Thinking Recursively

Part IV

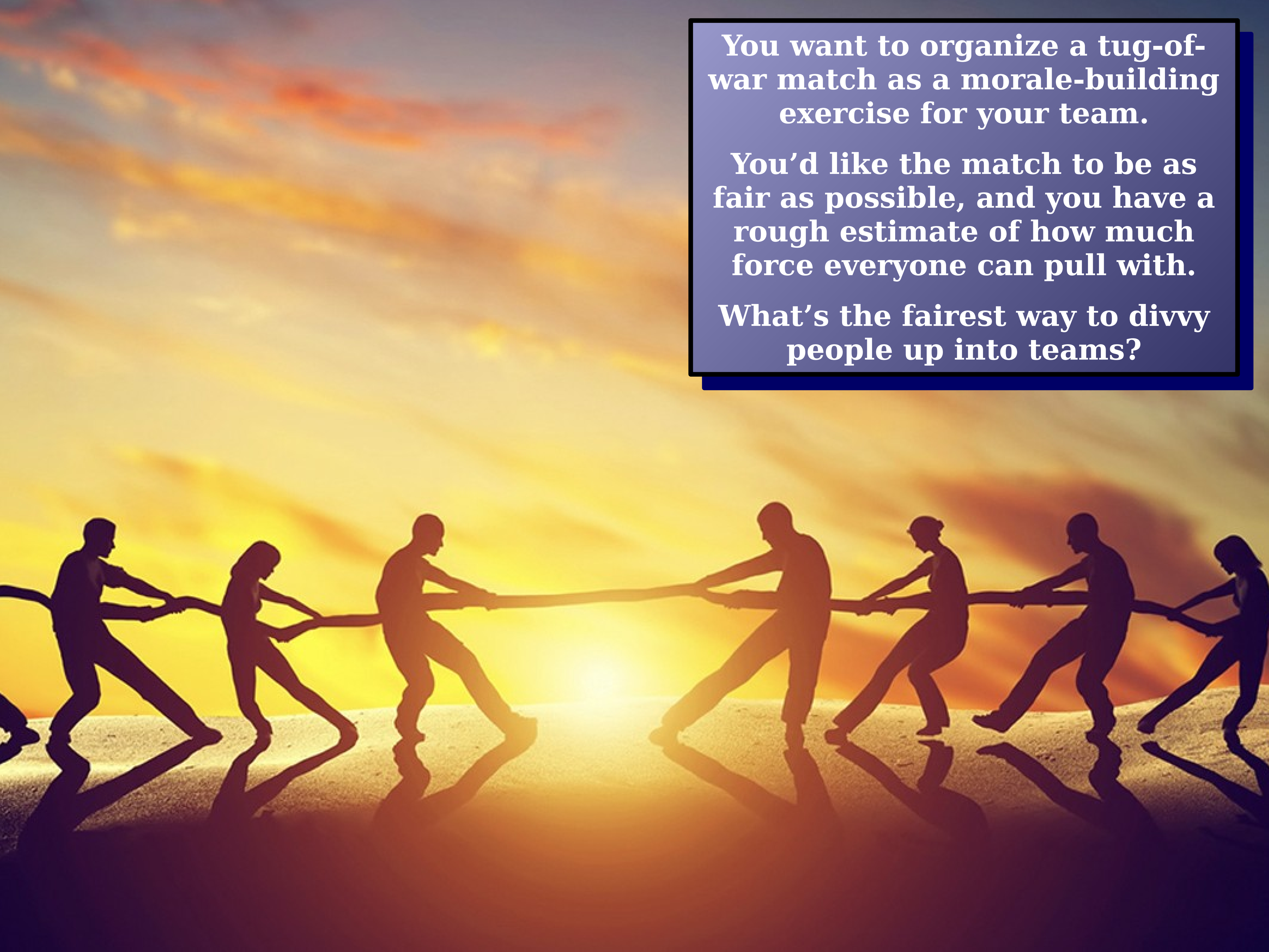
Outline for Today

- ***Recap From Last Time***
  - Where are we, again?
- ***More on Tug-of-War***
  - Addressing some points from last time.
- ***Shrinkable Words***
  - A little word puzzle!

Recap from Last Time

Enumeration and Optimization

- An ***enumeration*** problem is one where the goal is to list all objects of some type.
- An ***optimization*** problem is one where the goal is to find the best object of some type.
- If you can enumerate all solutions to a problem, with a few quick code tweaks you can convert what you have into a solution to an optimization problem.

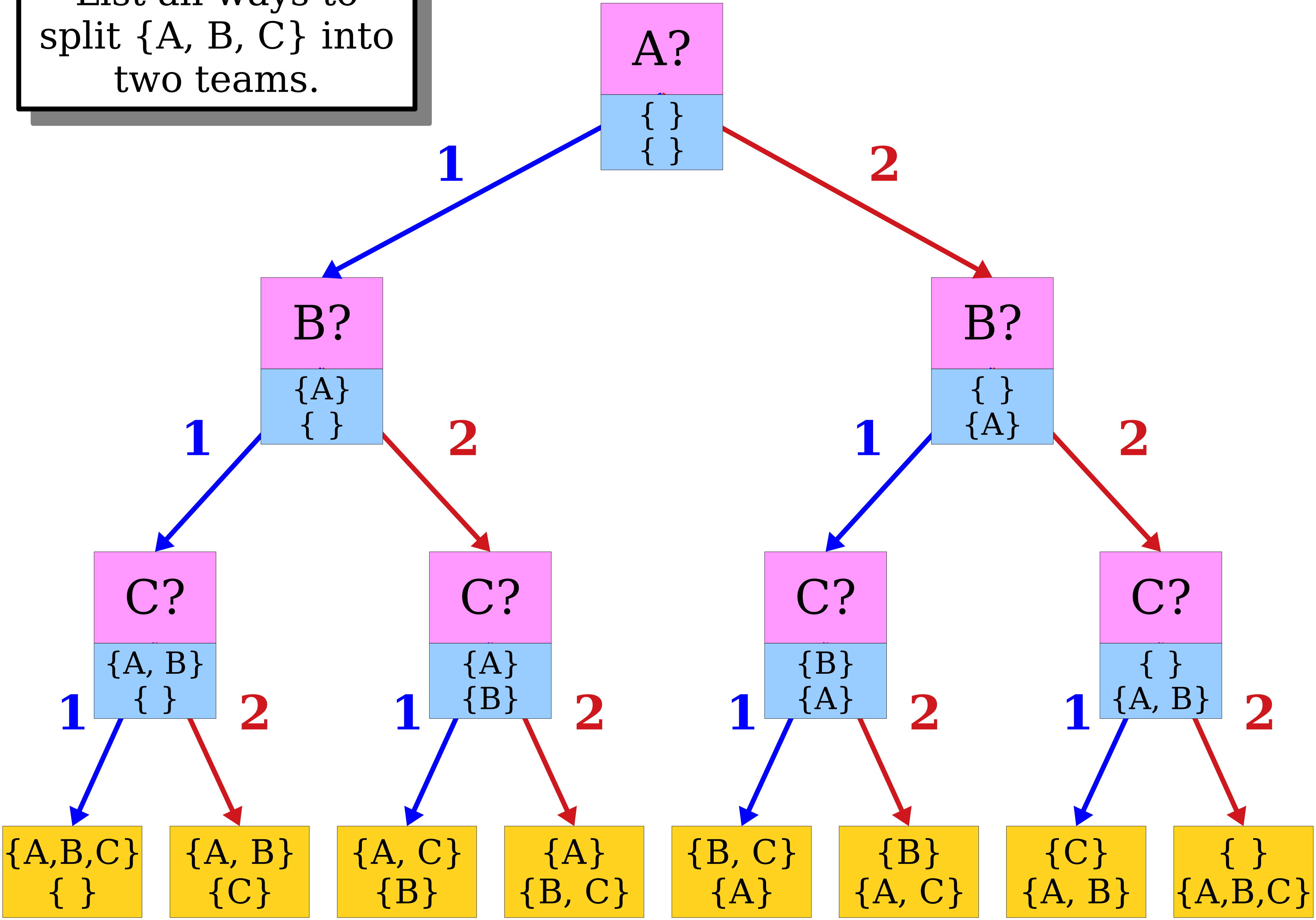


You want to organize a tug-of-war match as a morale-building exercise for your team.

You'd like the match to be as fair as possible, and you have a rough estimate of how much force everyone can pull with.

What's the fairest way to divvy people up into teams?

List all ways to split {A, B, C} into two teams.



New Stuff!

Answering Your Questions

Question 1:

