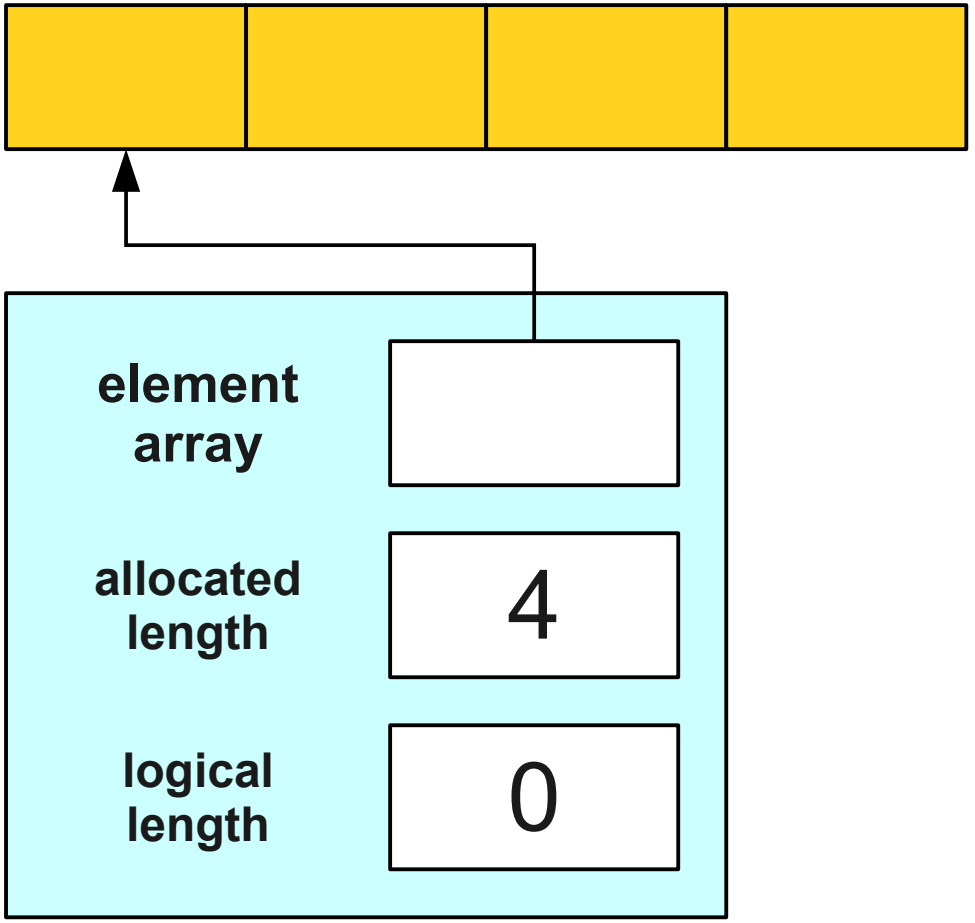
Implementing Abstractions

Part Two

Friday Four Square!
4:15PM, Outside Gates

An Initial Idea



