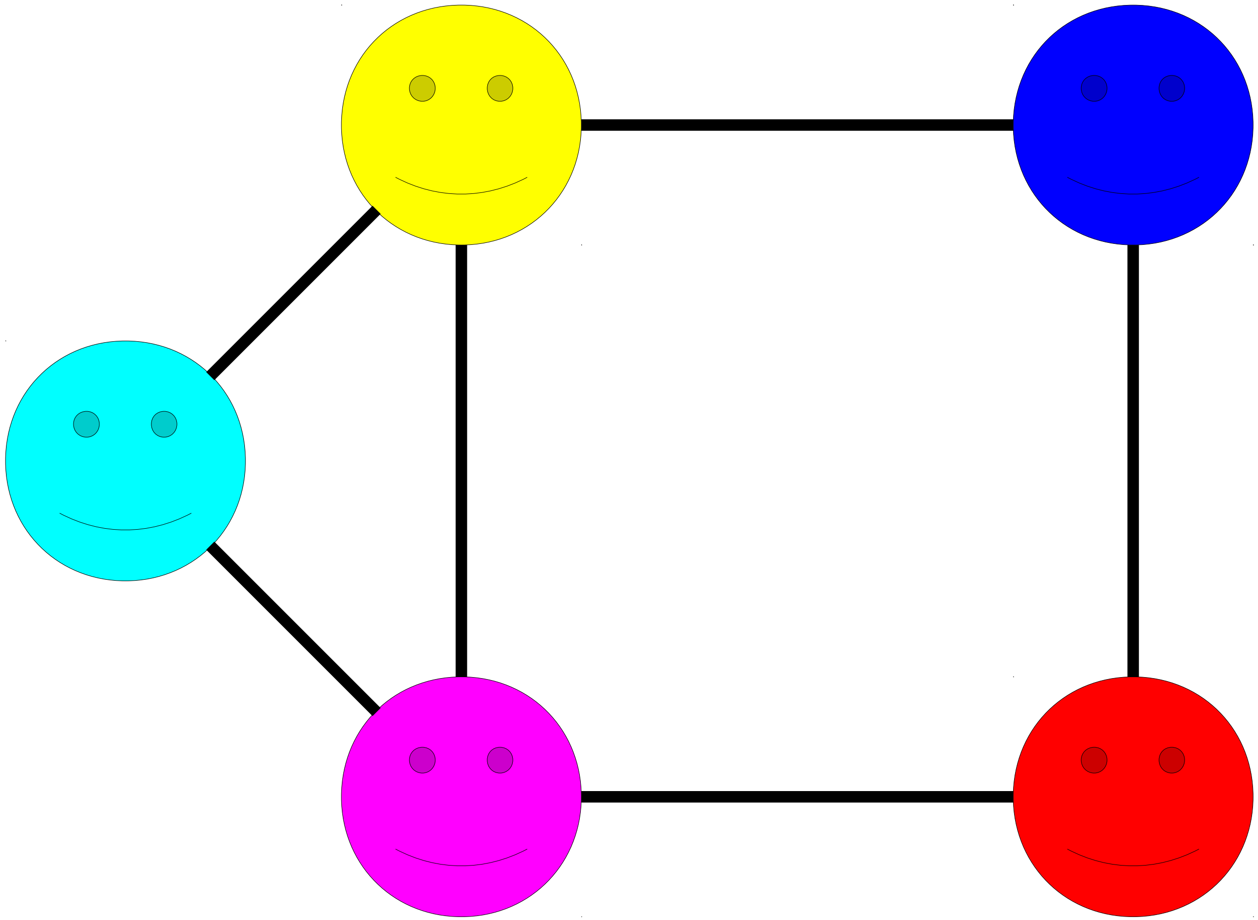
Shortest Paths

Part One

Recap from Last Time

A ***graph*** is a mathematical structure for representing relationships.



A graph consists of a set of ***nodes*** connected by ***edges***.