What happens if we make a bad decision early on? Won’t we be stuck committed to the wrong solution?

```
Teams bestTeamsRec(const Set<Person>& remaining,
                   const Teams& soFar) {
  if (remaining.isEmpty()) {
    return soFar;
  } else {
    Person curr = remaining.first();

    /* Option 1: Put this person on Team 1. */
    Teams best1 = bestTeamsRec(remaining - curr,
                               { soFar.one + curr, soFar.two });

    /* Option 2: Put this person on Team 2. */
    Teams best2 = bestTeamsRec(remaining - curr,
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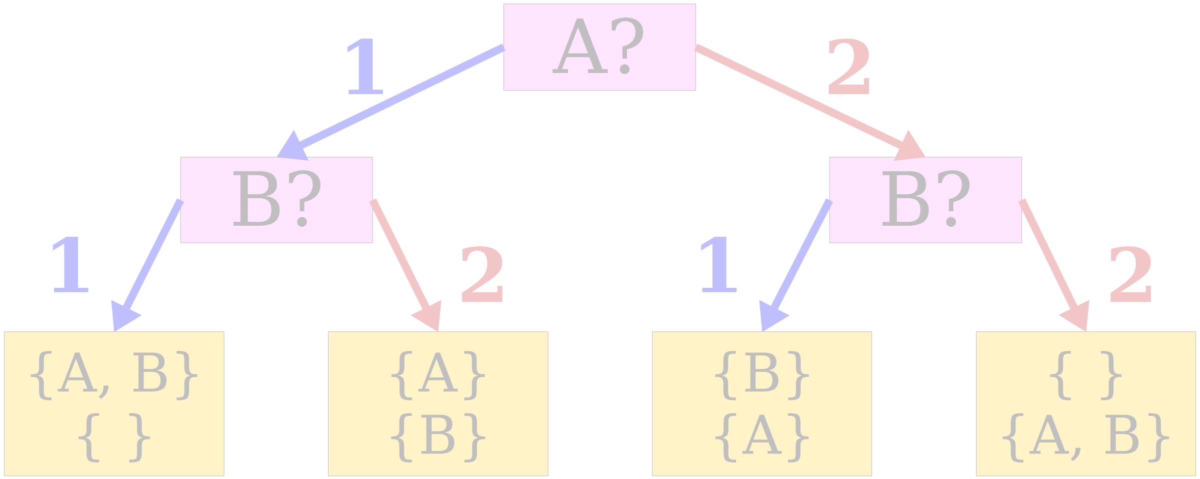
Perspective 1: ***Trace the Recursion***


A?
1
2
B?
B?
1
2
1
2
{A, B}
{ }
{A}
{B}
{B}
{A}
{ }
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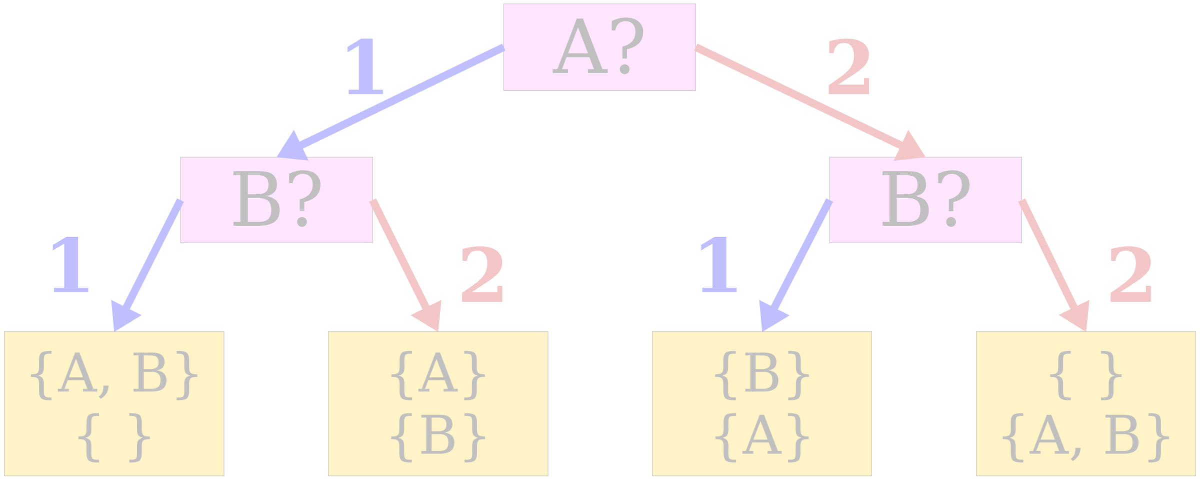
remaining

{A, B}

soFar

{ }
{ }



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{A, B}

soFar

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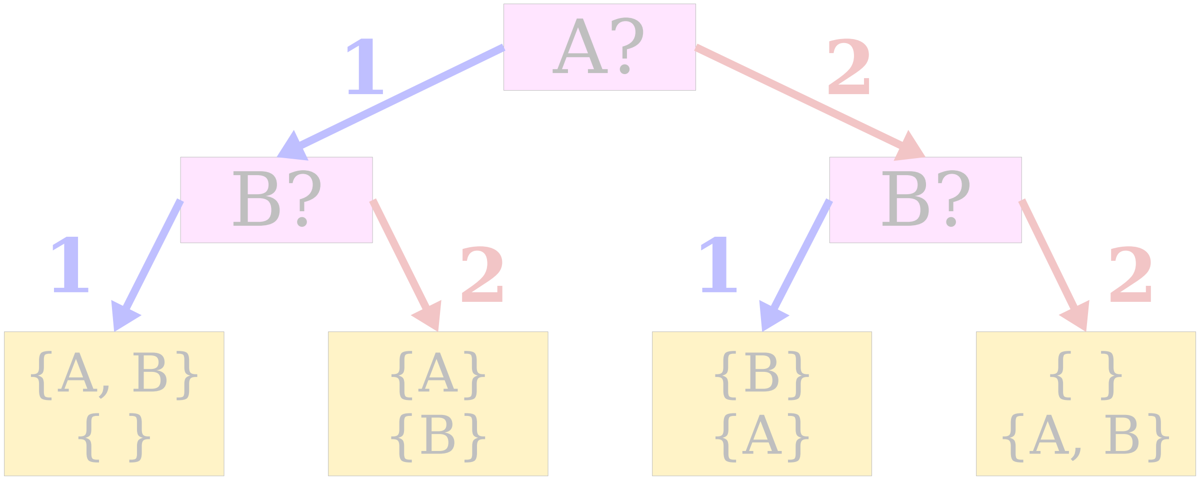

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soFar

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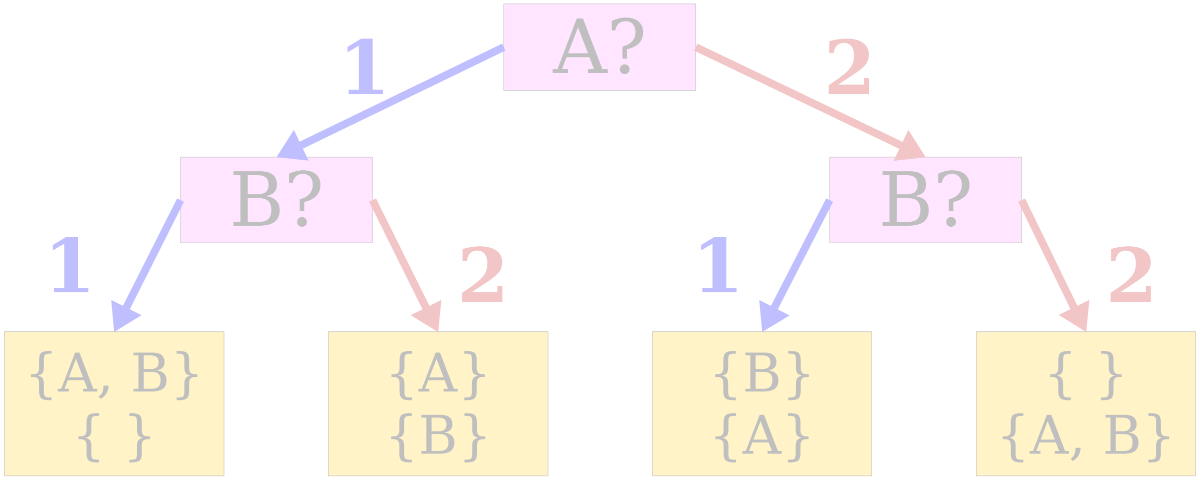

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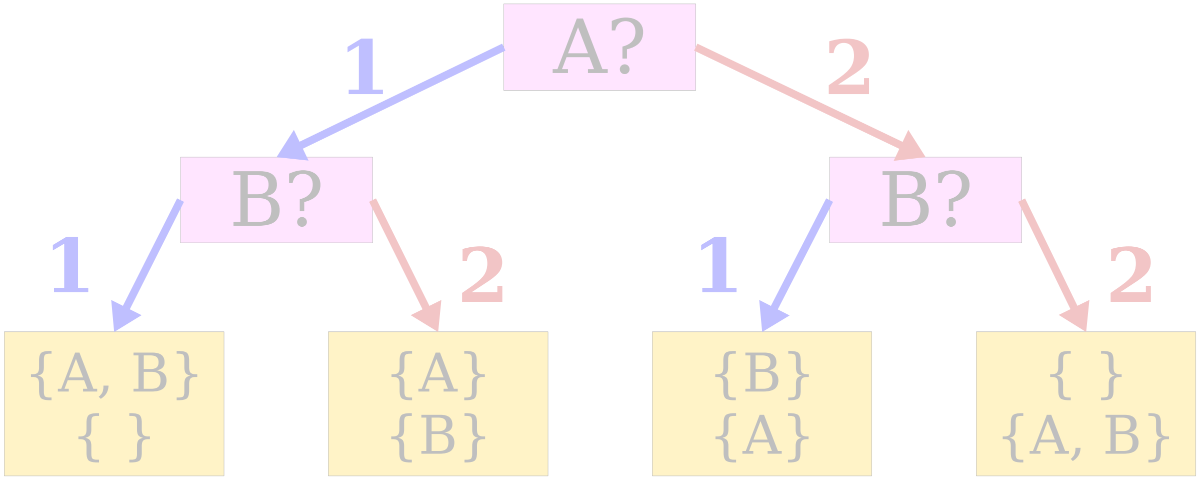

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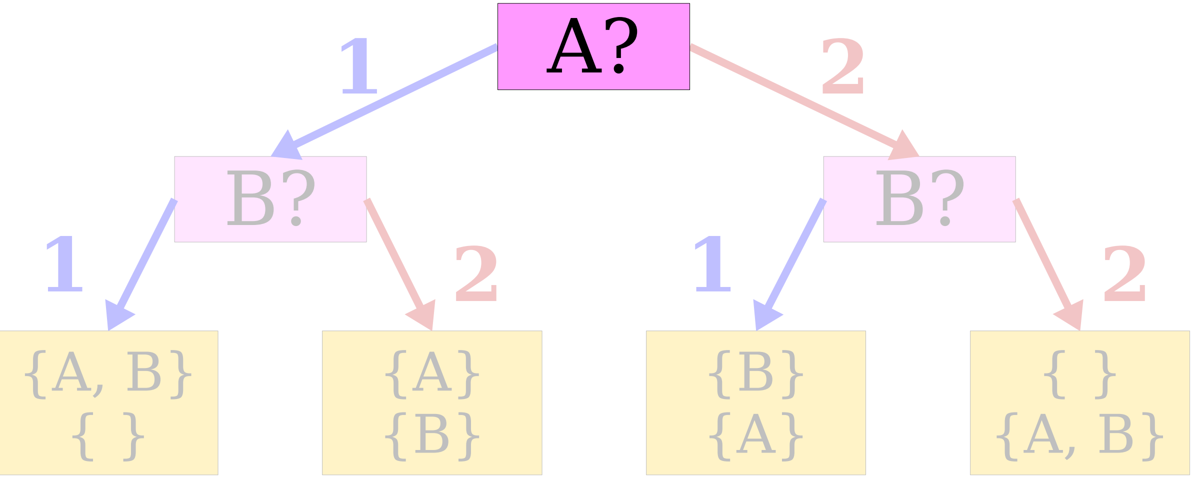

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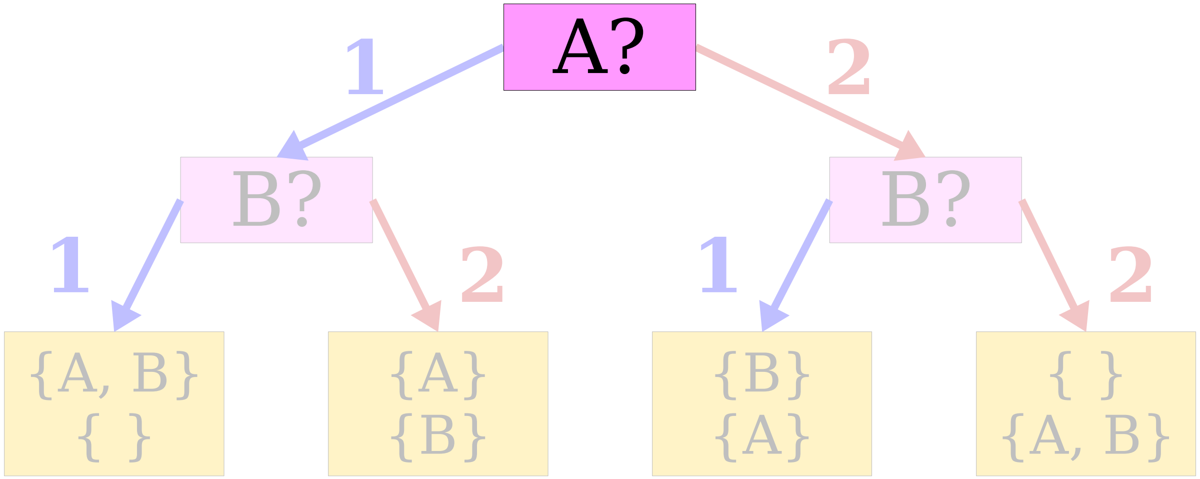

