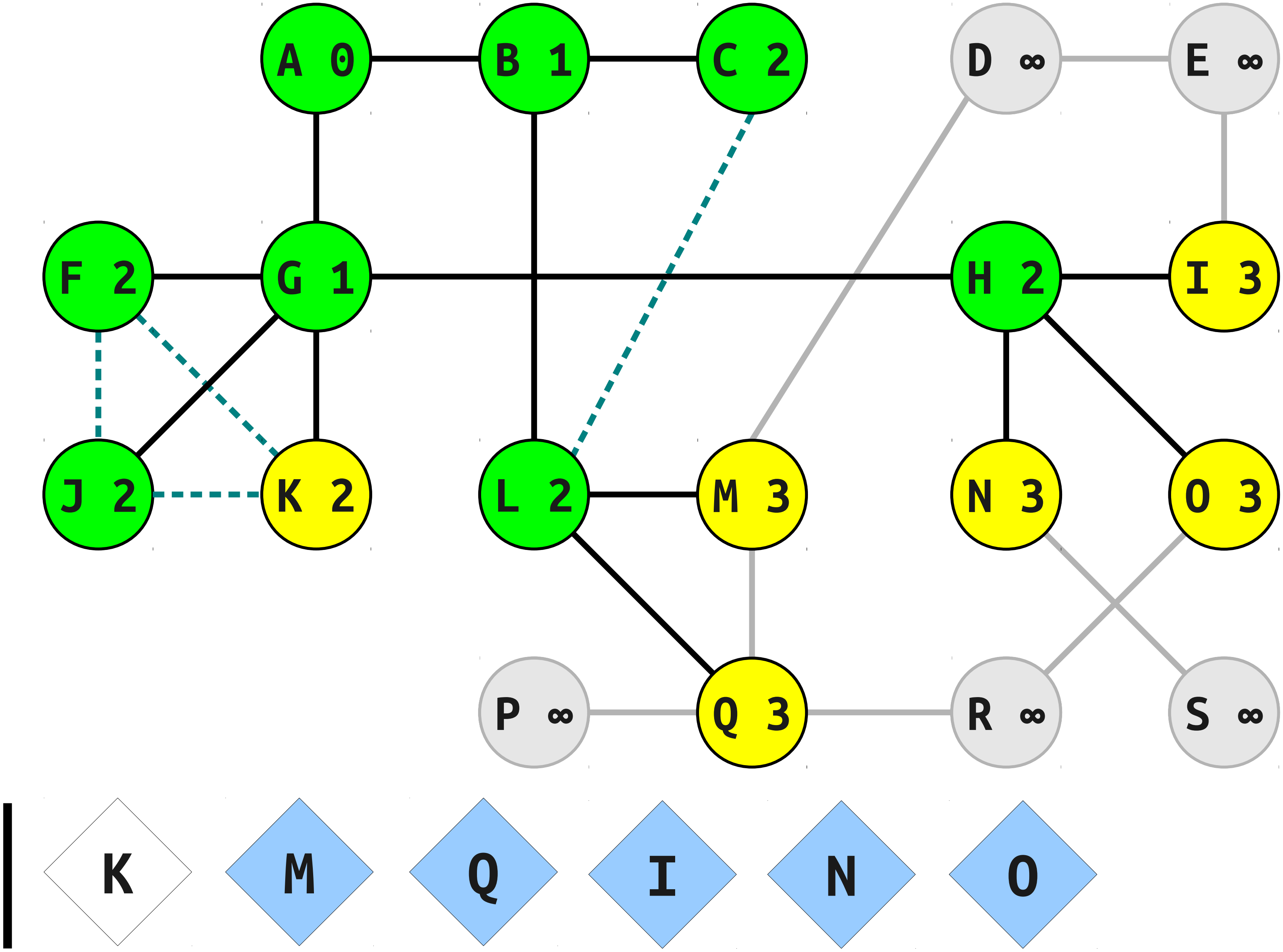

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