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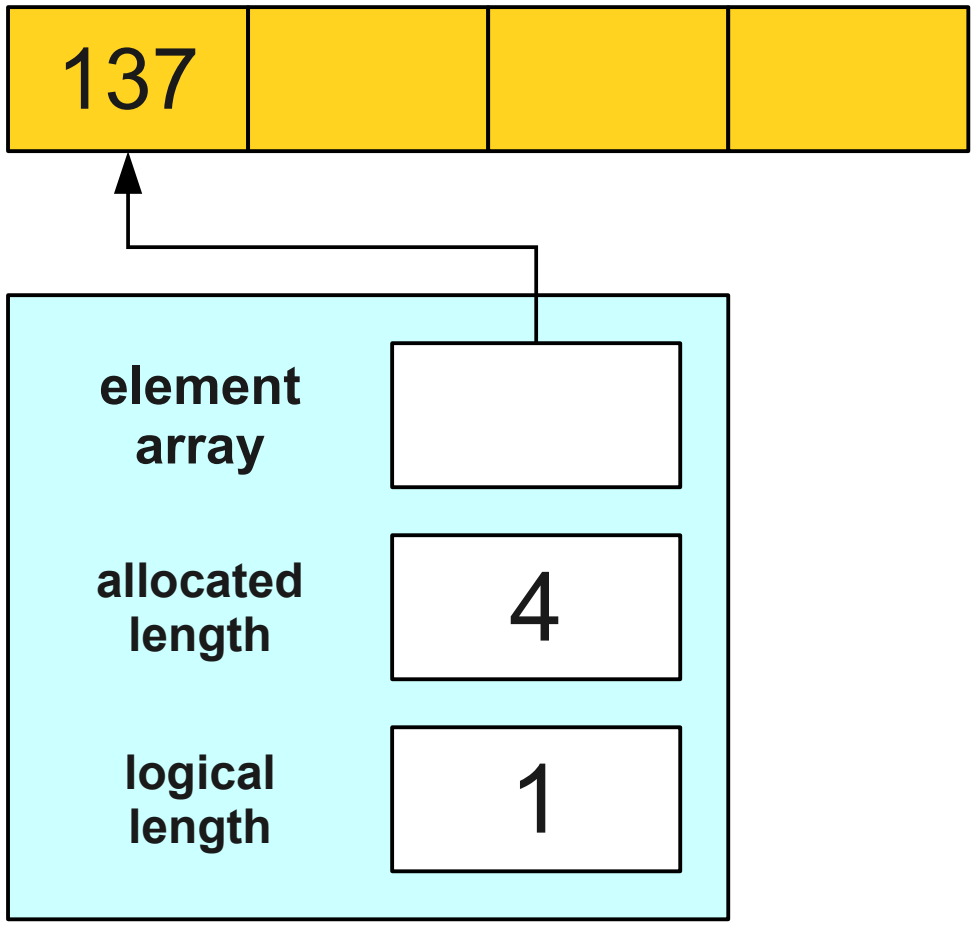

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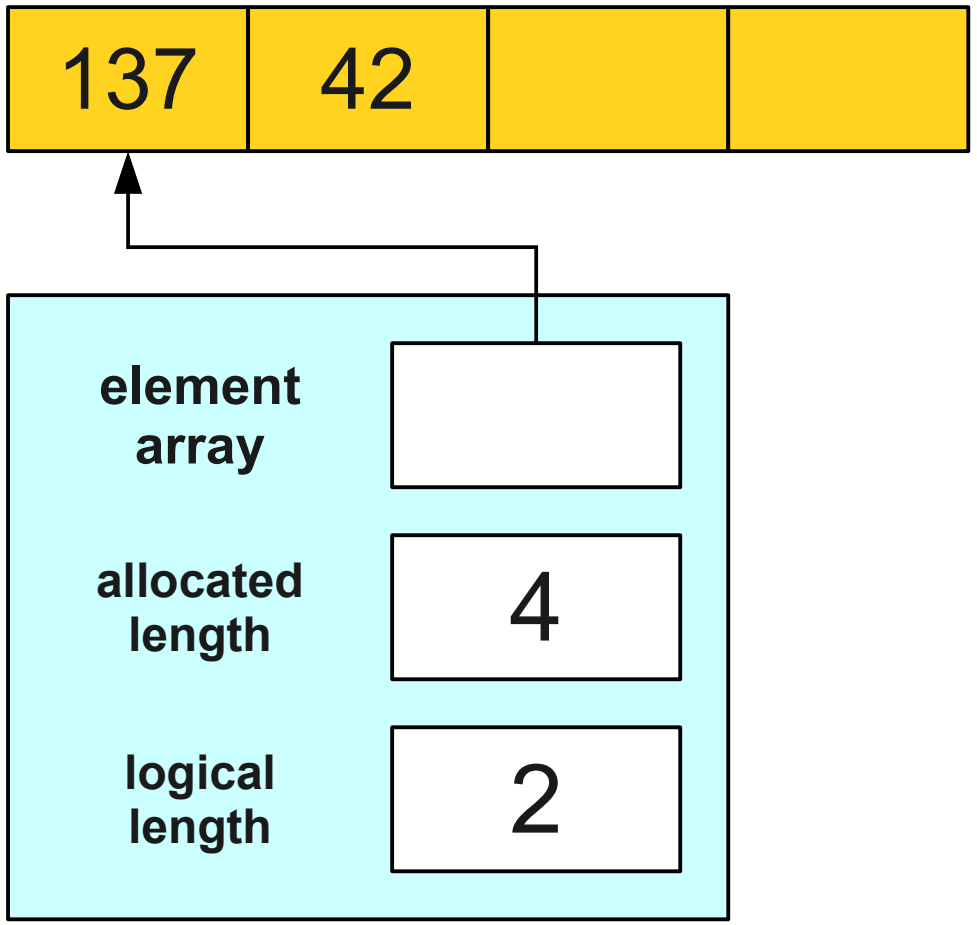