Breadth-First Search

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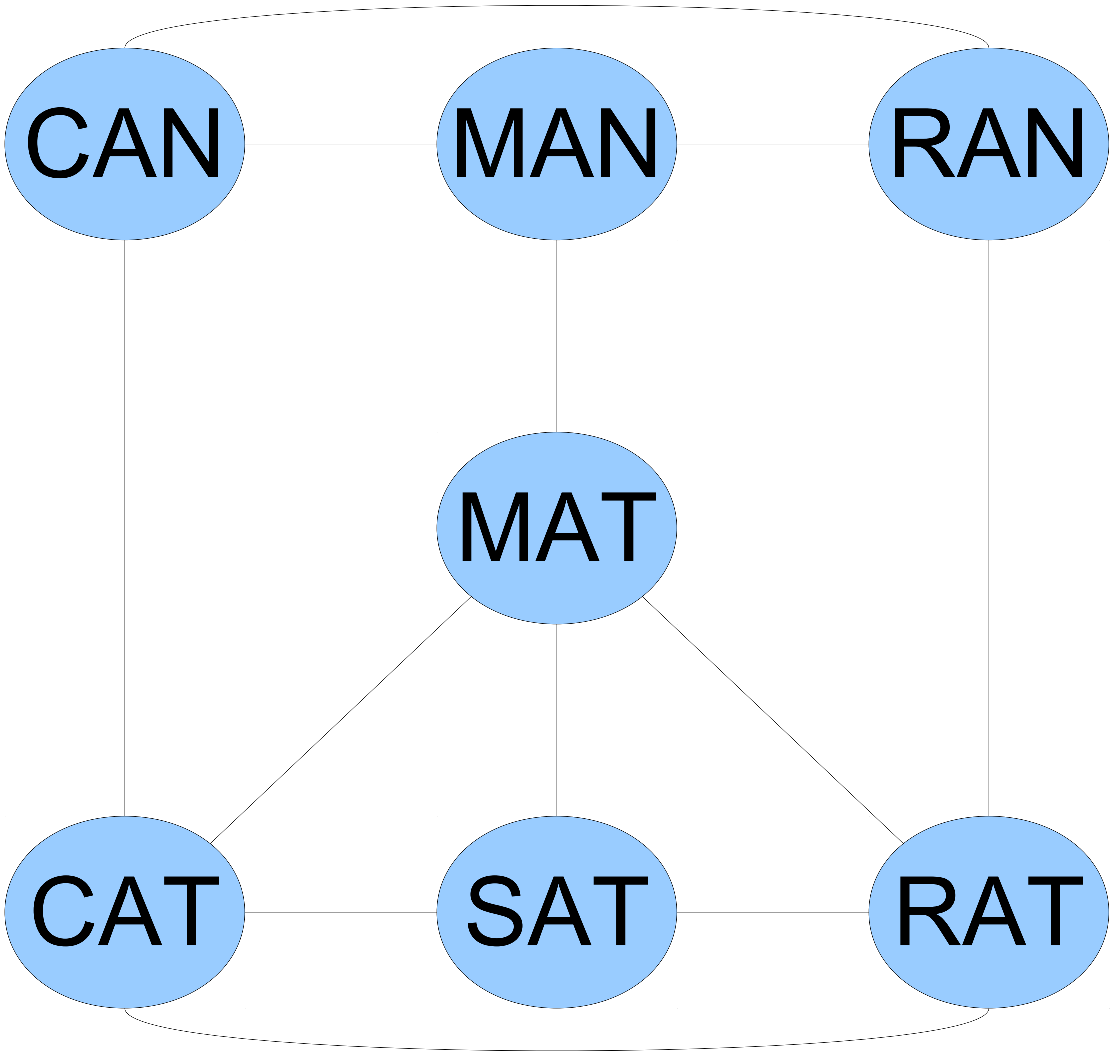