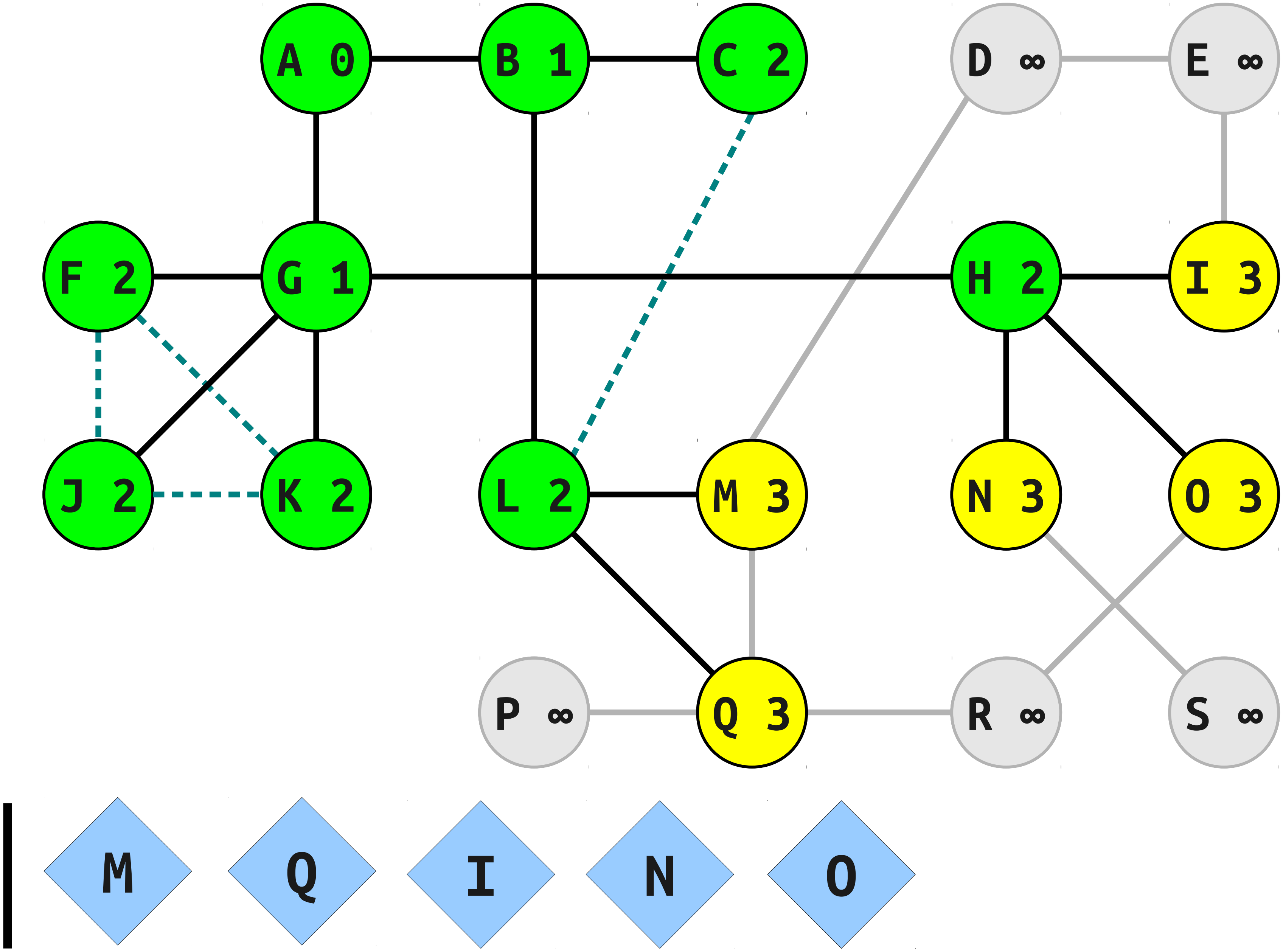
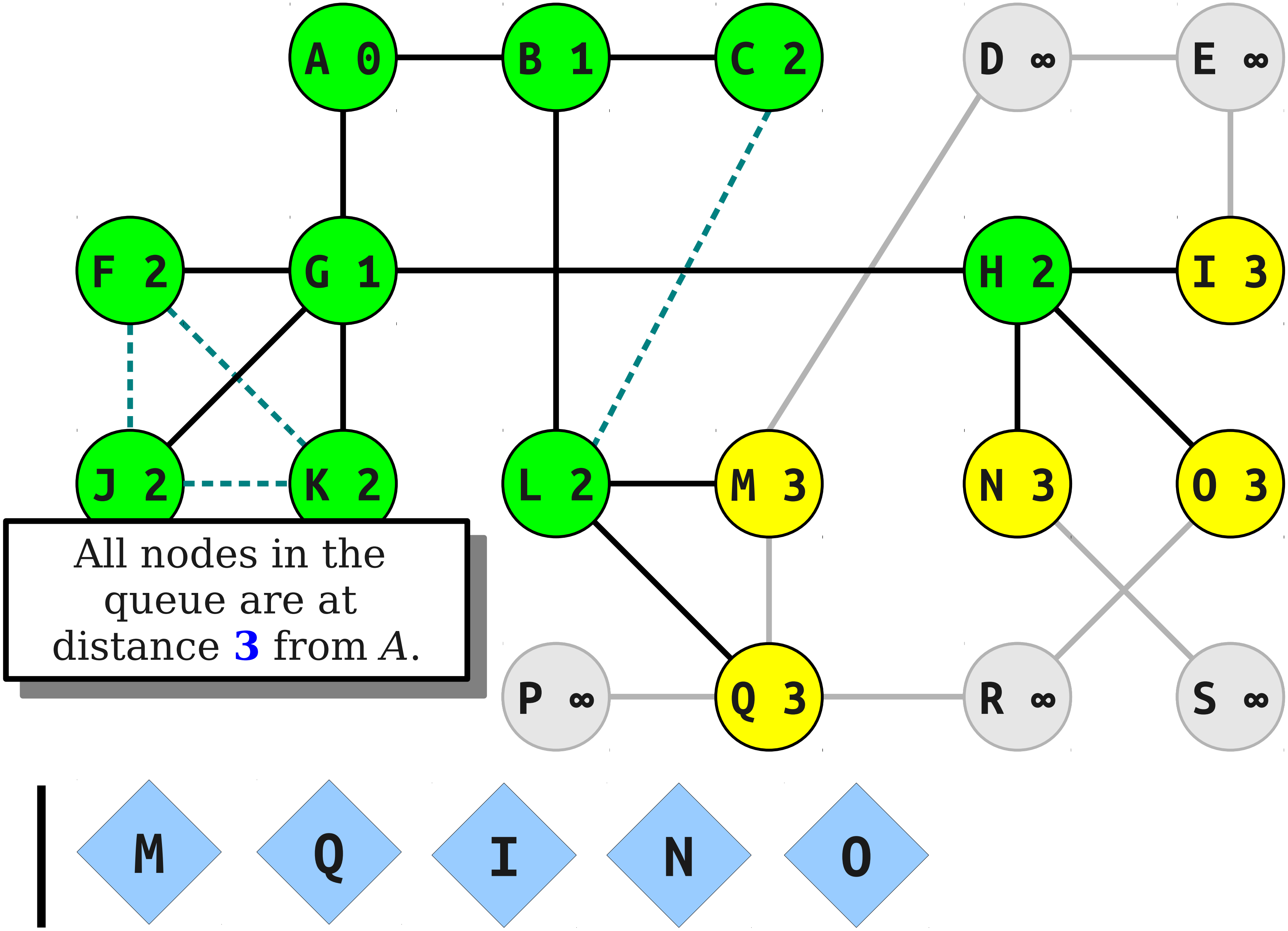
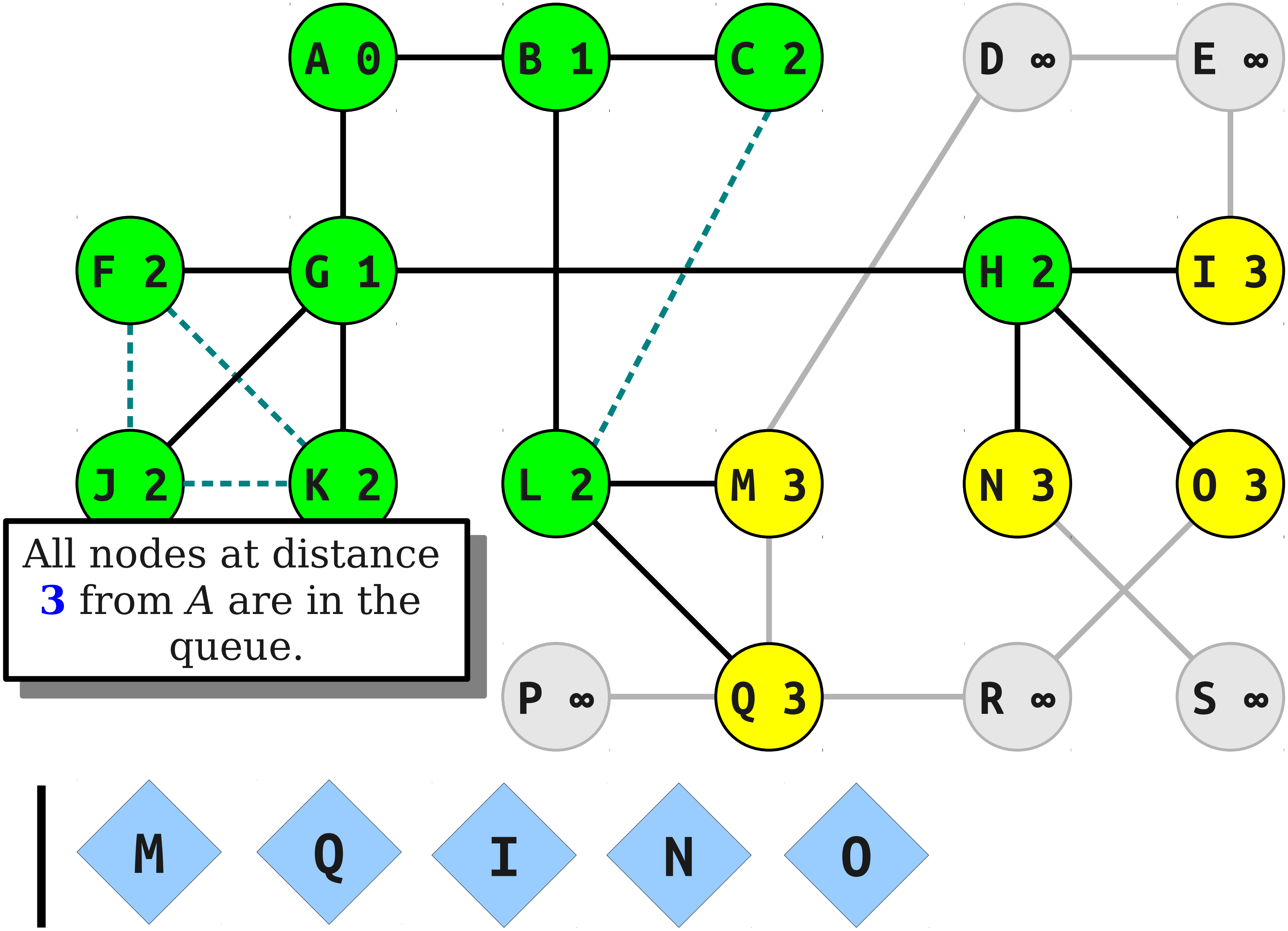