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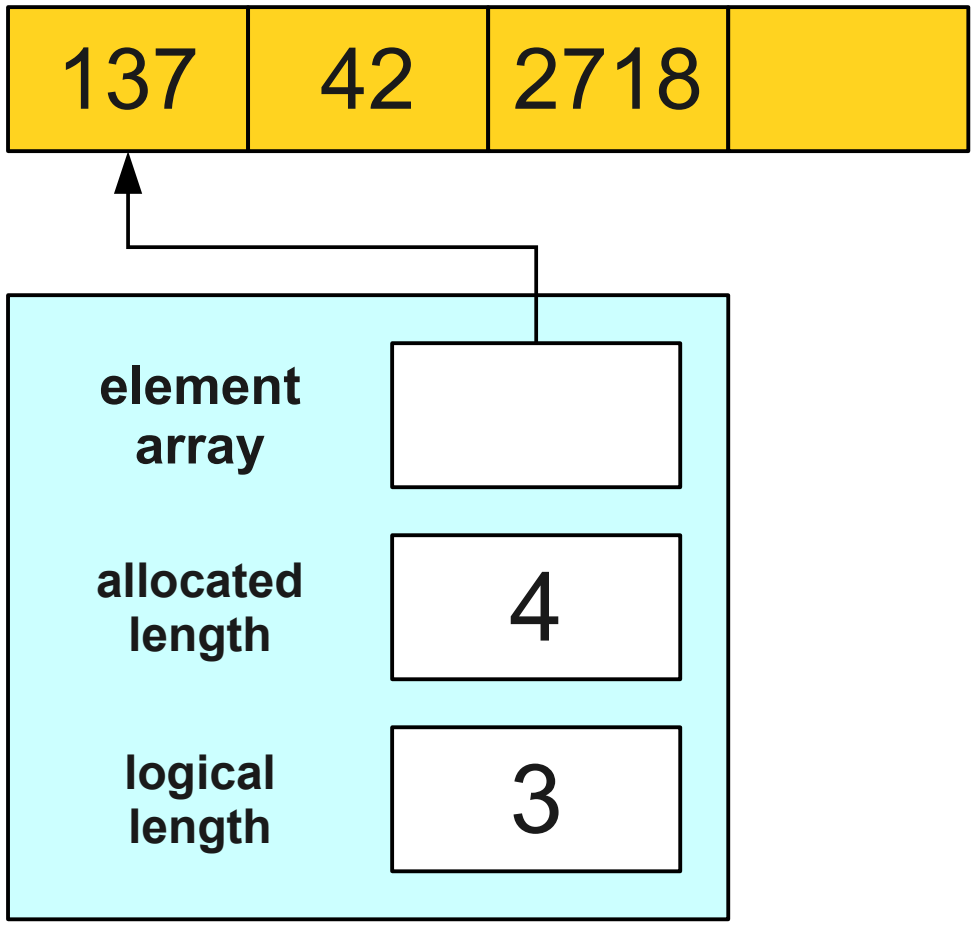