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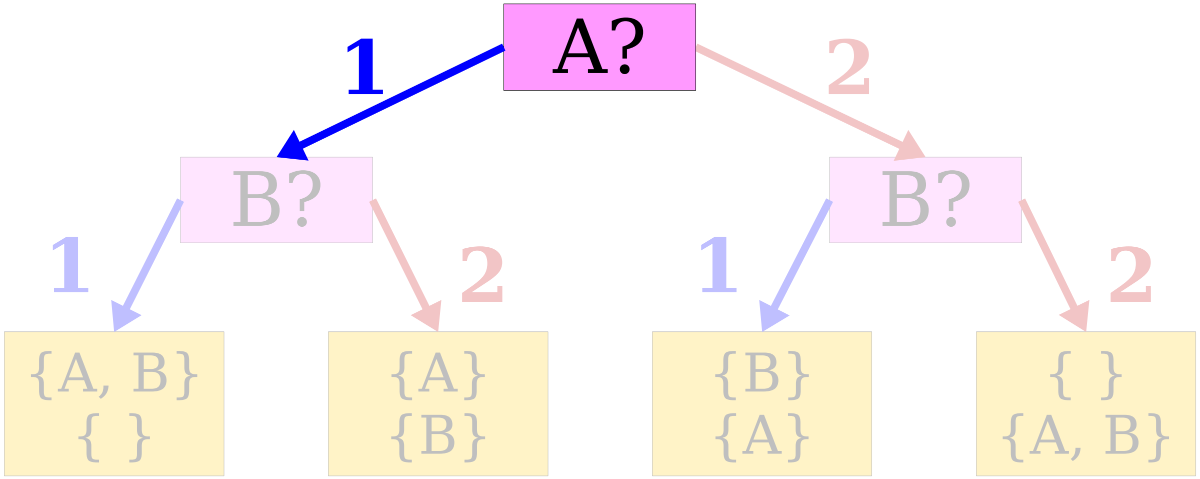
remaining

{B}

soFar

{A}
{ }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{B}

soFar

{A}
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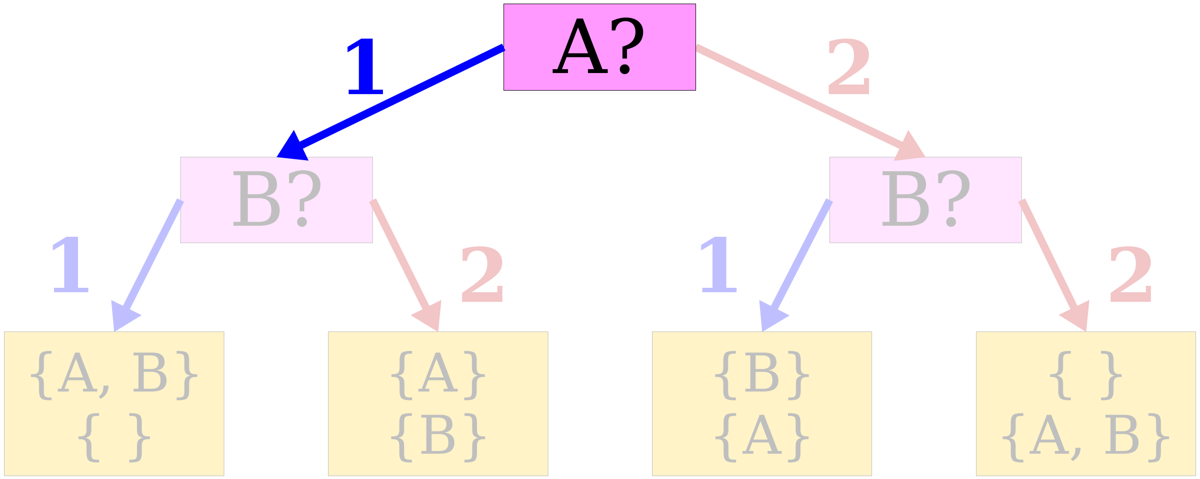

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{B}

soFar

{A}
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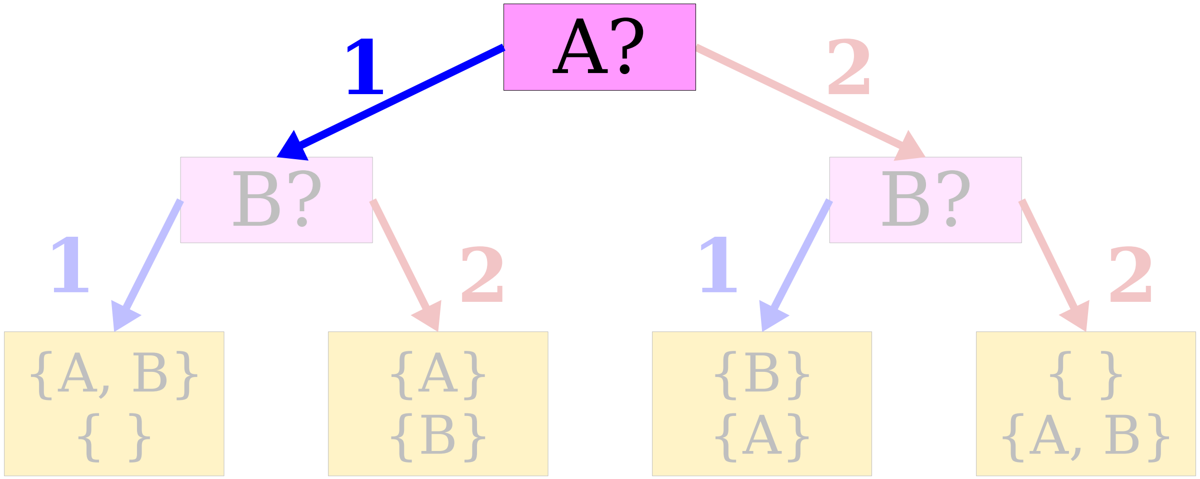

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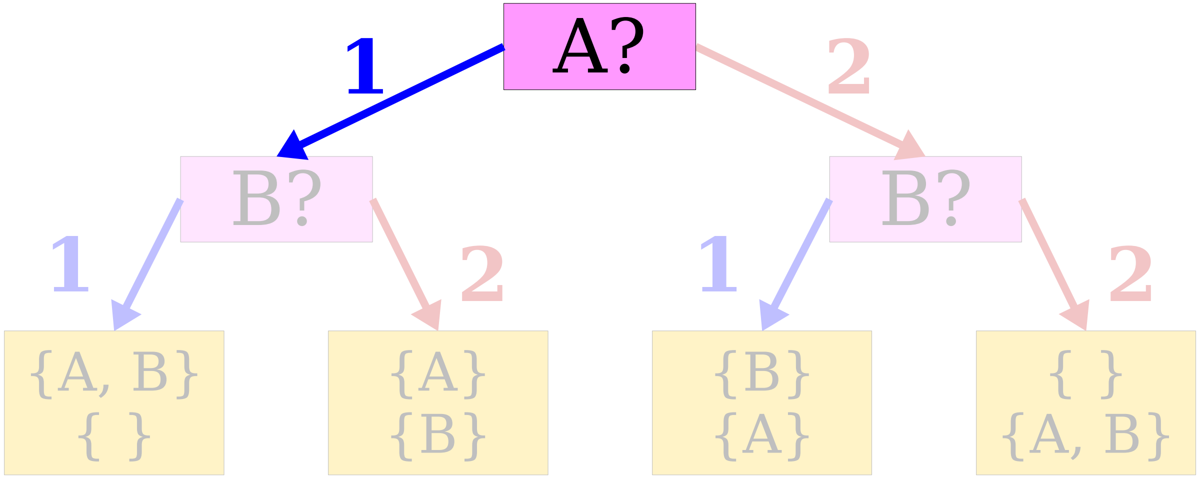
remaining	curr	soFar

{B}	B	{A}
		{ }



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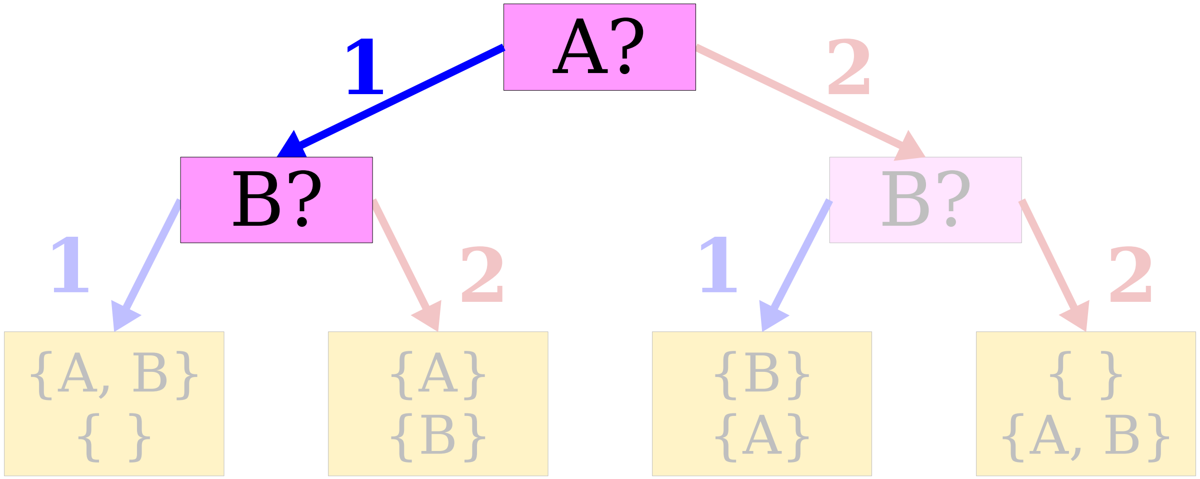
remaining
{B}

curr
B

soFar
{A}
{ }



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{B}	B	{A}
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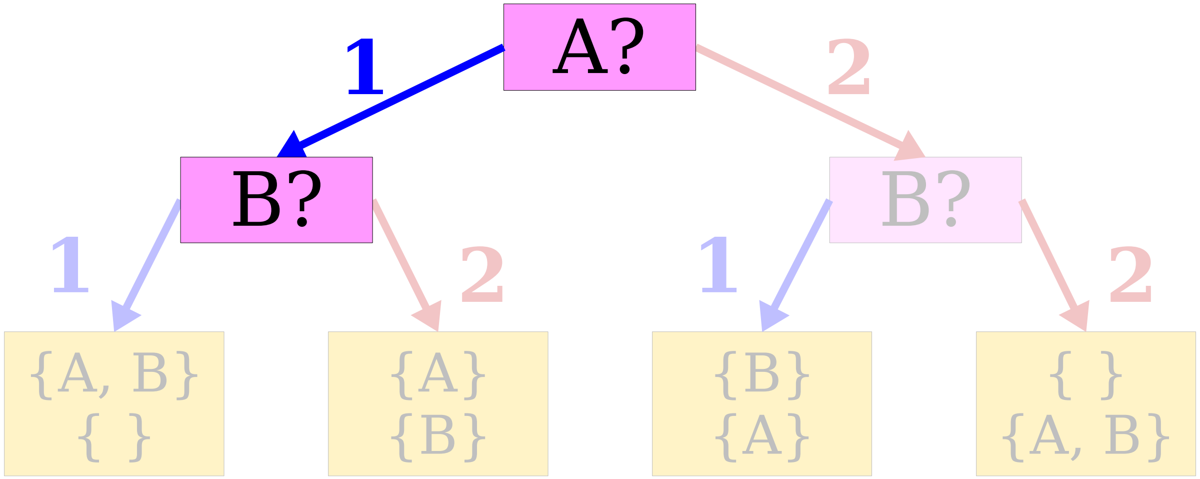

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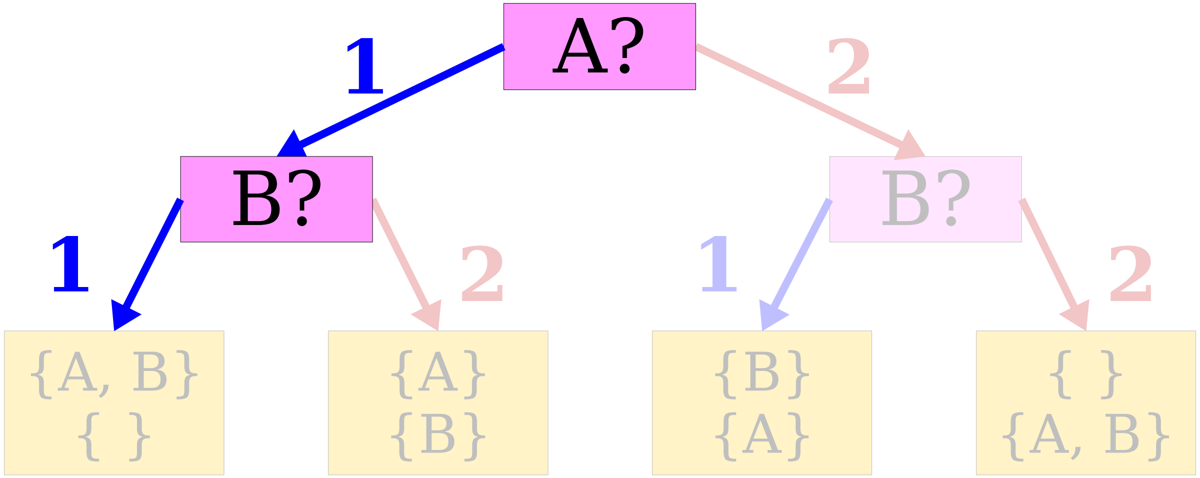
{ }

soFar

{A, B}
{ }



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soFar

{A, B}
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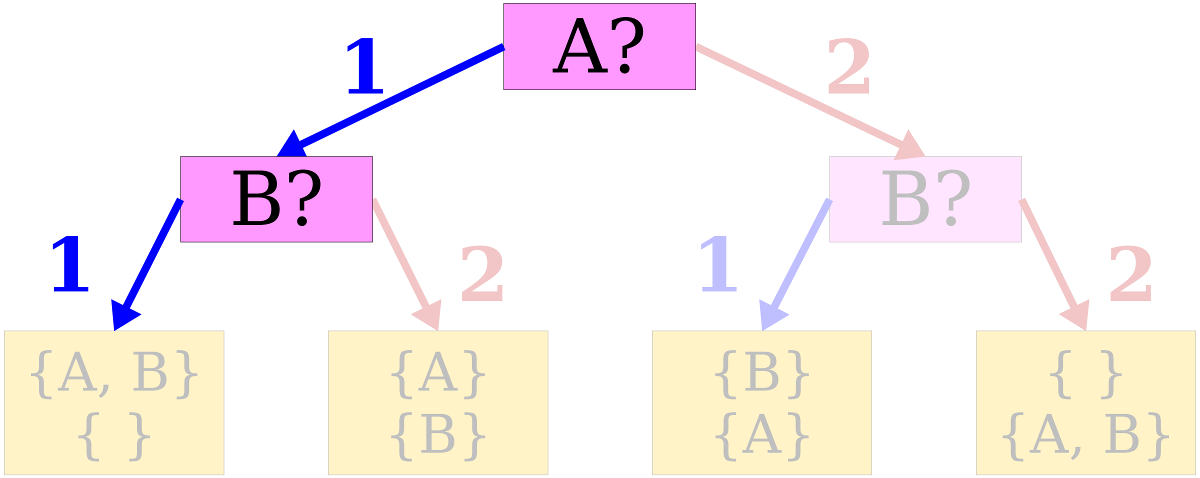

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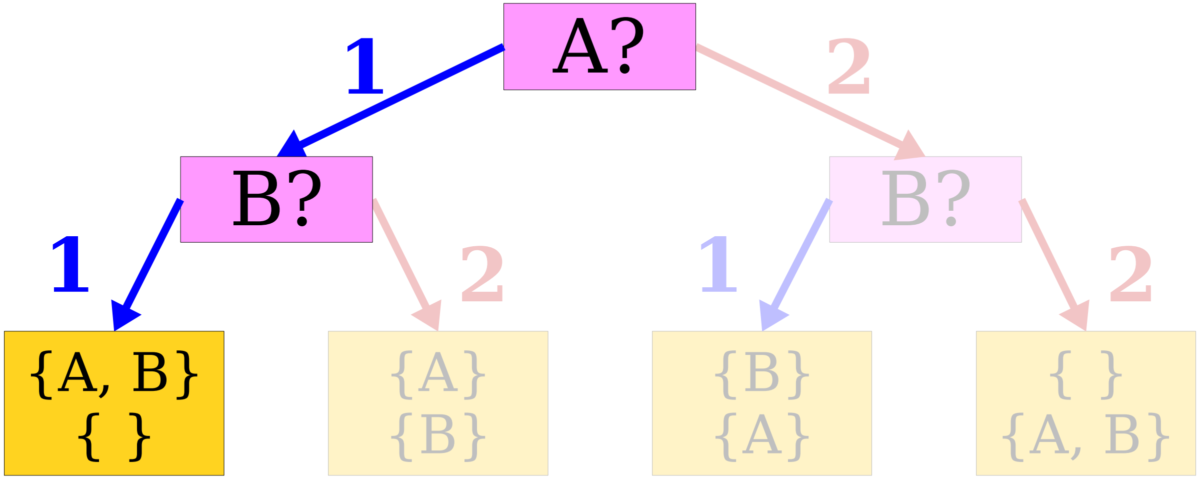
{ }

soFar

{A, B}
{ }



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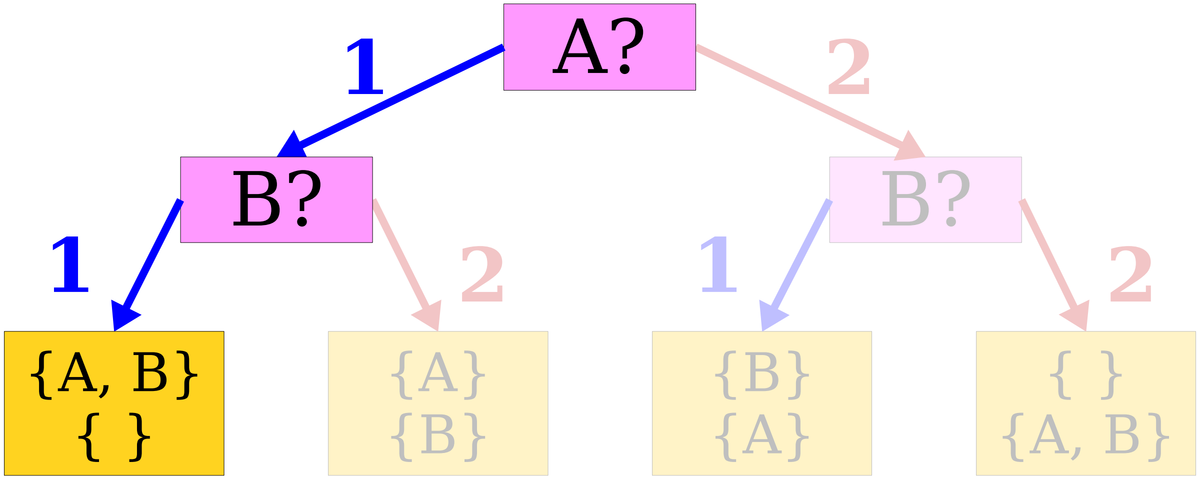
```