Breadth-First Search



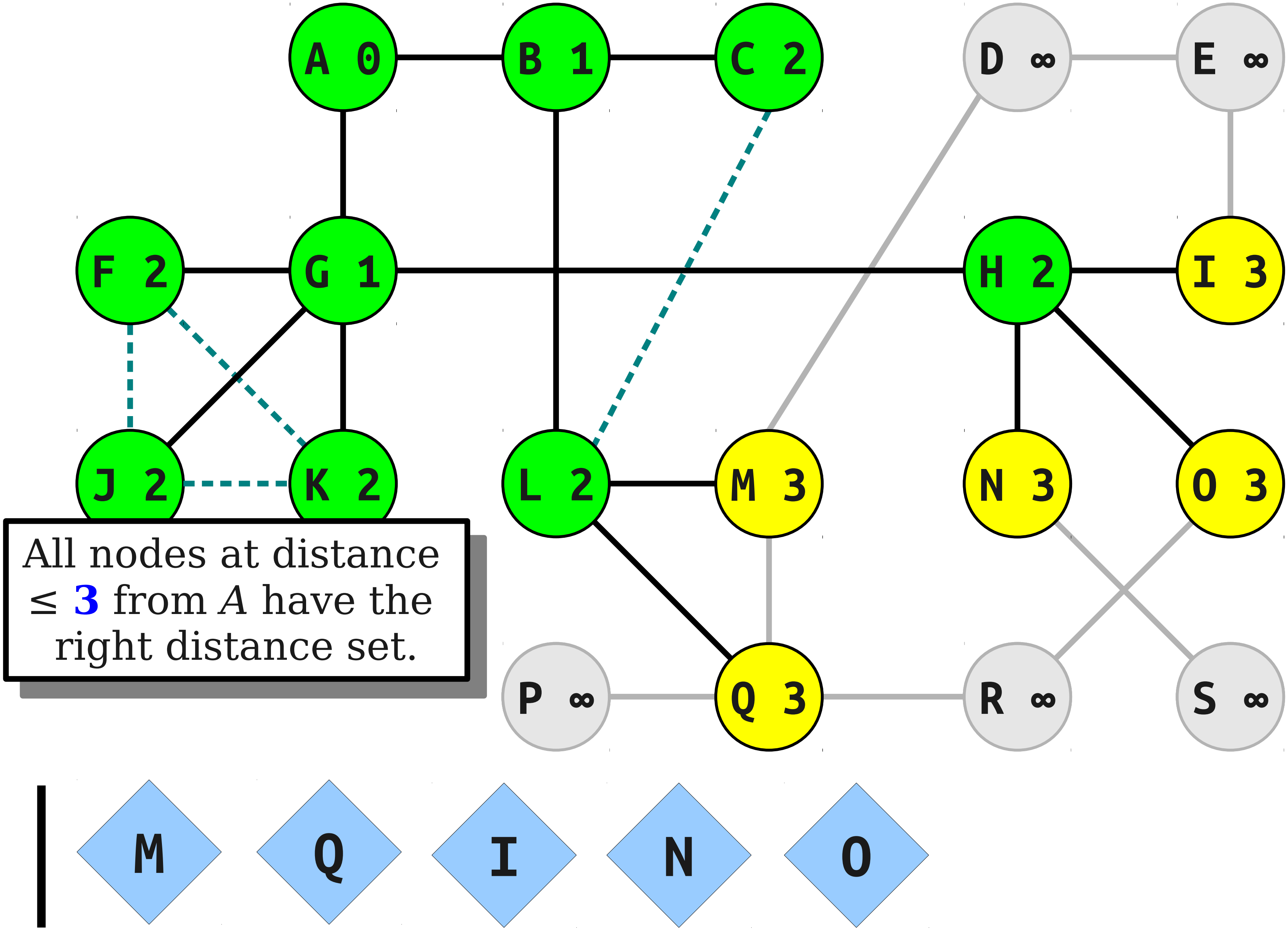
All nodes in the queue are at distance **3** from *A*.

Breadth-First Search



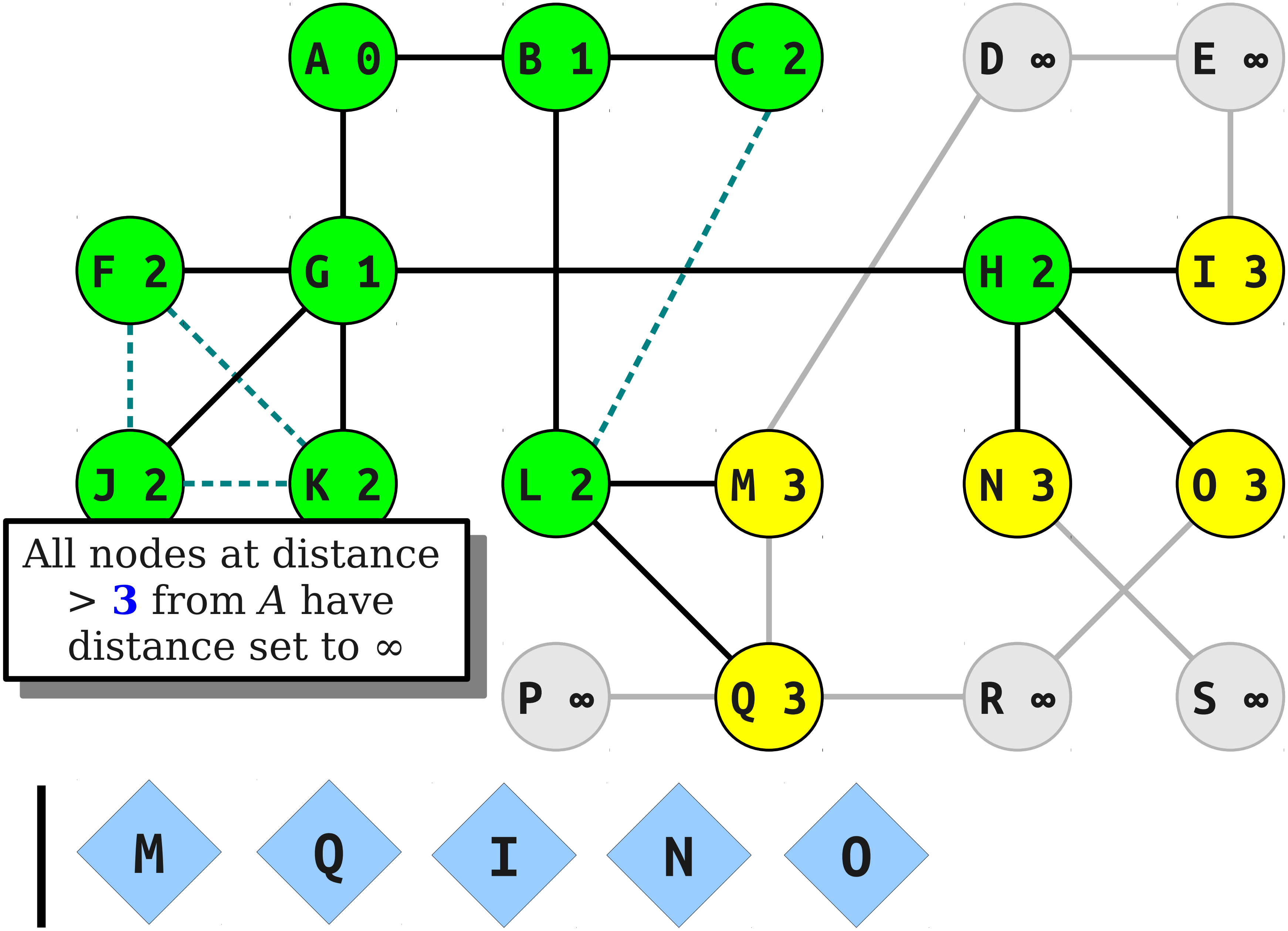
All nodes at distance 3 from *A* are in the queue.

Breadth-First Search



All nodes at distance ≤ 3 from *A* have the right distance set.

Breadth-First Search



All nodes at distance > **3** from *A* have distance set to ∞

Theorem: Breadth-first search always terminates with dist[v] = d(s, v) for all $v \in V$.

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Proof: Define "round n" of BFS to be an instance where at the start of the loop, all nodes v in the queue satisfy dist[v] = n.

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Proof: Define "round n" of BFS to be an instance where at the start of the loop, all nodes v in the queue satisfy dist[v] = n. We will prove in an lemma the following are always true after the first n rounds:

(1) For any node v, d(s, v) = n iff v is in the queue.
(2) All nodes v where d(s, v) $\leq n$ have dist[v] = d(s, v).
(3) All nodes v where d(s, v) $> n$ have dist[v] = ∞

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Let k be the maximum finite distance of any node from node s. Note the following:

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(2) All nodes v where d(s, v) ≤ n have dist[v] = d(s, v).
(3) All nodes v where d(s, v) > n have dist[v] = ∞

Let k be the maximum finite distance of any node from node s. Note the following:

- Any node v where d(s, v) is finite satisfies d(s, v) ≤ k, and any node v where d(s, v) > k satisfies d(s, v) = ∞. This follows from the fact that we picked the maximum possible finite k.

Theorem: Breadth-first search always terminates with dist[v] = d(s, v) for all $v \in V$.