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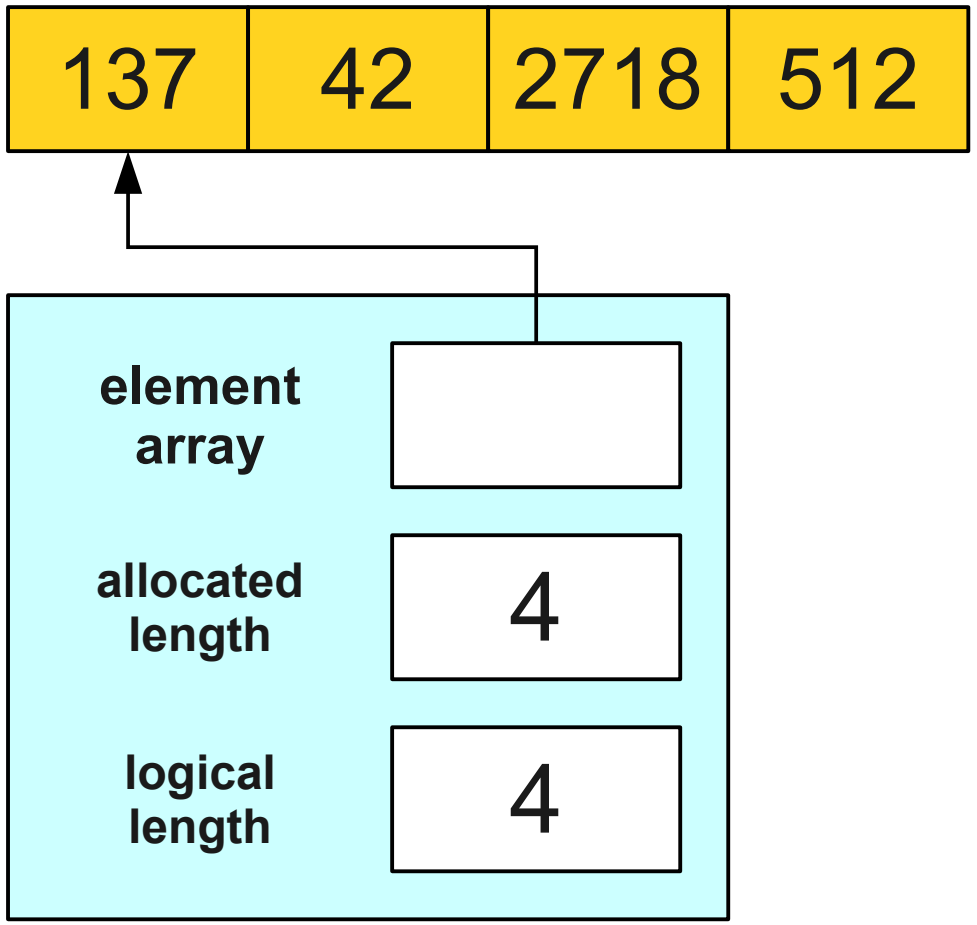

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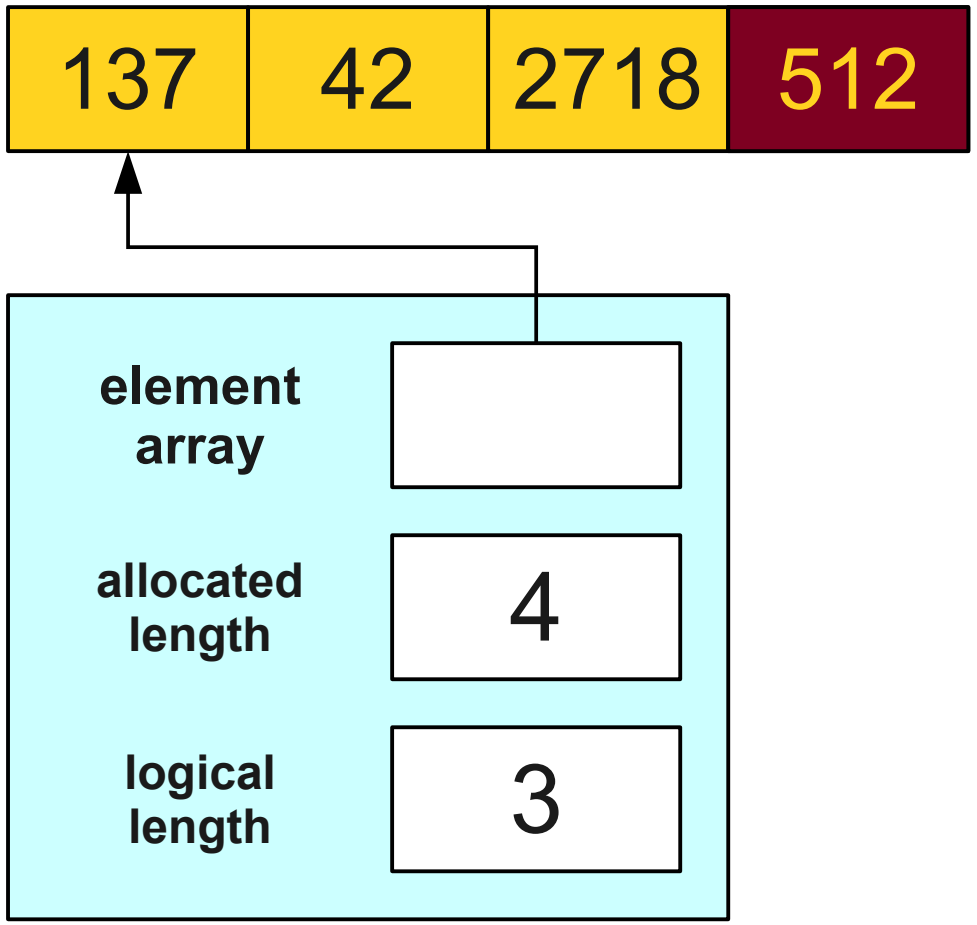