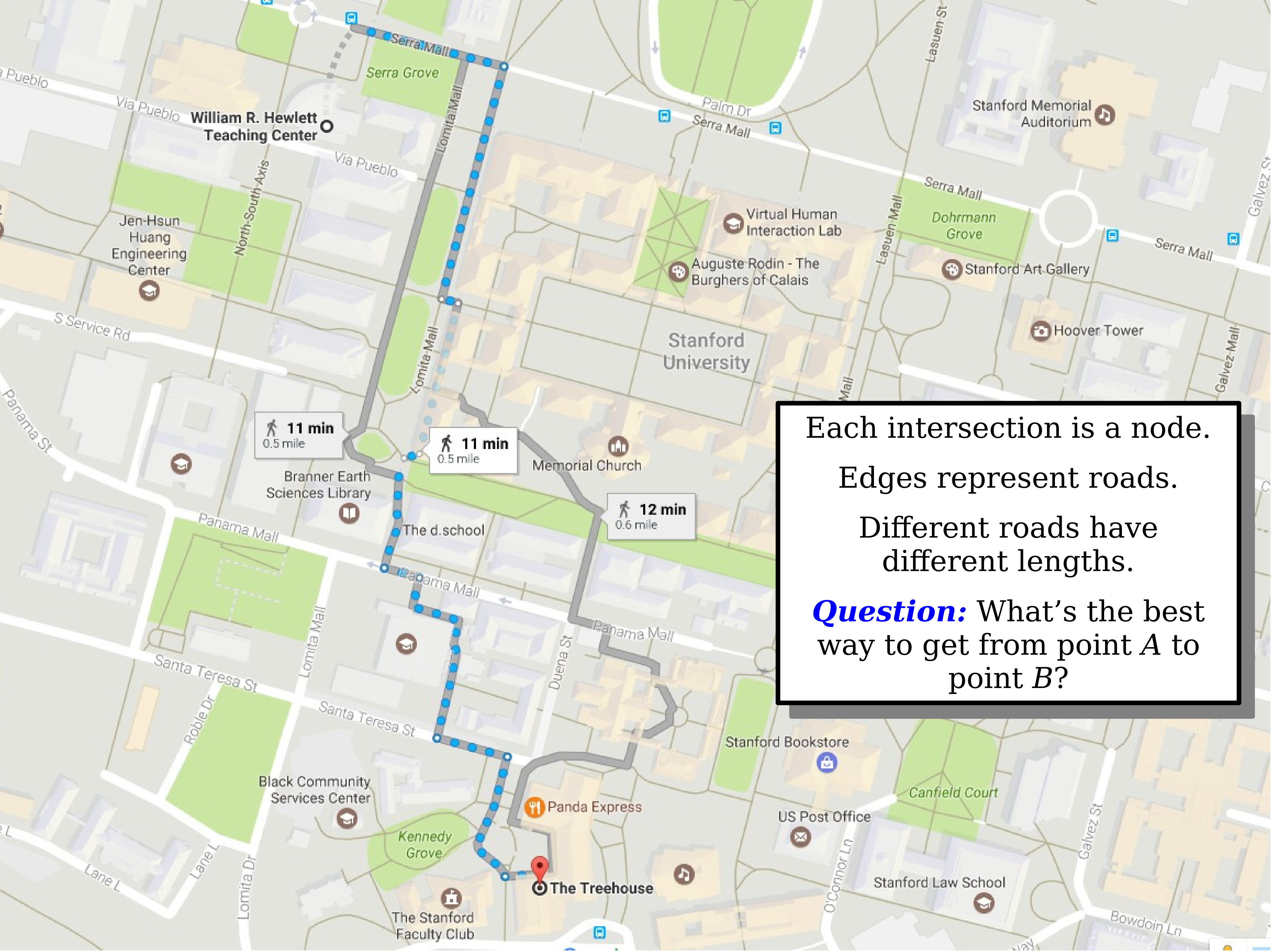
BFS Pseudocode

```
breadth-first-search() {
  make a queue of nodes.
  enqueue start node.
  color the start node yellow.

  while (the queue is not empty) {
    dequeue a node from the queue.
    color that node green.

    for (each neighboring node) {
      if (that node is gray) {
        color the node yellow.
        enqueue it.
      }
    }
  }
}
```

The Limits of Breadth-First Search



Each intersection is a node.