Proof: Define "round n" of BFS to be an instance where at the start of the loop, all nodes v in the queue satisfy dist[v] = n. We will prove in an lemma the following are always true after the first n rounds:

(1) For any node v, d(s, v) = n iff v is in the queue.
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Let k be the maximum finite distance of any node from node s. Note the following:

- Any node v where d(s, v) is finite satisfies d(s, v) $\leq k$, and any node v where d(s, v) $> k$ satisfies d(s, v) = ∞. This follows from the fact that we picked the maximum possible finite k.
- There must be nodes at distances 0, 1, 2, …, k from s. A simple inductive argument using property (1) shows that there will be exactly $k + 1$ rounds, corresponding to distances 0, 1, …, k.

Theorem: Breadth-first search always terminates with $\mathrm{dist}[v] = \mathrm{d}(s, v)$ for all $v \in V$.

Proof: Define "round n" of BFS to be an instance where at the start of the loop, all nodes v in the queue satisfy $\mathrm{dist}[v] = n$. We will prove in an lemma the following are always true after the first n rounds:

(1) For any node v, $\mathrm{d}(s, v) = n$ iff v is in the queue.
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Lemma: After n rounds, the following hold:

(1) For any node v, $d(s, v) = n$ iff v is in the queue.
(2) All nodes v where $d(s, v) \leq n$ have $dist[v] = d(s, v)$.
(3) All nodes v where $d(s, v) > n$ have $dist[v] = \infty$

Proof: By induction n. After 0 rounds, $dist[s] = 0$, $dist[v] = \infty$ for any $v \neq s$, and the queue holds only s. Since s is the only node at distance 0, (1) – (3) hold.

For the inductive step, assume for some n that (1) – (3) hold after n rounds. We will prove (1) – (3) hold after $n + 1$ rounds. We need to show the following:

(a) For any node v, $d(s, v) = n + 1$ iff v is in the queue.
(b) All nodes v where $d(s, v) \leq n + 1$ have $dist[v] = d(s, v)$.
(c) All nodes v where $d(s, v) > n + 1$ have $dist[v] = \infty$

To prove (a), note that at the end of round n, all nodes of distance n will have been dequeued, so we need to show all nodes v where $d(s, v) = n + 1$ are enqueued and nothing else is. Note that if a node u is enqueued in round $n + 1$, then at the start of round $n + 1$ $dist[u] = \infty$ (so by (2) and (3), its distance is at least $n + 1$) and u must have been adjacent to a node v in the queue (by (1), $d(s, v) = n$). Thus there is a path of length $n + 1$ to u (take the path of length n to v, then follow the edge to u), and there is no shorter path, so this is the shortest path to u. Thus, $d(s, u) = n + 1$. Also note that if a node u satisfies $d(s, u) = n + 1$, then by (3) at the start of round $n + 1$ it must have $dist[u] = \infty$. Also, it must be adjacent to some node at distance n, which by (1) must be in the queue at the start of the round. Thus at the end of round $n + 1$, u will be enqueued and $dist[u]$ set to $n + 1$.

By our above argument, we know that (a) must hold. Since we didn't change any dist values for nodes at distance n or less, and we set dist values for all enqueued nodes to $n + 1$, (b) holds. Finally, since we only changed labels for nodes at distance $n + 1$, (c) holds as well. This completes the induction. ■

Question 1: How do we prove this always finds the right distances?

Question 2: How *efficiently* does this find the right distances?

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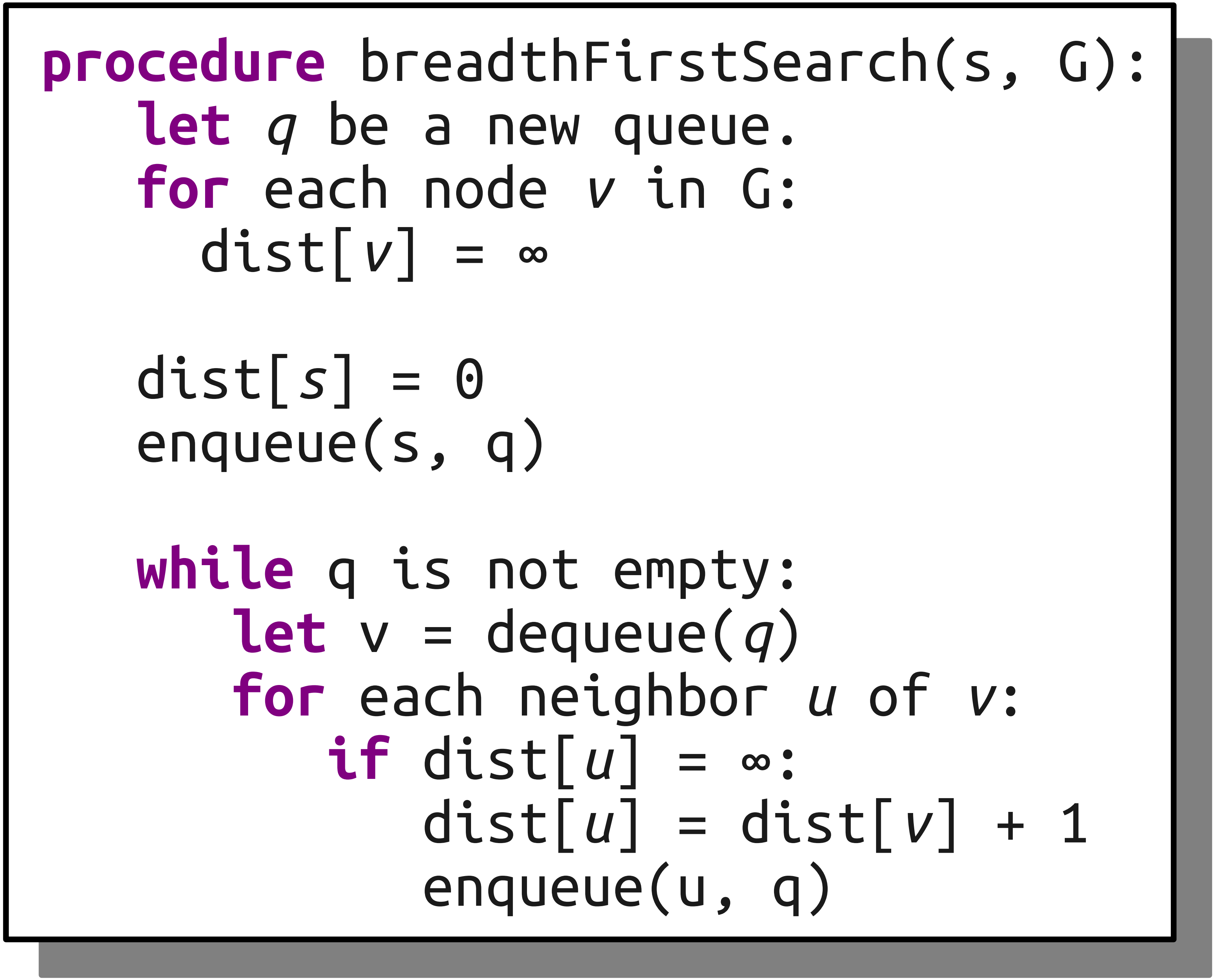
Graph Terminology

- When analyzing algorithms on a graph, there are (usually) two parameters we care about:
  - The number of nodes, denoted ***n***. ($n = |V|$)
  - The number of edges, denoted ***m***. ($m = |E|$)
- Note that $m = O(n^2)$. *(Why?)*
- A graph is called **dense** if $m = \Theta(n^2)$. A graph is called **sparse** if it is not dense.



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procedure breadthFirstSearch(s, G):
   let q be a new queue.
   for each node v in G:
      dist[v] = ∞

   dist[s] = 0
   enqueue(s, q)

   while q is not empty:
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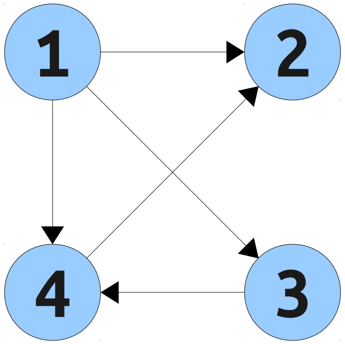
How are our graphs represented?