remaining
{B}

curr
B

soFar
{A}
{ }

best1
{A, B}
{ }



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remaining

{B}

curr

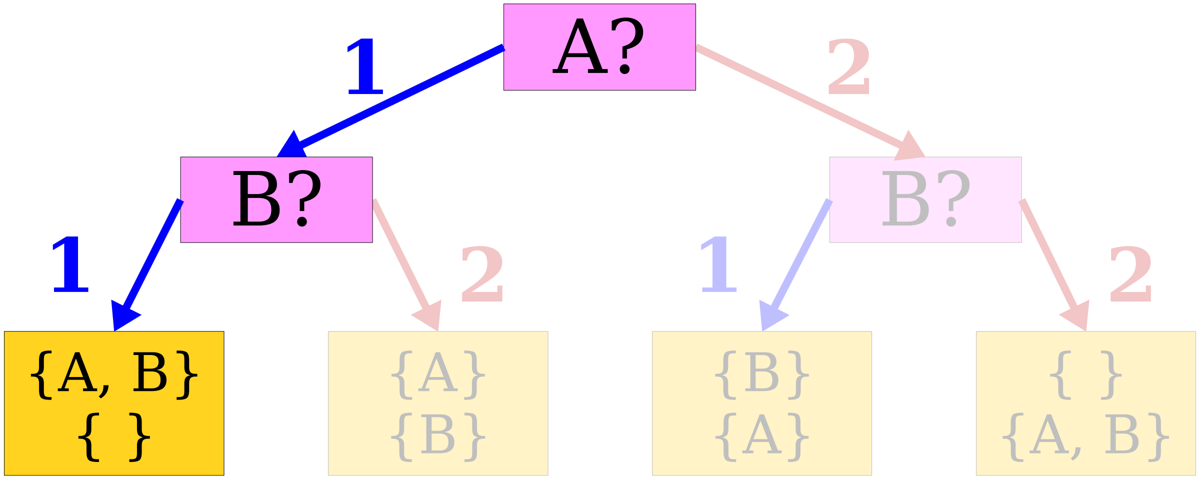
B

soFar

{A}
{ }

best1

{A, B}
{ }



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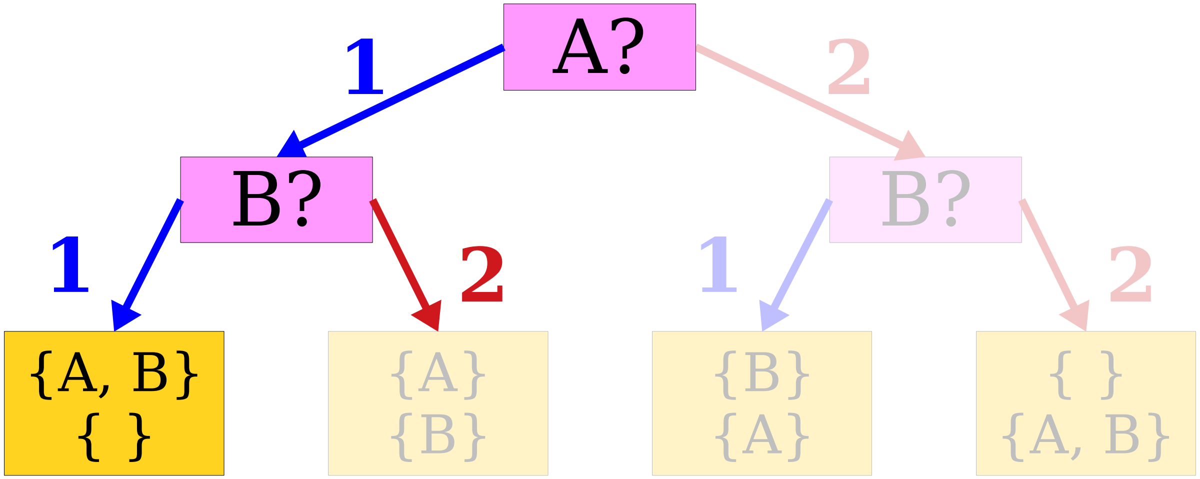
remaining	soFar
{ }	{A} {B}



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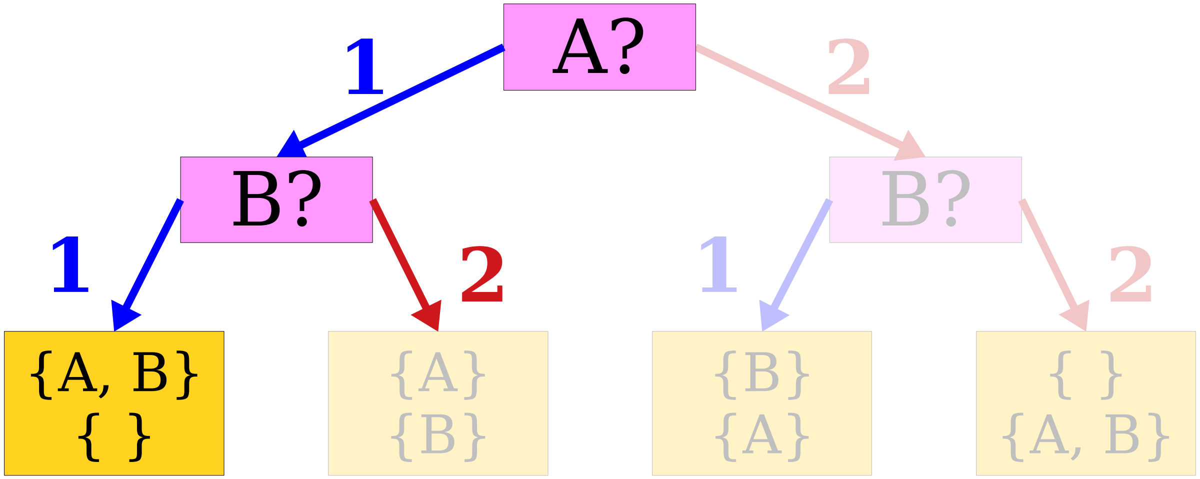
remaining