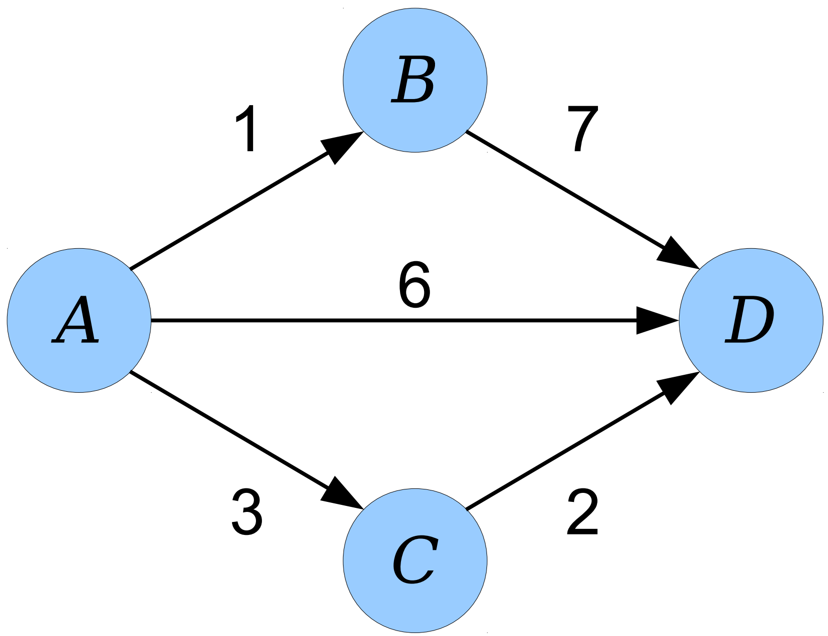
Edges represent roads.

Different roads have different lengths.

Question: What's the best way to get from point A to point B?

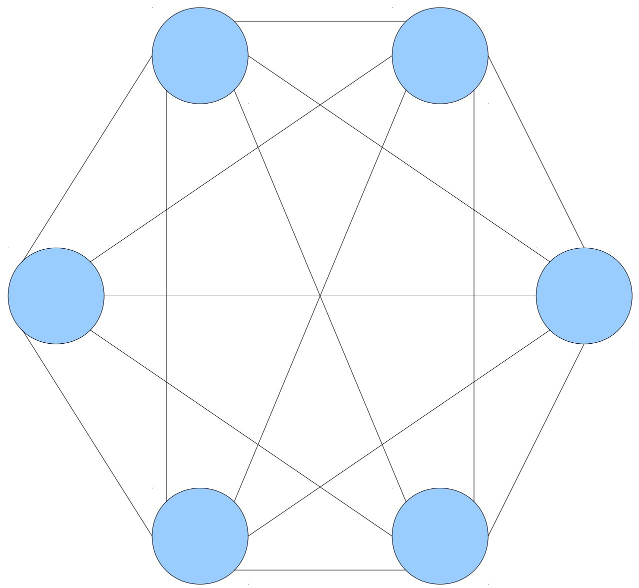
The Model

- We have a graph in which each edge has a nonnegative ***cost*** or ***weight*** associated with it.
- We want to find the lowest-cost path from point A to point B.
- BFS does not take edge weights into account.
- How might we go about solving this problem?



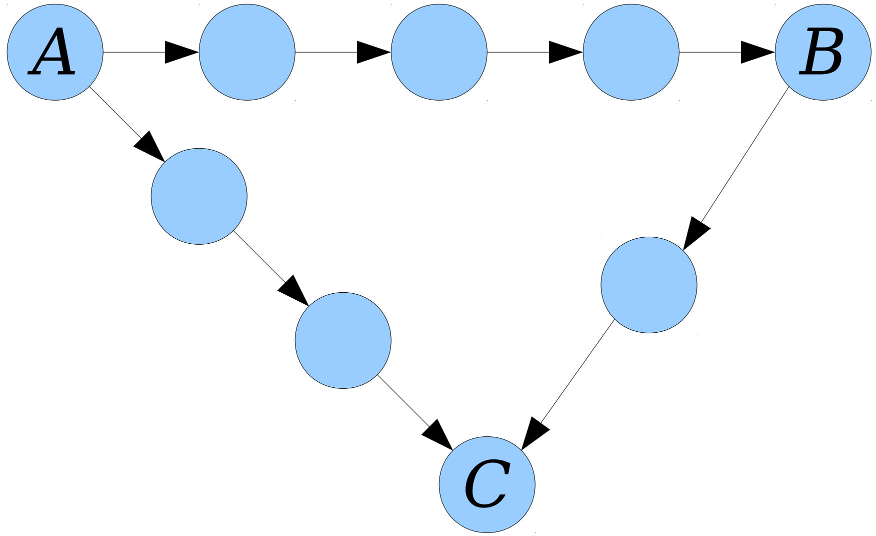
Option 1: Brute-Force!