Adjacency Matrices

- An **adjacency matrix** is a representation of a graph as an $n \times n$ matrix M of 0s and 1s, where
  - $M_{uv} = 1$ if $(u, v) \in E$.
  - $M_{uv} = 0$ otherwise.



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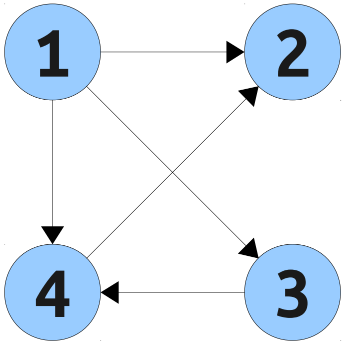

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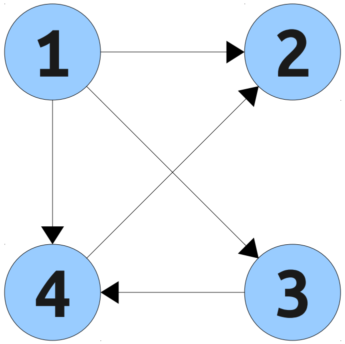

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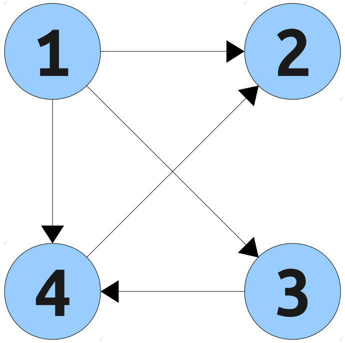

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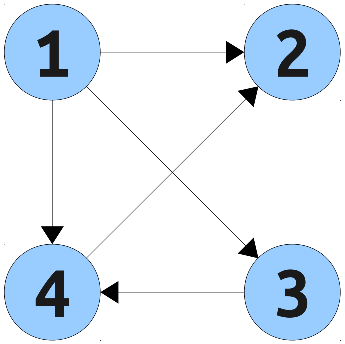

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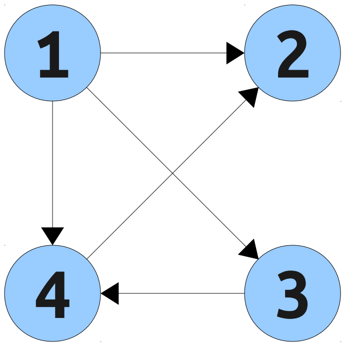

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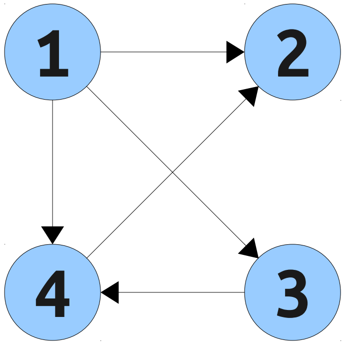

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$O(1)$

$O(n)$

$O(1)$

$+O(n^2)$

$O(n^2)$

$\Theta(n)$

Why isn't the runtime $\Theta(n^2)$?

Linear Time on Graphs

- With an adjacency matrix, BFS runs in time **$O(n^2)$**. Is that efficient?
- In a graph with n nodes and m edges, we say that an algorithm runs in **linear time** iff the algorithm runs in time $O(m + n)$.
  - This is linear in the number of “pieces” of the graph, which is the number of nodes plus the number of edges.
- On a dense graph, this implementation of BFS runs in linear time:

$$O(n^2) = O(n^2 + n) = O(m + n)$$

- On sparser graphs (say, $m = O(n)$), though, this is not linear time:

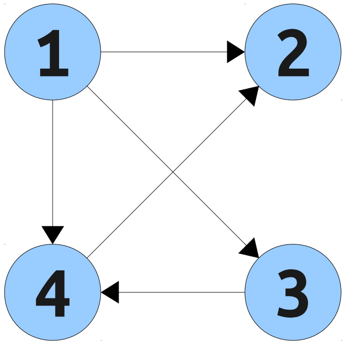
$$O(n^2) \neq O(n) = O(m + n)$$

The Issue

- Our algorithm is slow because this step always takes Θ(n) time:

```
for each neighbor u of v:
```

- Can we refine our data structure for storing the graph so that we can easily find all edges incident to a node?



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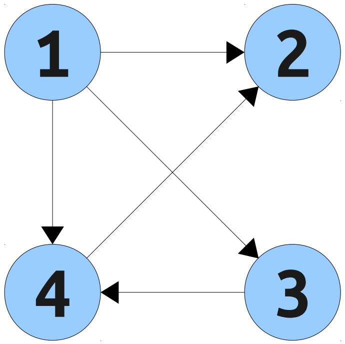
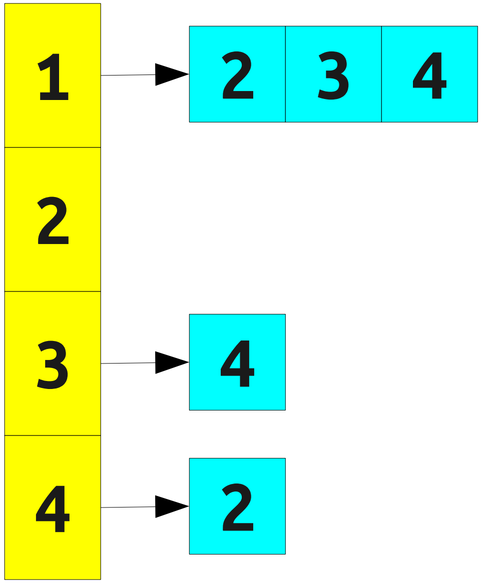
Adjacency Lists

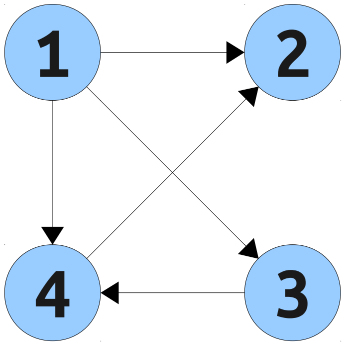
- An **adjacency list** is a representation of a graph as an array A of n lists. The list $A[u]$ holds all nodes v where (u, v) is an edge.

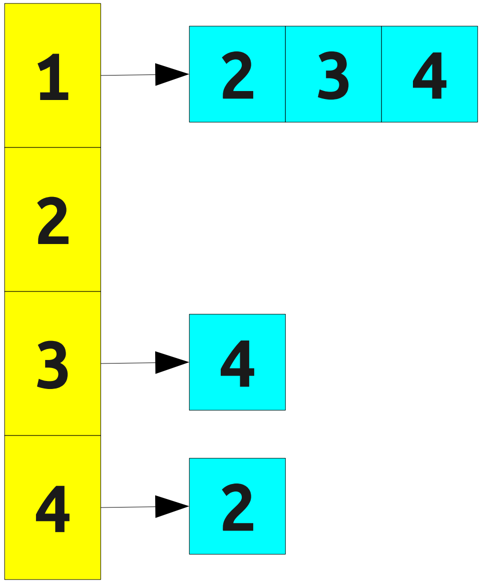




Adjacency Lists

- An **adjacency list** is a representation of a graph as an array *A* of *n* lists. The list *A*[*u*] holds all nodes *v* where (*u*, *v*) is an edge.





- Memory usage:

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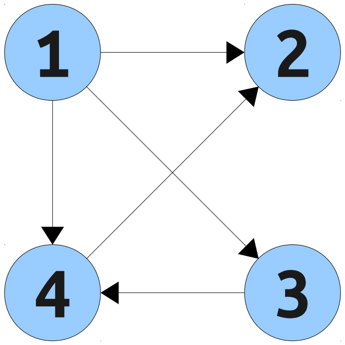

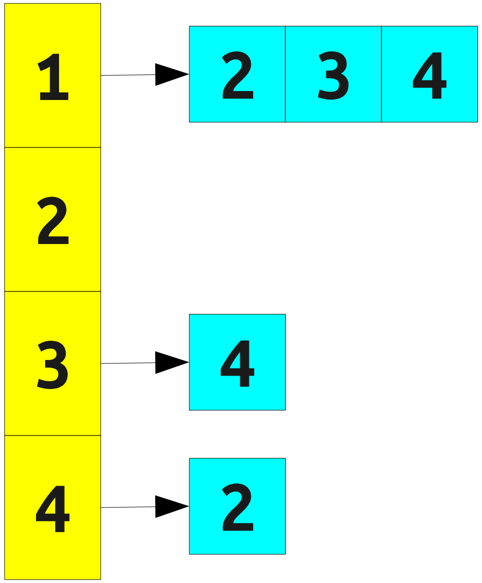

- Memory usage: **Θ(*n* + *m*)**.