{ }

soFar

{A}
{B}



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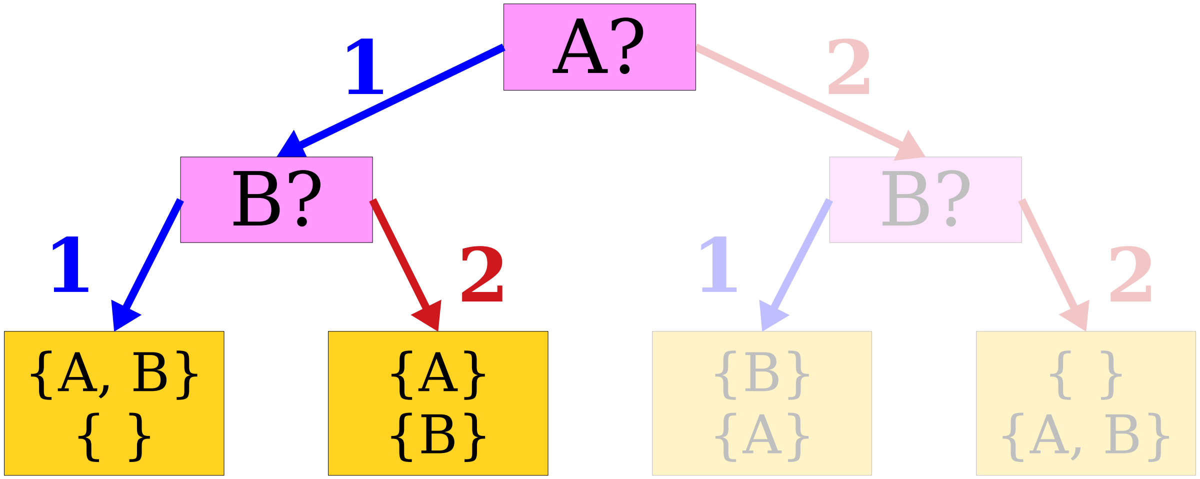
remaining

{ }

soFar

{A}
{B}



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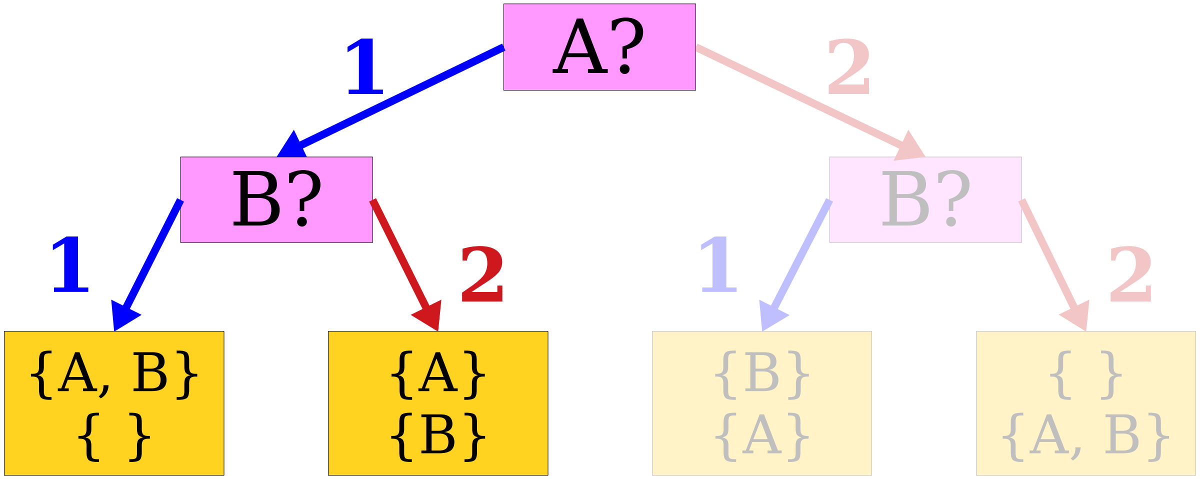
remaining
{B}

curr
B

soFar
{A}
{ }

best1
{A, B}
{ }

best2
{A}
{B}



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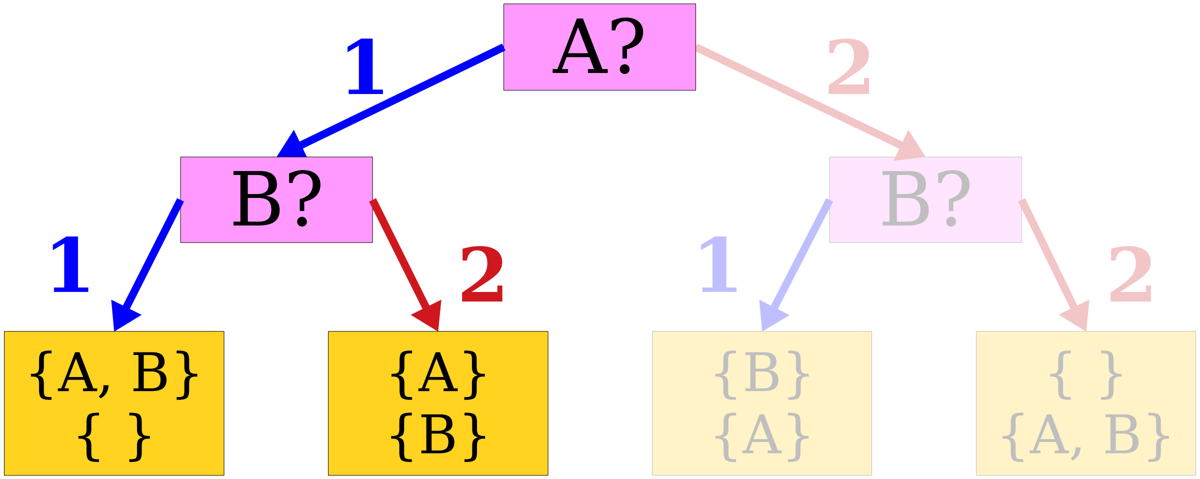
remaining	curr	soFar

{B}	B	{A}
		{ }

best1	best2

{A, B}
{ }
	{A}
	{B}



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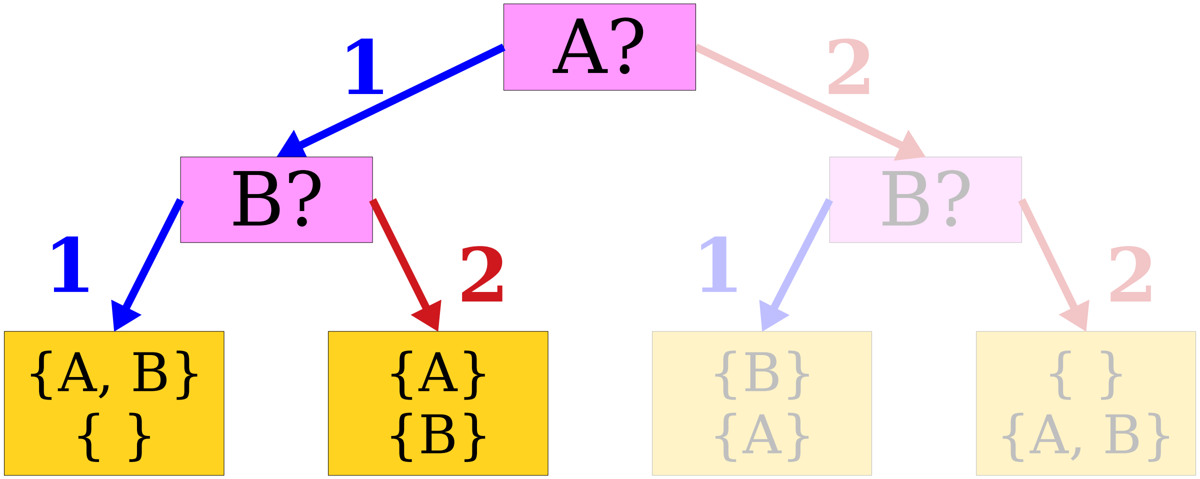
remaining
{B}

curr
B

soFar
{A}
{ }

best1
{A, B}
{ }

best2
{A}
{B}



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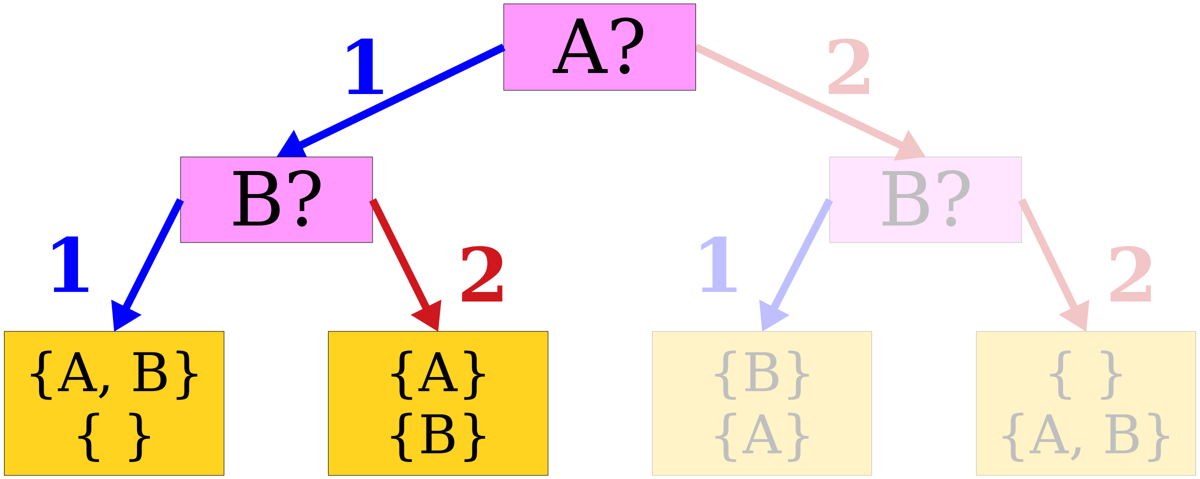
    return best1;
  } else {
    return best2;
  }
}
```

remaining	curr	soFar

{B}	B	{A}
		{ }

best1	best2

{A, B}
{ }
	{A}
	{B}



```
   if (remaining.isEmpty()) {
     return soFar;
   } else {
     Person curr = remaining.first();

⇨    Teams best1 = bestTeamsRec(remaining - curr,
                                { soFar.one + curr,  soFar.two });