- We could conceivably solve this problem using brute force and a backtracking recursion.
- ***Problem:*** There can be a *lot* of different paths in a graph!
- This is way too inefficient to use in practice.



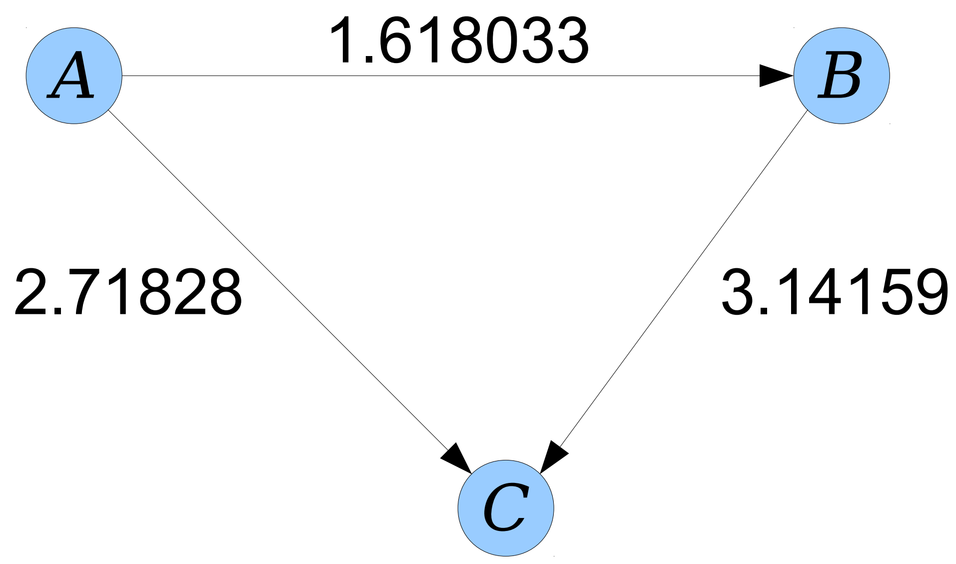
Option 2: Expand the Graph

- BFS works in the case where each edge has equal weight.
- ***Idea:*** What if we split each edge of length k into k smaller edges?



Option 2: Expand the Graph

- BFS works in the case where each edge has equal weight.
- ***Idea:*** What if we split each edge of length k into k smaller edges?
- What if there are fractional edges? Or large weights?