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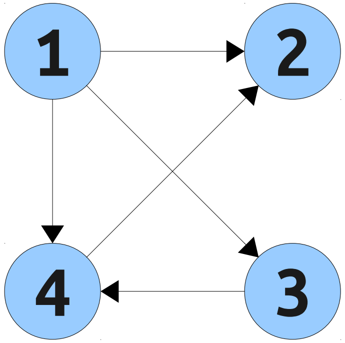

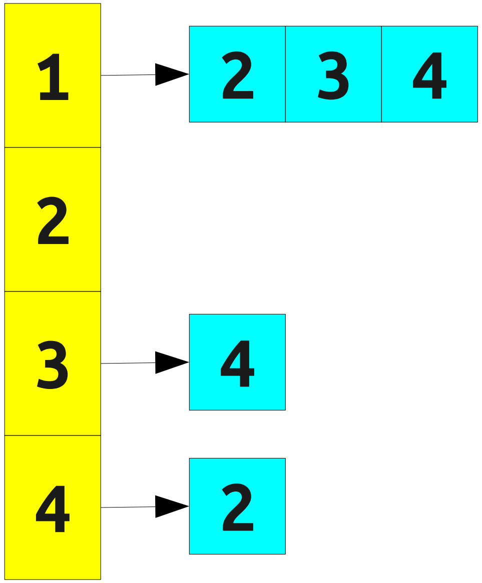

- Memory usage: $\boldsymbol{\Theta(n + m)}$.
- Time to check if edge (u, v) exists:

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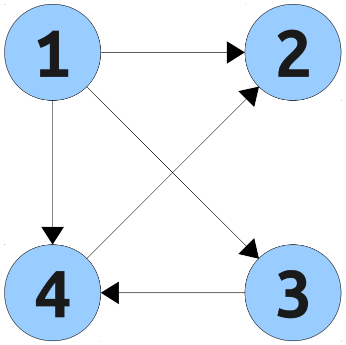

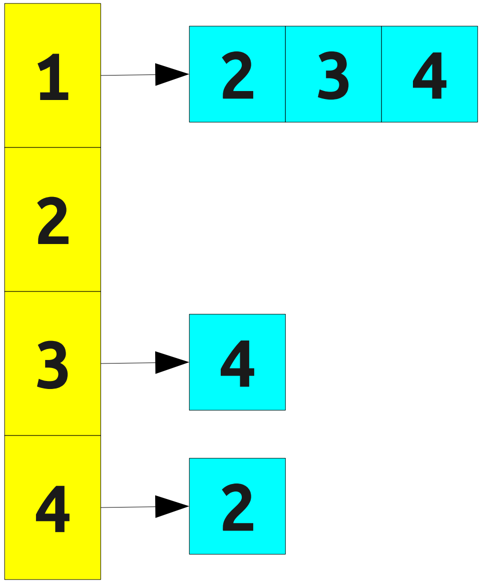

- Memory usage: **$\Theta(n + m)$**.
- Time to check if edge (*u*, *v*) exists: **$O(\deg^+(u))$**

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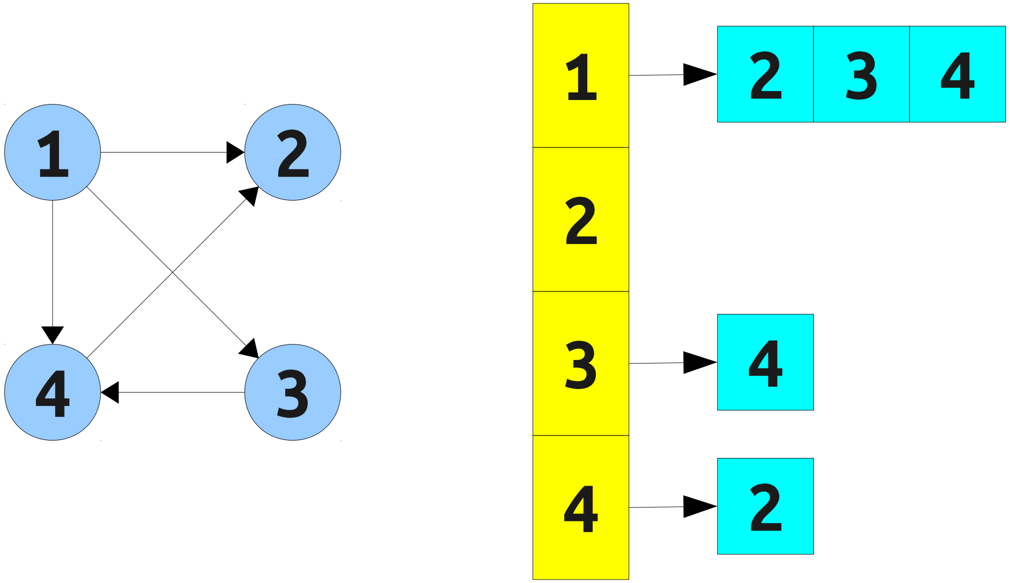

- Memory usage: $\boldsymbol{\Theta(n + m)}$.
- Time to check if edge (u, v) exists: $\mathbf{O(deg^+(}\boldsymbol{u}\mathbf{))}$
- Time to find all outgoing edges for a node u:

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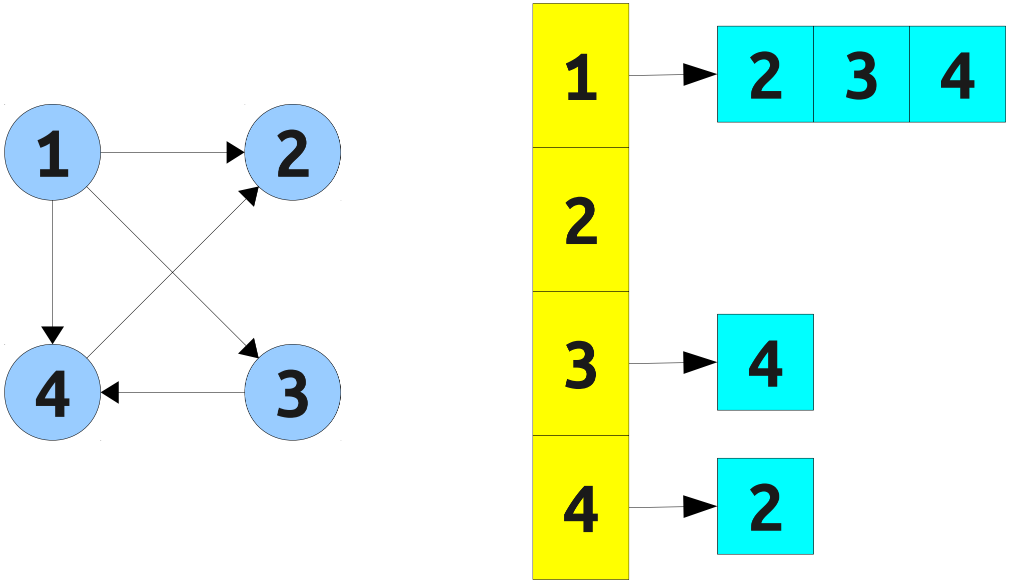

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       procedure breadthFirstSearch(s, G):
O(1)      let q be a new queue.
O(n)      for each node v in G:
            dist[v] = ∞

O(1)      dist[s] = 0
          enqueue(s, q)

          while q is not empty:
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A Better Analysis