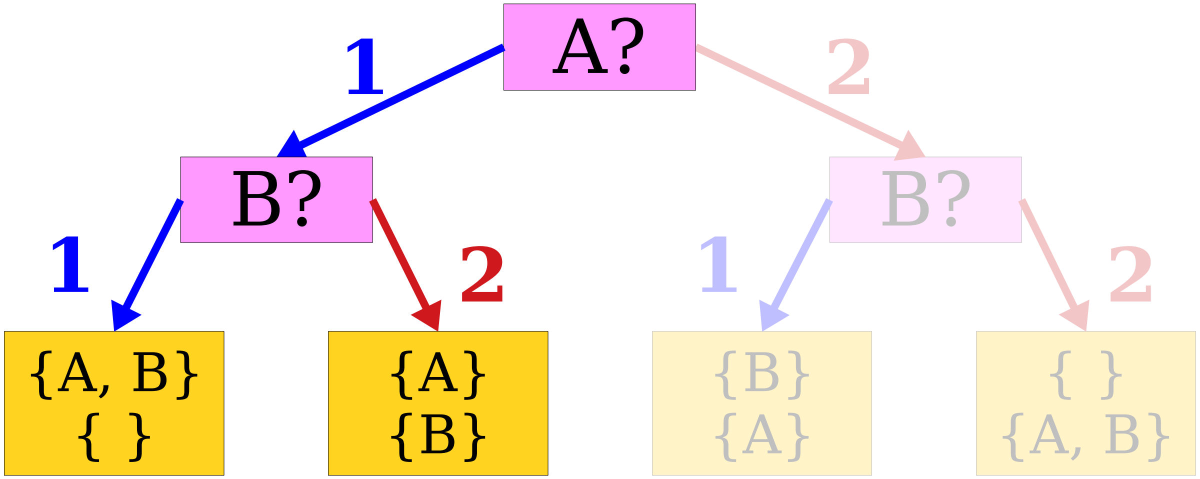
     Teams best2 = bestTeamsRec(remaining - curr,
                                { soFar.one, soFar.two +  curr });

     if (imbalanceOf(best1) < imbalanceOf(best2)) {
       return best1;
     } else {
       return best2;
     }
   }
```

remaining	curr	soFar
{A, B}	A	{ } { }

best1

{A}
{B}



```
Teams bestTeamsRec(const Set<Person>& remaining,
                   const Teams& soFar) {
  if (remaining.isEmpty()) {
    return soFar;
  } else {
    Person curr = remaining.first();

    /* Option 1: Put this person on Team 1. */
    Teams best1 = bestTeamsRec(remaining - curr,
                               { soFar.one + curr, soFar.two });

    /* Option 2: Put this person on Team 2. */
    Teams best2 = bestTeamsRec(remaining - curr,
                               { soFar.one, soFar.two + curr });

    if (imbalanceOf(best1) < imbalanceOf(best2)) {
      return best1;
    } else {
      return best2;
    }
  }
}
```

```
Teams bestTeamsRec(const Set<Person>& remaining,
                   const Teams& soFar) {
  if (remaining.isEmpty()) {
    return soFar;
  } else {
    Person curr = remaining.first();

    /* Option 1: Put this person on T
    Teams best1 = bestTeamsRec(remai
                               { soFa                            ;

    /* Option 2: Put this person on Team 2. */
    Teams best2 = bestTeamsRec(remaining - curr,
                               { soFar.one, soFar.two + curr });

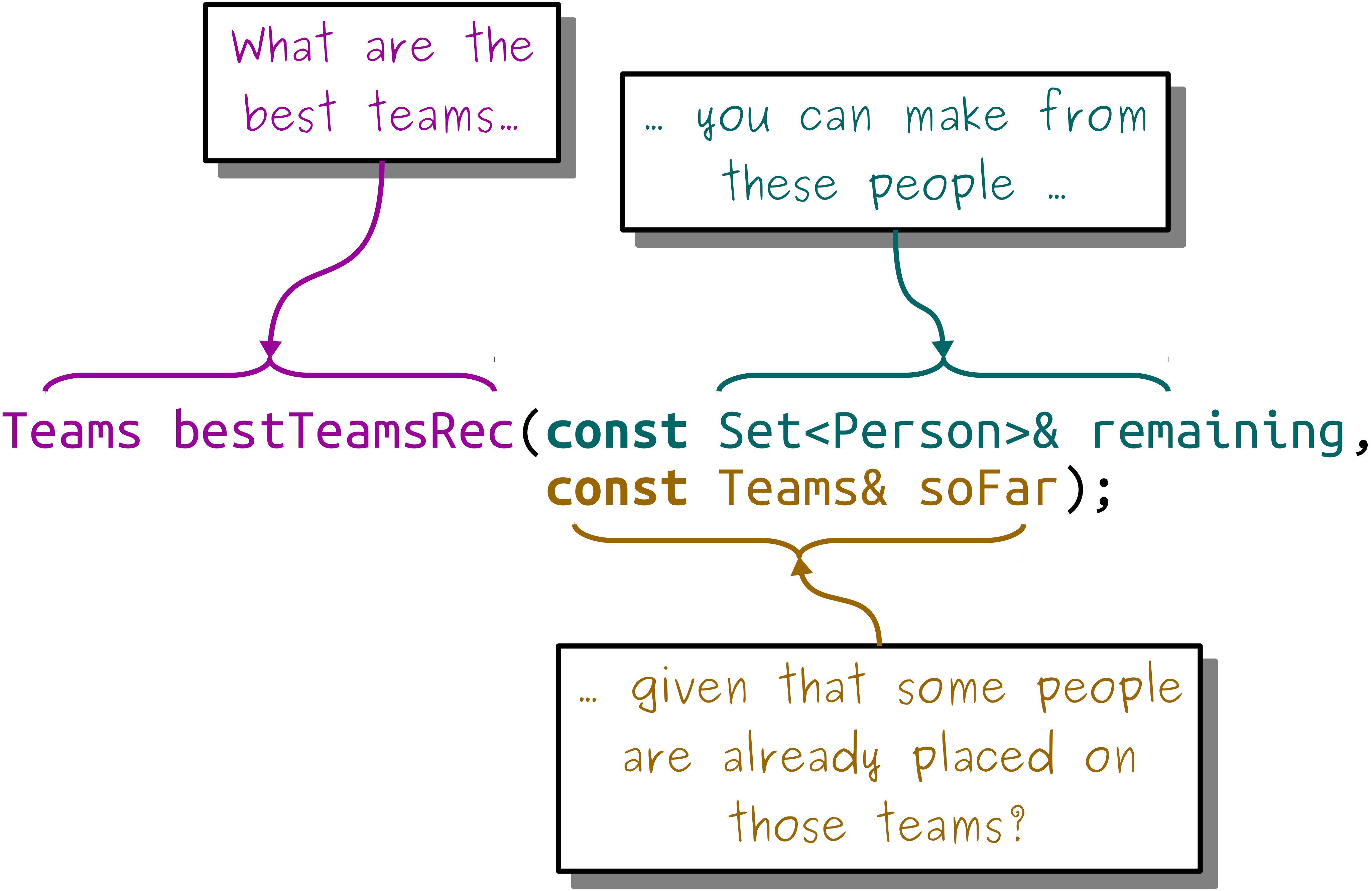
    if (imbalanceOf(best1) < imbalanceOf(best2)) {
      return best1;
    } else {
      return best2;
    }
  }
}
```

This just kicks the answer one level higher up. It doesn’t end the recursive exploration.

Perspective 2: ***Think Abstractly***

```
Teams bestTeamsRec(const Set<Person>& remaining,
                   const Teams& soFar);
```

Without looking at the implementation, can you explain what this function does?



What are the best teams…

… you can make from these people …

```
Teams bestTeamsRec(const Set<Person>& remaining,
                   const Teams& soFar);
```

… given that some people are already placed on those teams?

```
Teams bestTeamsRec(const Set<Person>& remaining,
                   const Teams& soFar) {
  if (remaining.isEmpty()) {
    return soFar;
  } else {
    Person curr = remaining.first();

    /* Option 1: Put this person on Team 1. */
    Teams best1 = bestTeamsRec(remaining - curr,
                               { soFar.one + curr, soFar.two });

    /* Option 2: Put this person on Team 2. */
    Teams best2 = bestTeamsRec(remaining - curr,
                               { soFar.one, soFar.two + curr });

    if (imbalanceOf(best1) < imbalanceOf(best2)) {
      return best1;
    } else {
      return best2;
    }
  }
}
```

```
Teams bestTeamsRec(const Set<Person>& remaining,
                   const Teams& soFar) {
  if (remaining.isEmpty()) {
    return soFar;
  } else {
    Person curr = remaining.first();

    /* Option 1: Put this person on T
    Teams best1 = bestTeamsRec(remain
                               { soFa                            ;

    /* Option 2: Put this person on Team 2. */
    Teams best2 = bestTeamsRec(remaining - curr,
                               { soFar.one, soFar.two + curr });

    if (imbalanceOf(best1) < imbalanceOf(best2)) {
      return best1;
    } else {
      return best2;
    }
  }
}
```

What are the best teams you can form if everyone is already assigned to a team?

Thinking Recursively

- When writing recursive functions, always ask yourself this question:

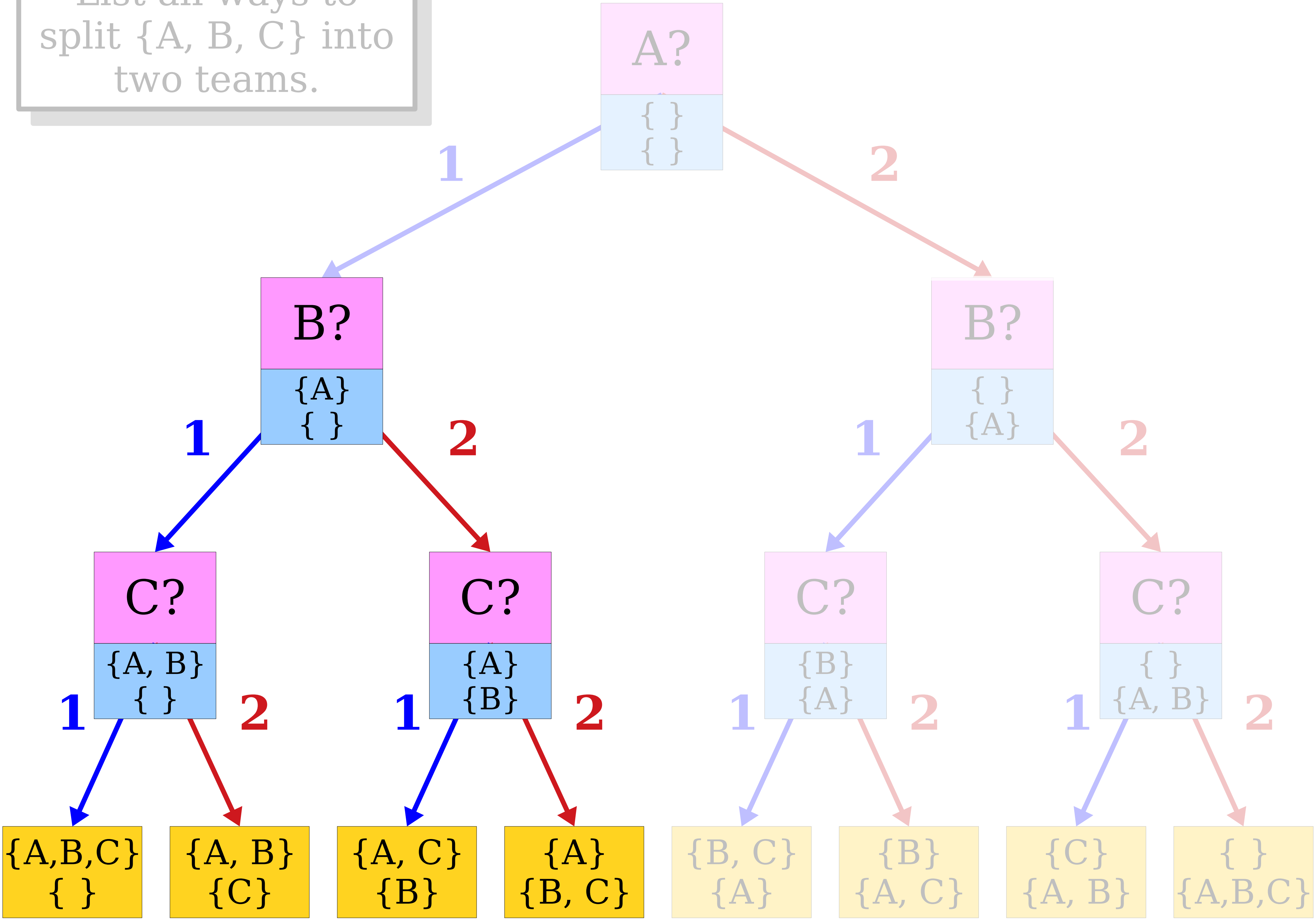
Can you describe what the function does purely by reading the signature?

- If so, great! That will guide your coding effort.
- If not, pause and think it through a bit. It’s hard to write a function correctly when you can’t explain what it’s supposed to do!

Question 2:

We’re generating duplicate solutions!
How do we fix that?

List all ways to split {A, B, C} into two teams.



Breaking Symmetry

- In many enumeration and optimization problems, there may be many solutions that are equivalent to one another.
  - Here, swapping Team 1 and Team 2 doesn't change anything.
- In some cases, you can break symmetries by committing to some fixed decision up front.
  - Here, forcing the first person to be on Team 1.
- In others, you'll need to rethink your recursive approach.
  - For example, finding a different decision tree.

Question 3:

That code is really long! Can we make it shorter and prettier?

The Wonderful `auto` Keyword

- In C++, you need to assign a type to each variable.
- In the case where you define a variable and give it an initial value, you can write `auto` instead of the name of a type to have C++ figure out the type for you.