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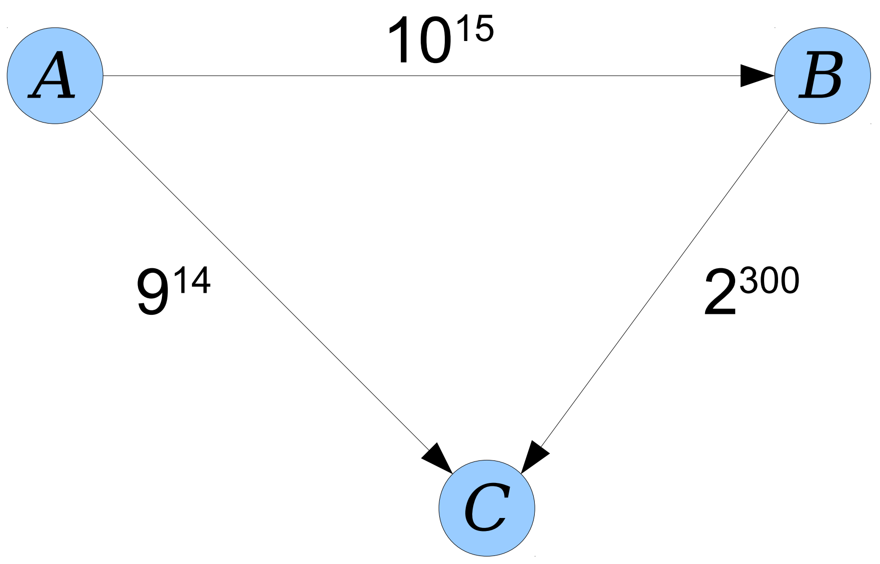