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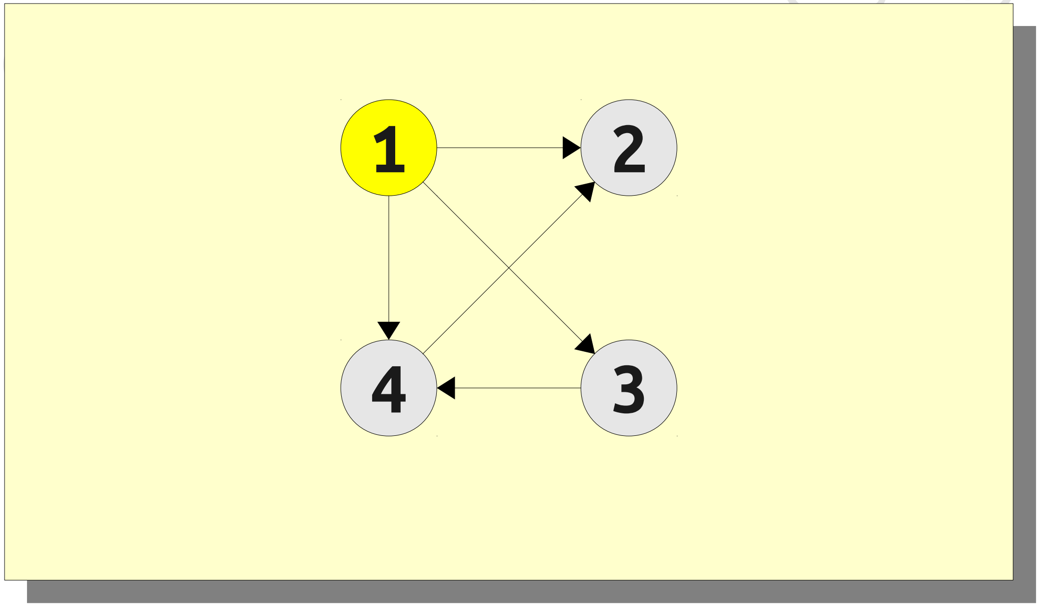

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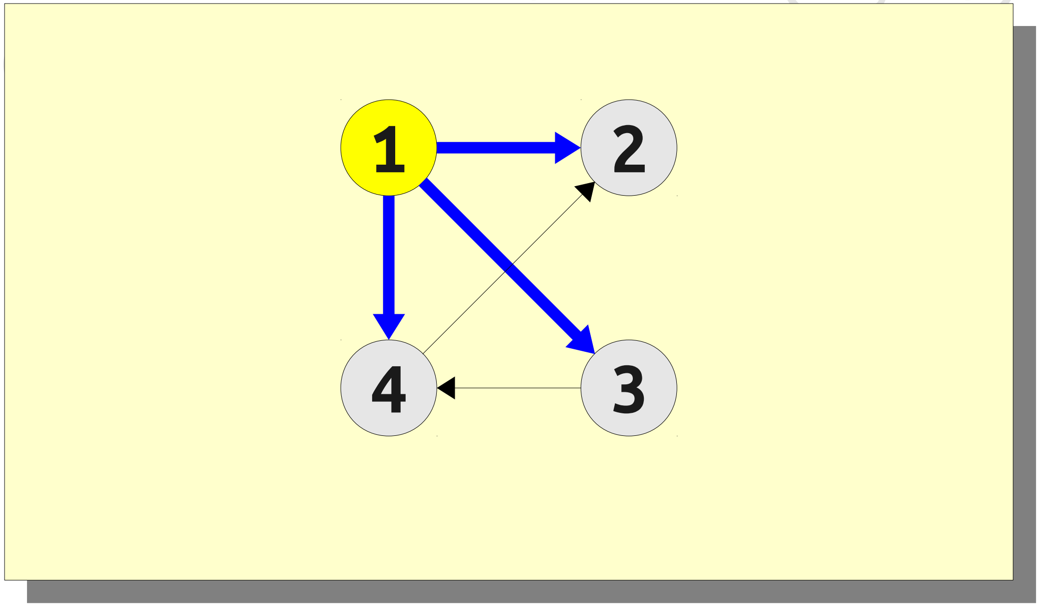

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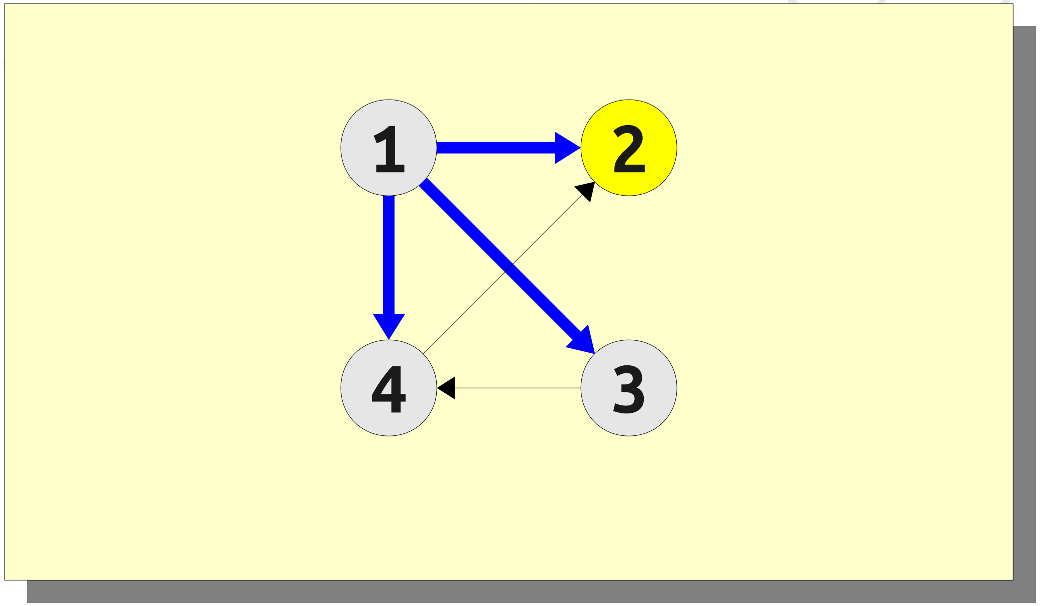

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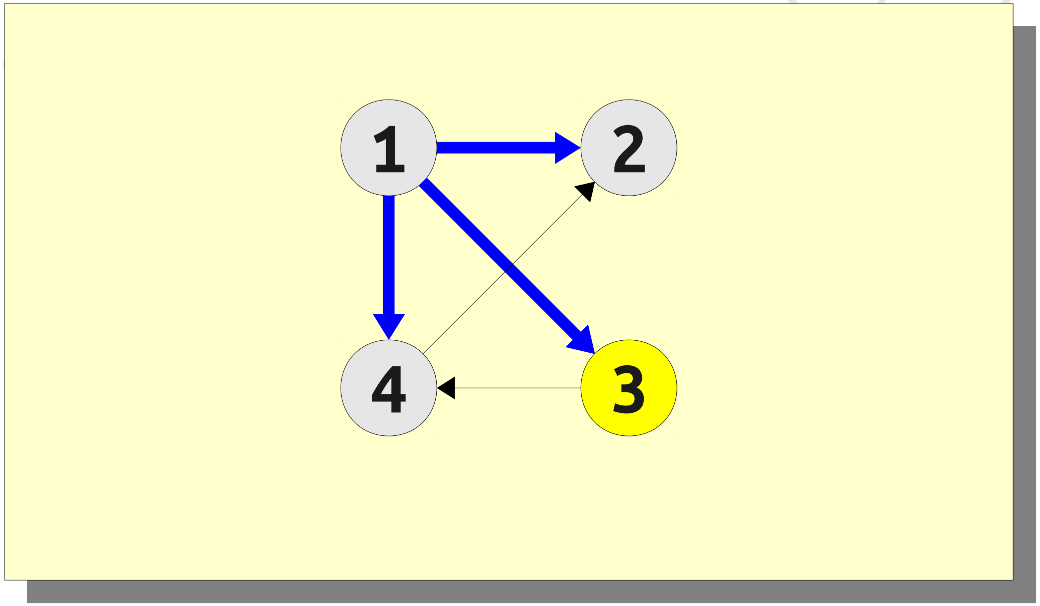