```
auto variable = expression;
```

- Use this when you are declaring a variable whose value can unambiguously be determined from the expression initializing it.

The Wonderful **?:** Operator

- In C++, the ***ternary conditional operator*** can be used to select one of two expressions.
- The syntax is

  expression? *if-true* : *if-false*

- This shows up all the time in recursive optimization problems.

Question 4:

Why do we even need recursion at all here? Can't we just iterate over the combinations and take the best?

Great exercise: Solve this problem without using recursion. How will you enumerate all the possible ways of splitting folks into teams?

Time-Out for Announcements!

Research Office Hours

- Two of our amazing PhD students – including one who’s a former section leader – are holding research office hours twice a week.
- Have questions about what it’s like to do research in CS? Head to ***Gates B02*** on

  Mondays, 1:30PM – 2:30PM

  or

  Fridays, 10:00AM – 11:00AM.

incluDE

- Stanford’s Society of Women Engineers (SWE) is hosting a conference on diversity in engineering.
- Includes a keynote by the Provost and a pretty impressive panel!
- It’s this upcoming Saturday, February 2nd from 10:00AM – 3:00PM in the d.school.
- RSVP using **this link**.

```
lecture.resume();
```

(The old, janky Java way of telling a thread that's been paused to start again. Basically no one uses this syntax any more.)

A Little Word Puzzle

“What nine-letter word can be reduced to a single-letter word one letter at a time by removing letters, leaving it a legal word at each step?”

The Startling Truth

S	T	A	R	T	L	I	N	G

The Startling Truth

S	T	A	R	T	I	N	G

The Startling Truth

S	T	A	R	I	N	G

The Startling Truth

S	T	R	I	N	G

The Startling Truth

S	T	I	N	G

The Startling Truth

S	I	N	G

The Startling Truth

S	I	N

The Startling Truth