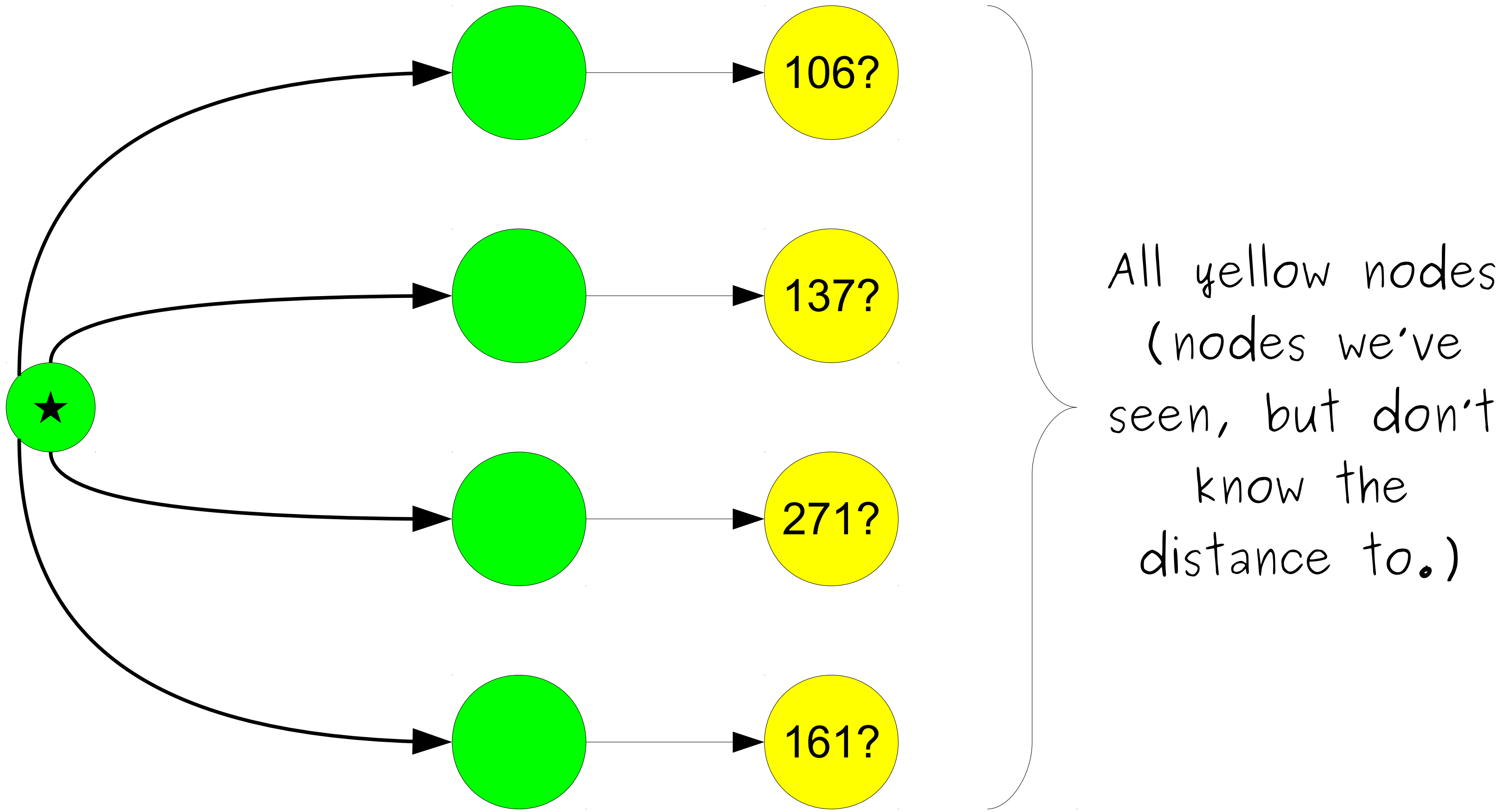
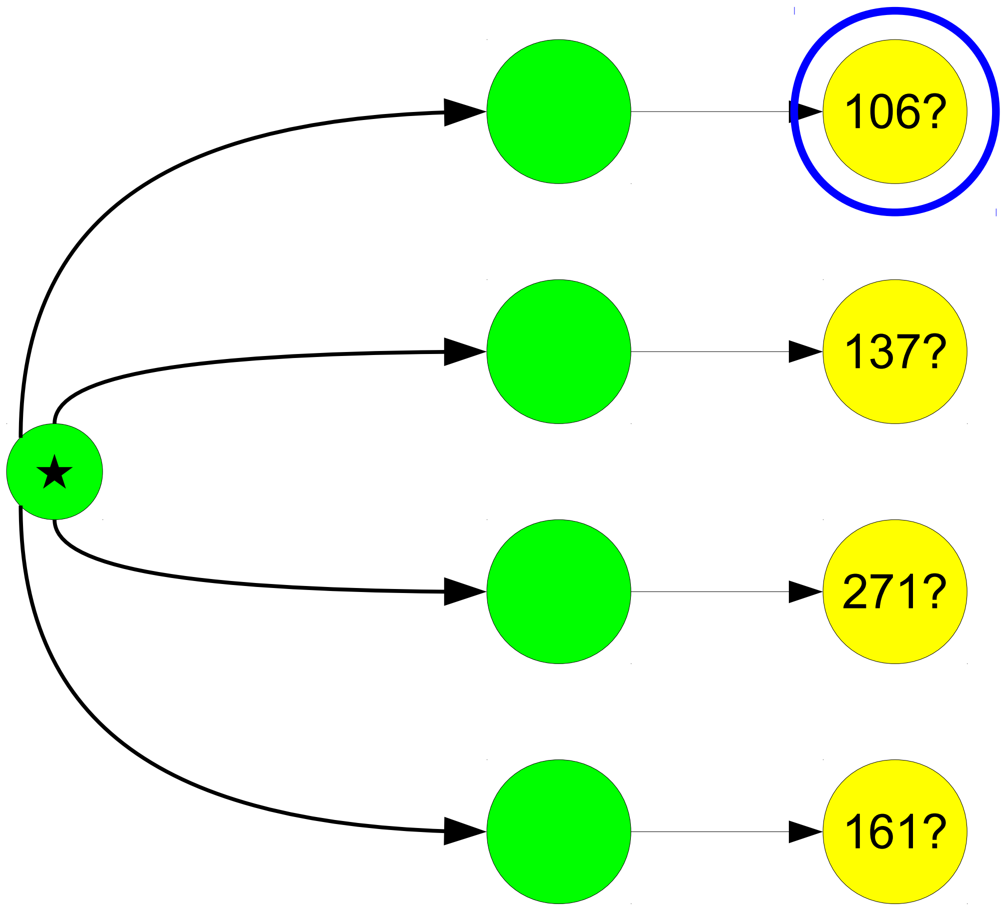