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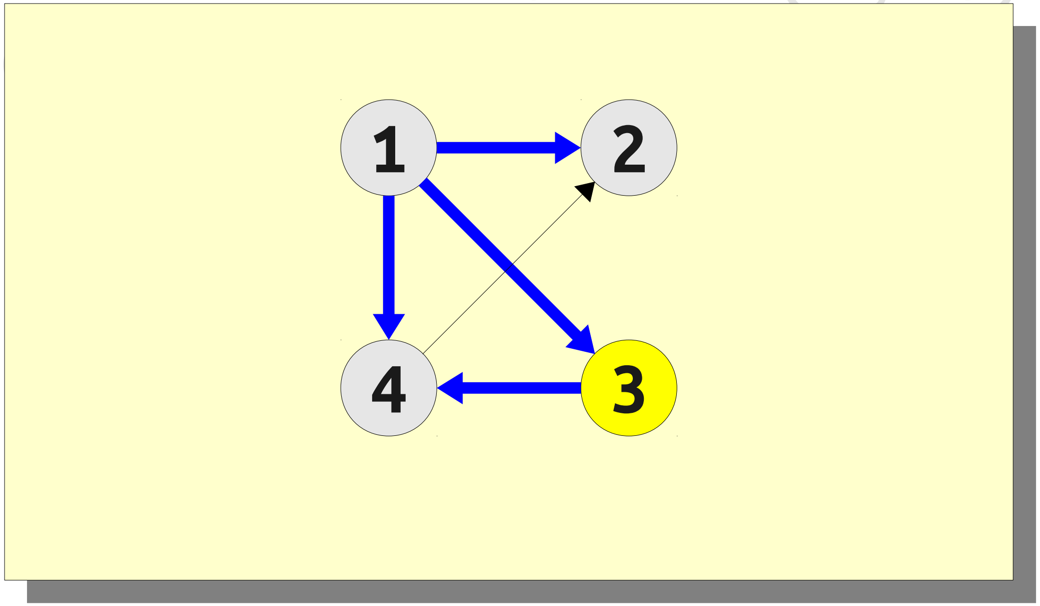

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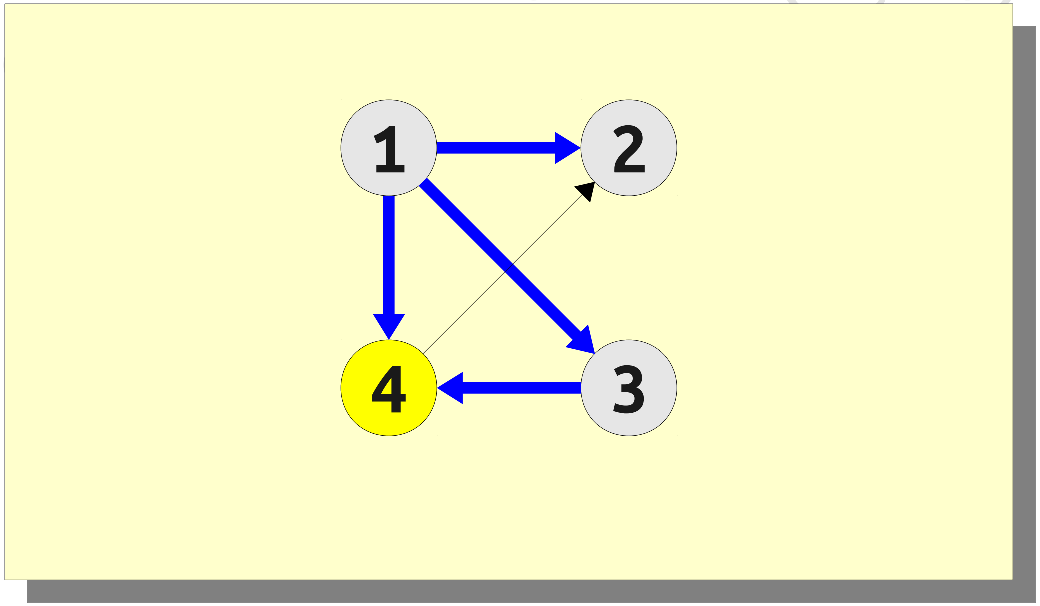

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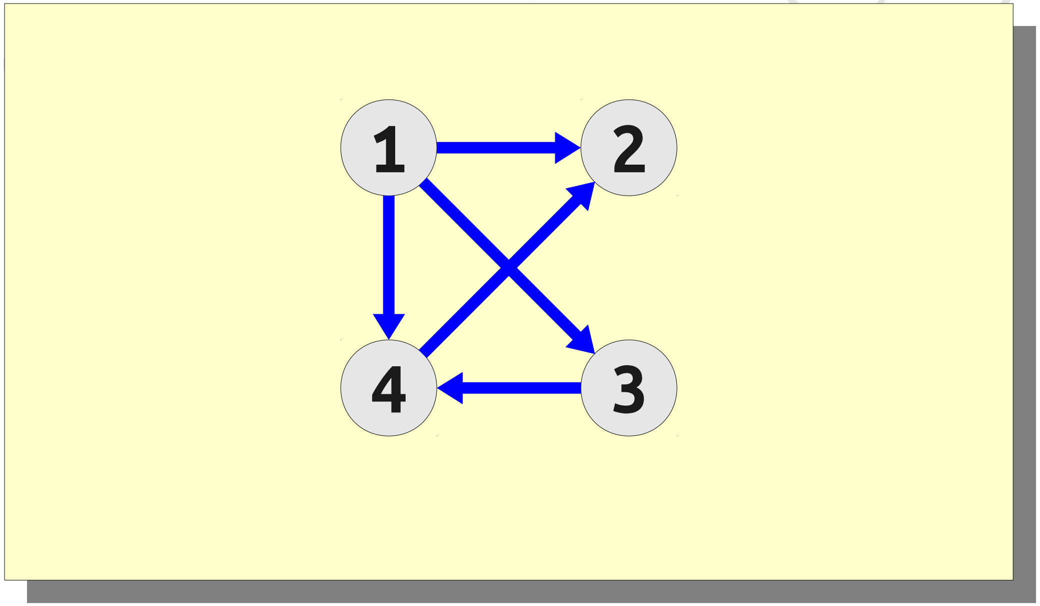

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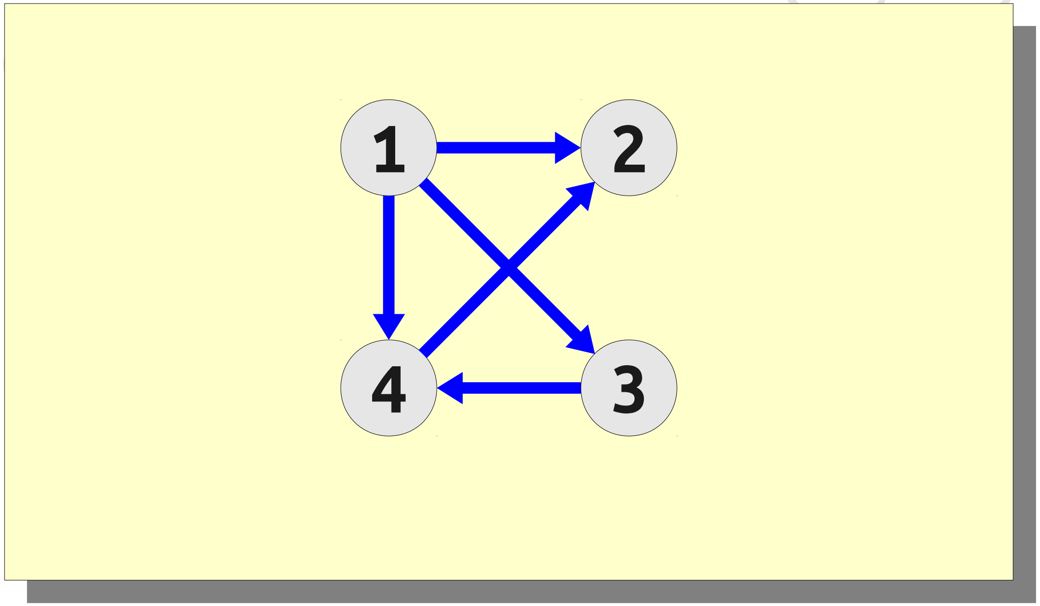

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A Better Analysis

- Using adjacency lists, BFS runs in time **O(m + n)**.
    - This is linear time!
- **Key Idea**: Do a more precise accounting of the work done by an algorithm.
    - Determine how much work is done *across all iterations* to determine total work.
    - Don't just find worst-case runtime and multiply by number of iterations.
- Going forward, we will use adjacency lists rather than adjacency matrices as our graph representation unless stated otherwise.

Next Time

- Dijkstra's Algorithm
- Depth-First Search
- Directed Acyclic Graphs