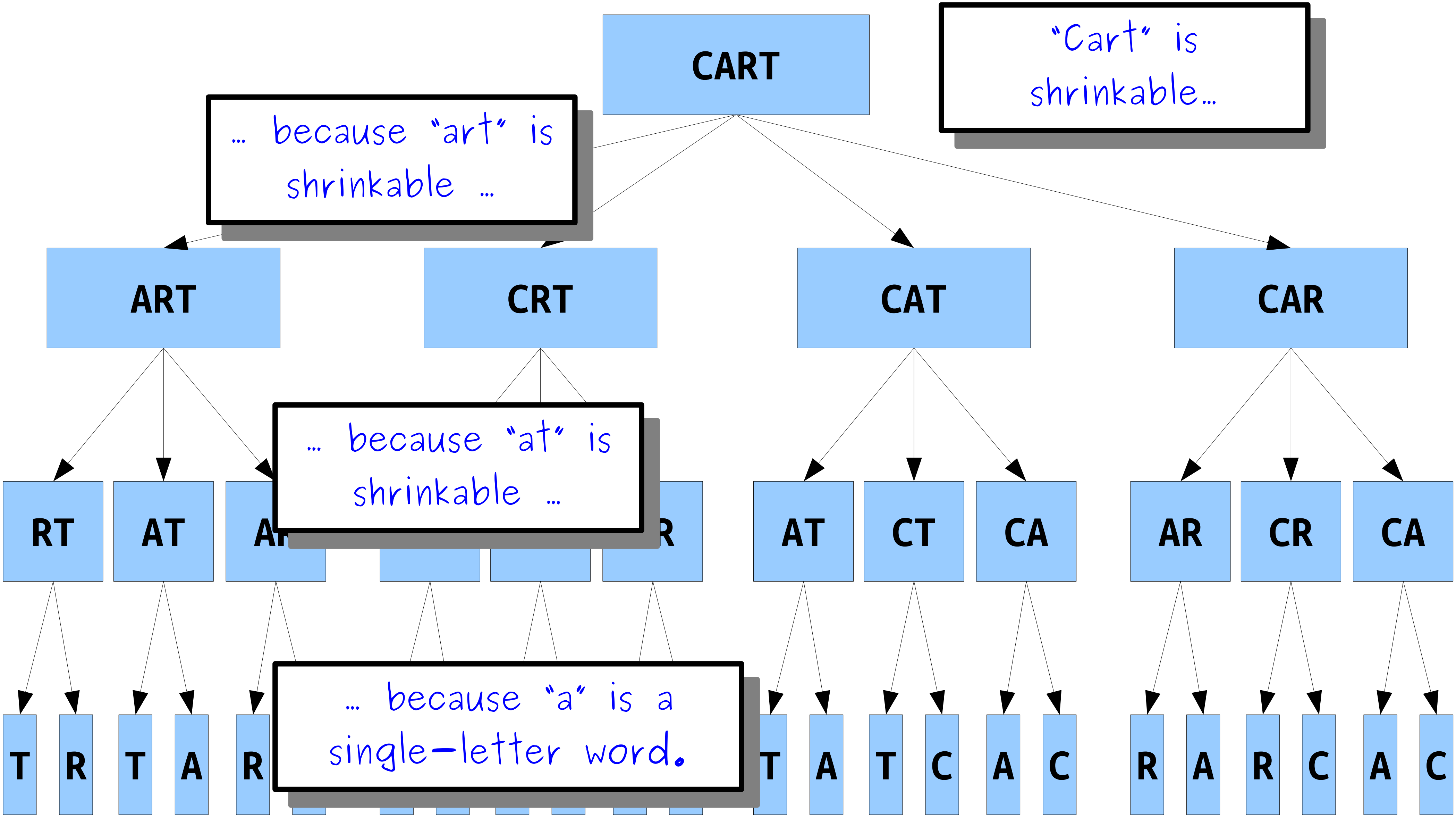
I	N

The Startling Truth

I

Is there ***really*** just one nine-letter word with this property?

All Possible Paths



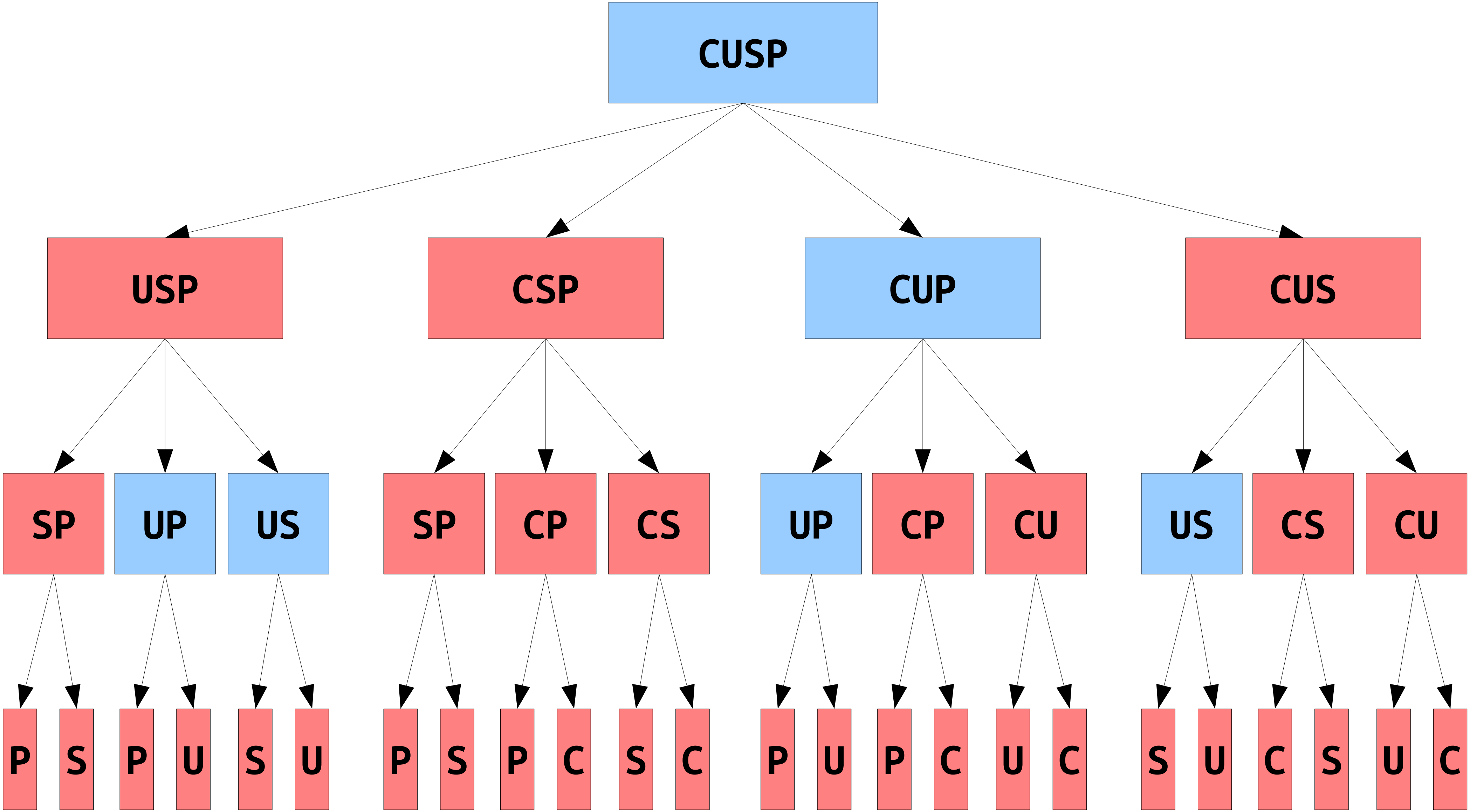
"Cart" is shrinkable…

… because "art" is shrinkable …

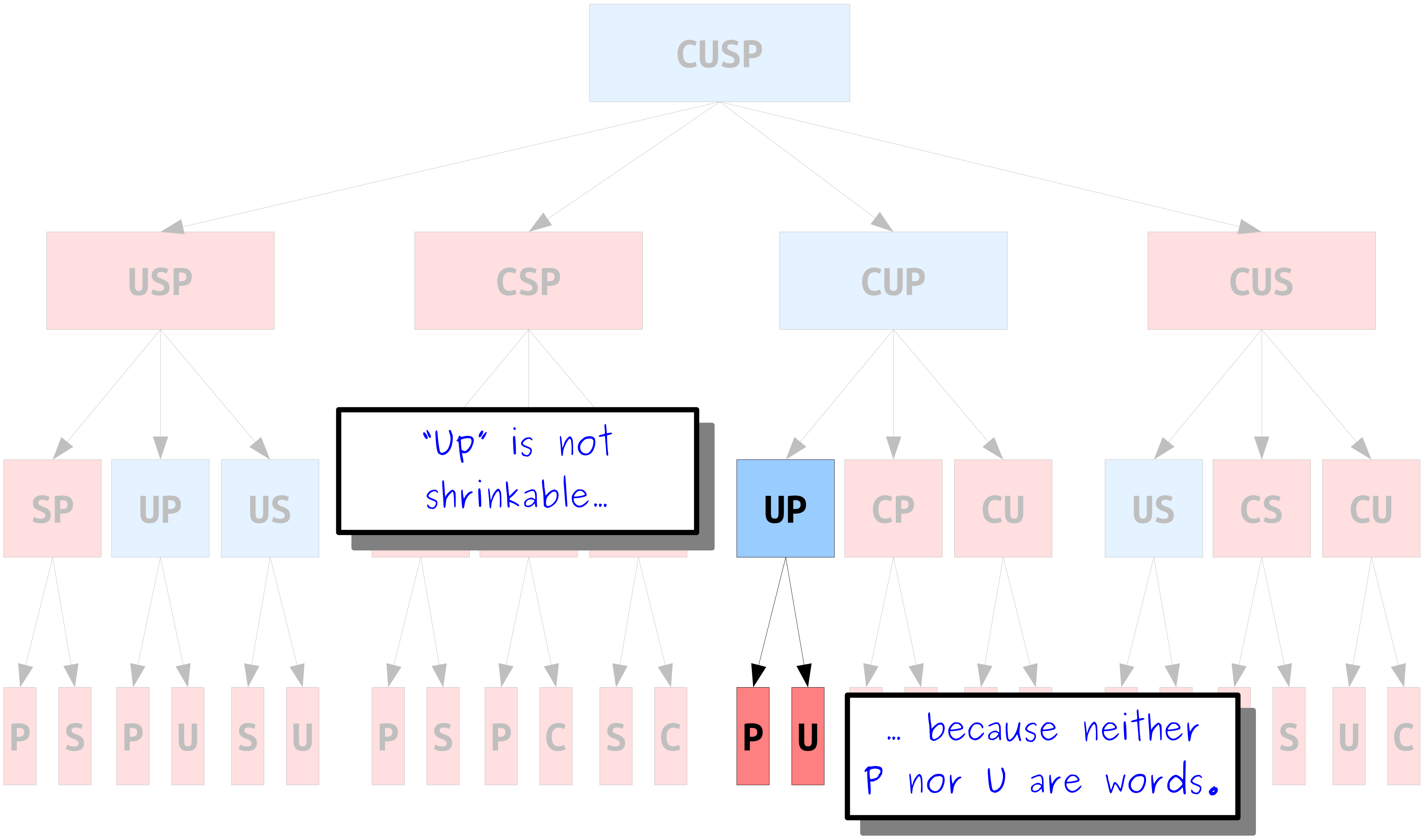
… because "at" is shrinkable …

… because "a" is a single-letter word.

All Possible Paths



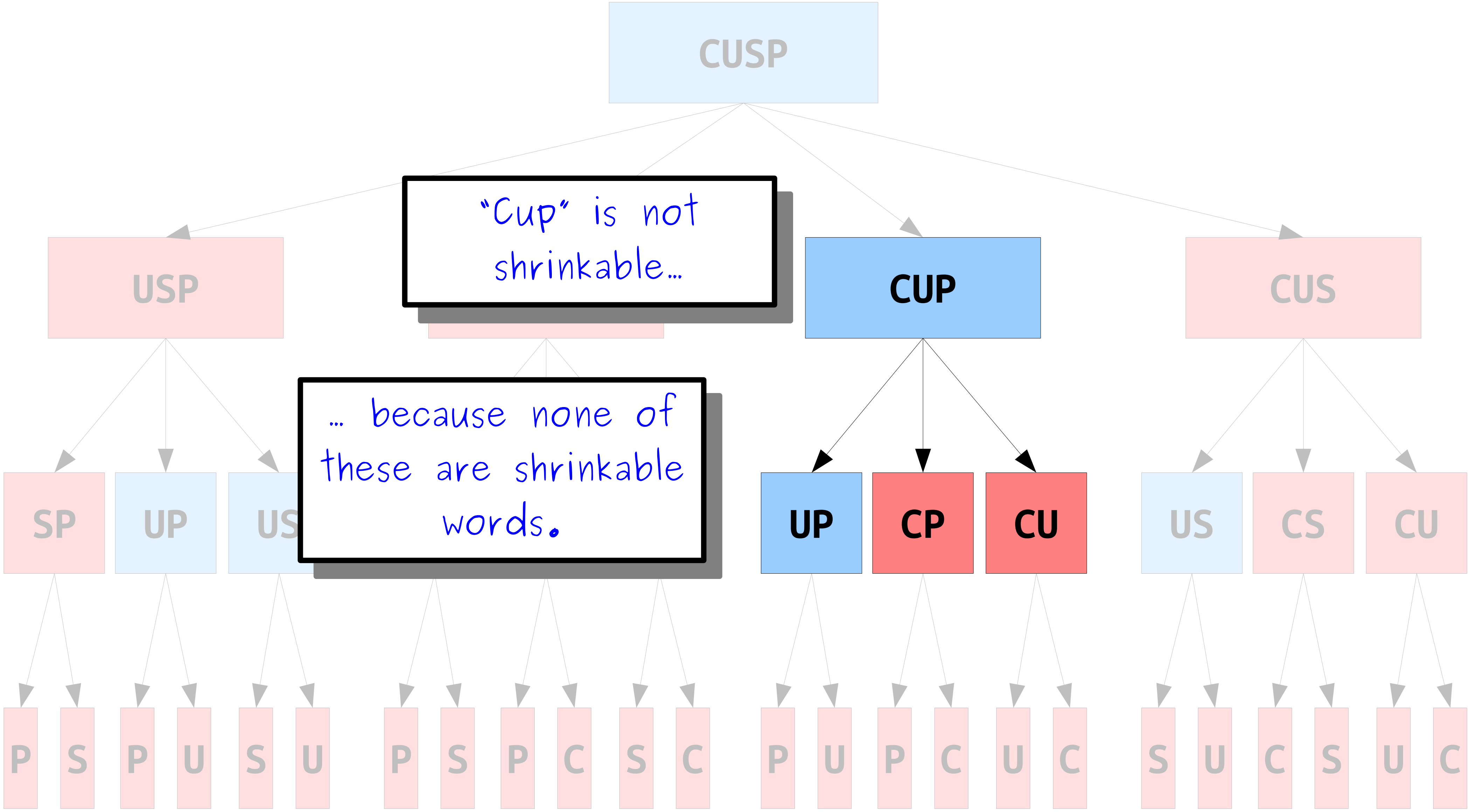
All Possible Paths



"Up" is not shrinkable...

... because neither P nor U are words.

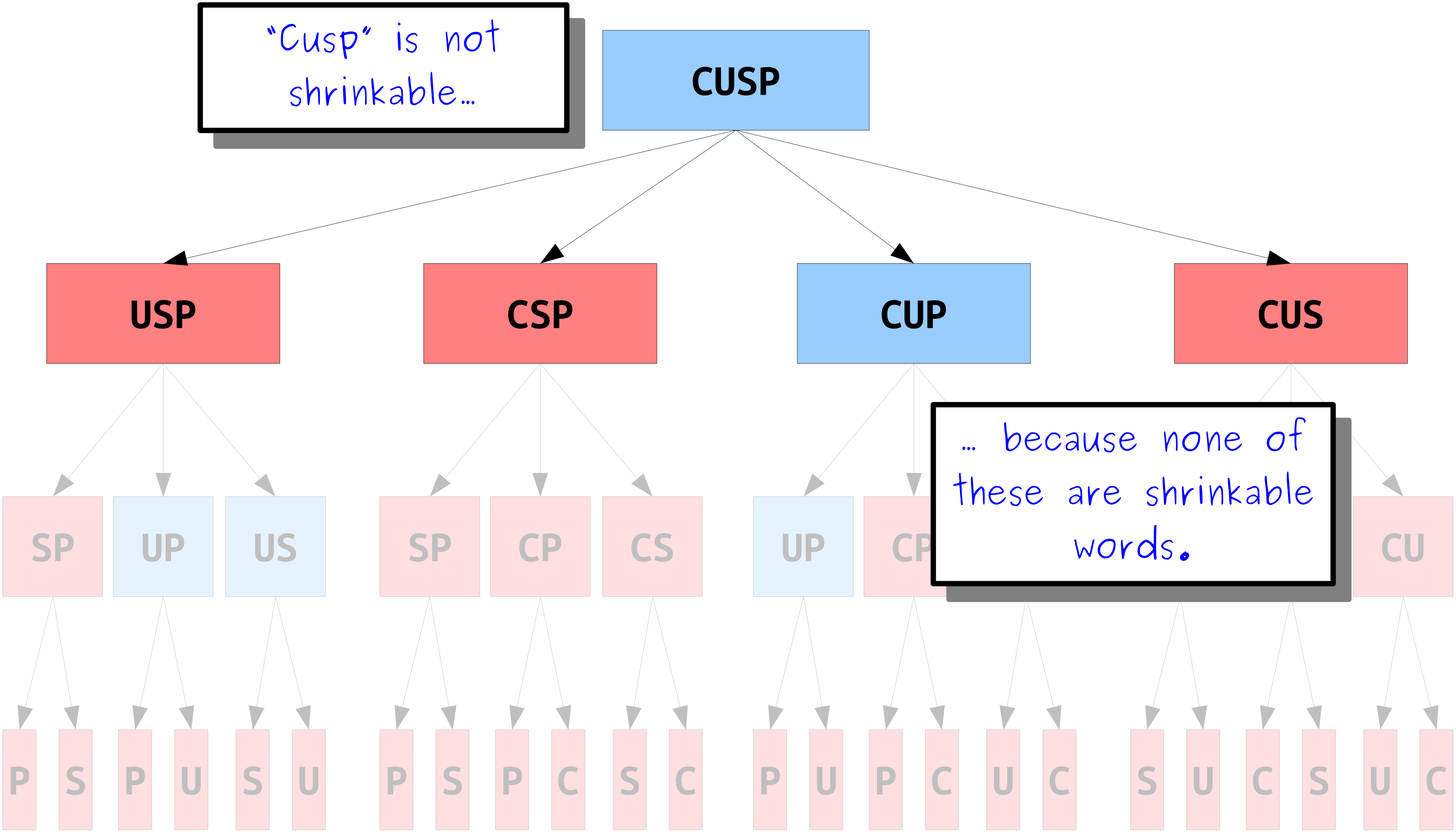
All Possible Paths



“Cup” is not shrinkable…

… because none of these are shrinkable words.

All Possible Paths



"Cusp" is not shrinkable…

… because none of these are shrinkable words.

Shrinkable Words

- Let's define a ***shrinkable word*** as a word that can be reduced down to one letter by removing one character at a time, leaving a word at each step.
- ***Base Cases*:**
  - A string that is not a word is not a shrinkable word.
  - Any single-letter word is shrinkable (A, I, and O).
- ***Recursive Step*:**
  - A multi-letter word is shrinkable if you can remove a letter to form a shrinkable word.
  - A multi-letter word is not shrinkable if no matter what letter you remove, it’s not shrinkable.

Your Action Items

- ***Read Chapter 9 of the textbook.***
  - There's tons of cool backtracking examples there, and it will help you prep for Friday.
- ***Keep working on Assignment 3.***
  - If you're following our timetable, you should be done with the Sierpinski triangle at this point and have started Human Pyramids.
  - Aim to complete Human Pyramids and to have started work on Shift Scheduling by Friday.

Next Time

- ***More Backtracking***
  - Techniques in searching for feasibility.
- ***Closing Thoughts on Recursion***
  - It’ll come back, but we’re going to focus on other things for a while!