Option 3: Look at the problem more closely

The Pattern


★
106?
137?
271?
161?
All yellow nodes (nodes we've seen, but don't know the distance to.)

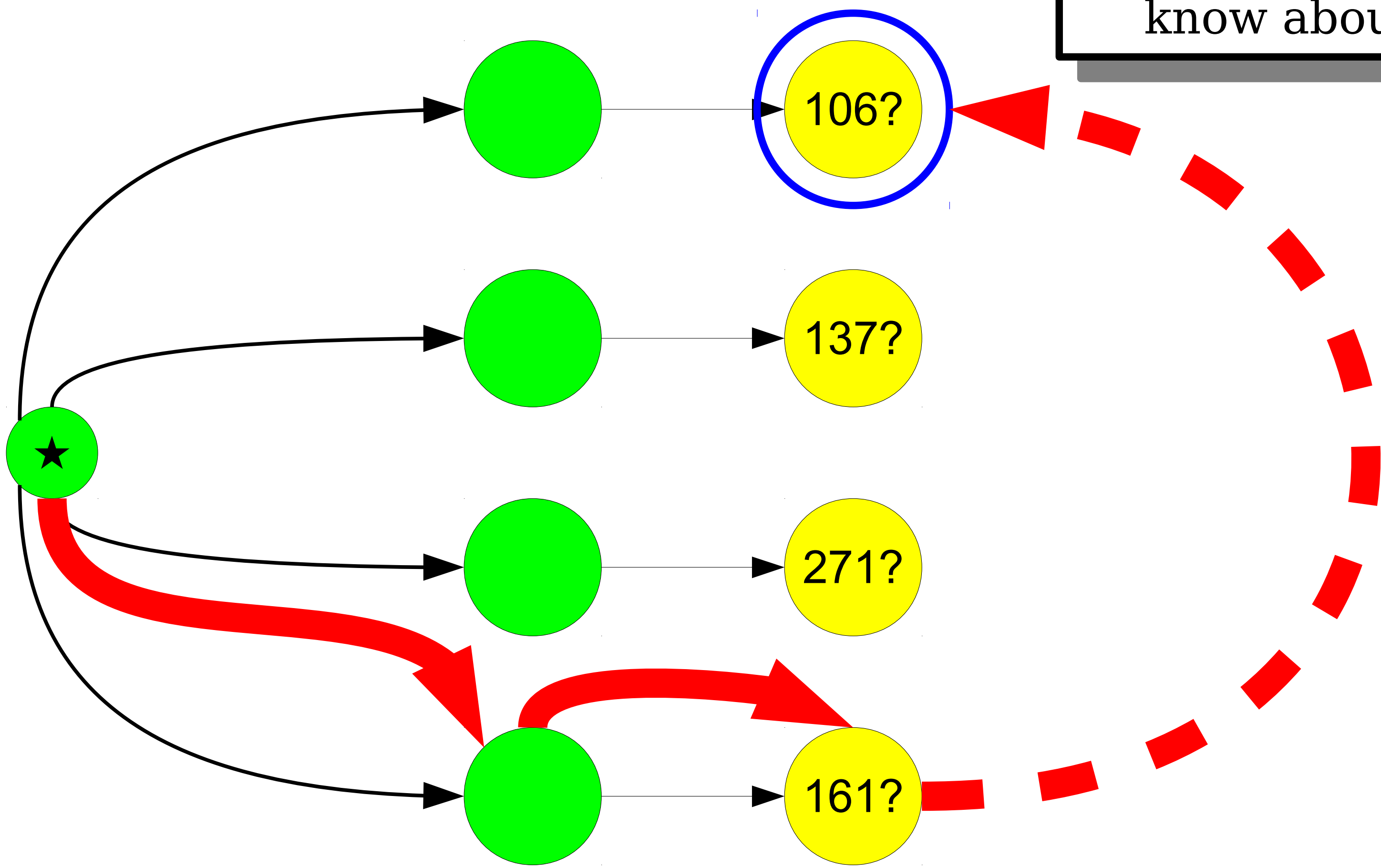
The Pattern



Look at the lowest-cost yellow node.

The Pattern

No other path to this node can be better than the one we already know about!



At a Glance

- The approach suggested here gives rise to ***Dijkstra's algorithm***, a fast, powerful, and famous algorithm for computing shortest paths.
- ***Key idea:*** As in BFS, split nodes into
  - ***gray nodes*** we haven't seen,
  - ***yellow nodes*** that are on the frontier, and
  - ***green nodes*** we have the best path to,

  then repeatedly turn the lowest-cost yellow node into a green node.

Implementing Dijkstra’s Algorithm

The Finished Product

```
dijkstra's-algorithm() {
  make a priority queue of nodes.
  enqueue start node at distance 0.
  color the start node yellow.

  while (the queue is not empty) {
    dequeue a node from the queue.
    if (that node isn't green) {
      color that node green.

      for (each neighboring node) {
        if (that node is not green) {
          color the node yellow.
          enqueue it at the new distance.
        }
      }
    }
  }
}
```

Use a priority queue rather than a standard queue to sort by distances, not number of hops.

The Finished Product

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  make a priority queue of nodes.
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  while (the queue is not empty) {
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    if (that node isn't green) {
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      for (each neighboring node) {
        if (that node is not green) {
          color the node yellow.
          enqueue it at the new distance.
        }
      }
    }
  }
}
```

Allow nodes to be enqueued multiple times. The first time we find the node might not be the best option.

The Finished Product

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  make a priority queue of nodes.
  enqueue start node at distance 0.
  color the start node yellow.

  while (the queue is not empty) {
    dequeue a node from the queue.
    if (that node isn't green) {
      color that node green.

      for (each neighboring node) {
        if (that node is not green) {
          color the node yellow.
          enqueue it at the new distance.
        }
      }
    }
  }
}
```

As a consequence, when dequeuing nodes, make sure we're not visiting something we've already processed.