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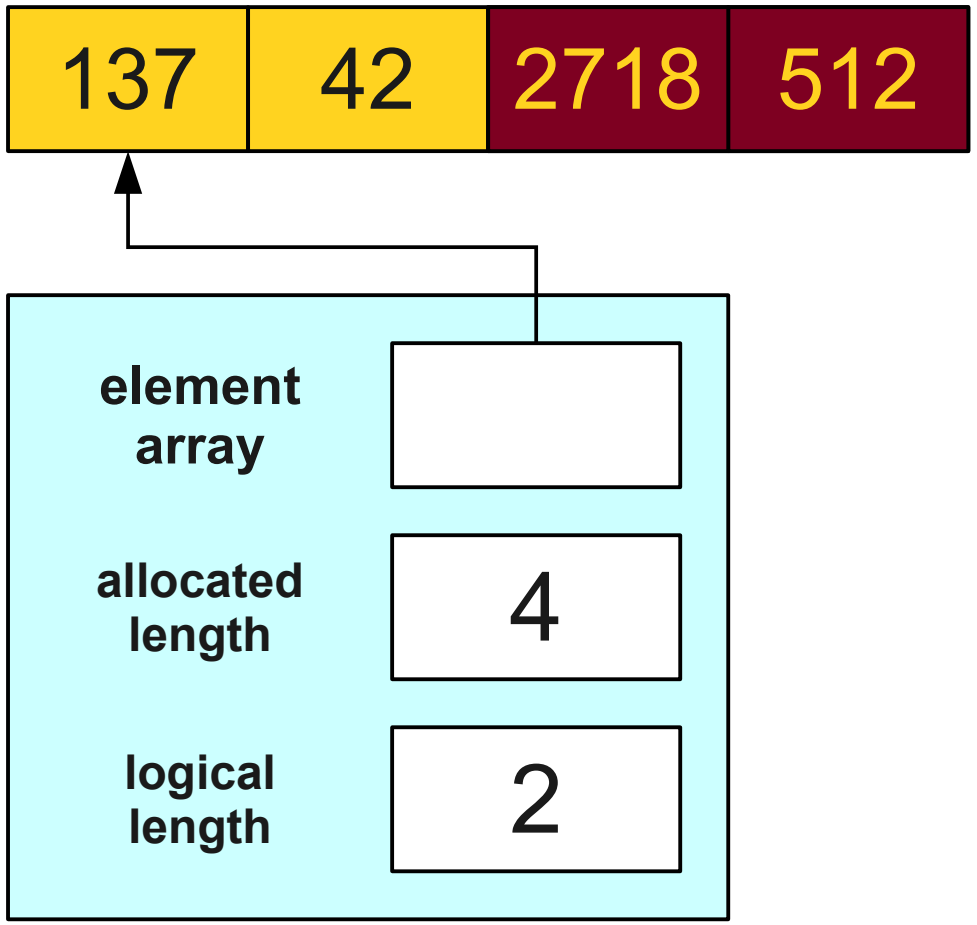

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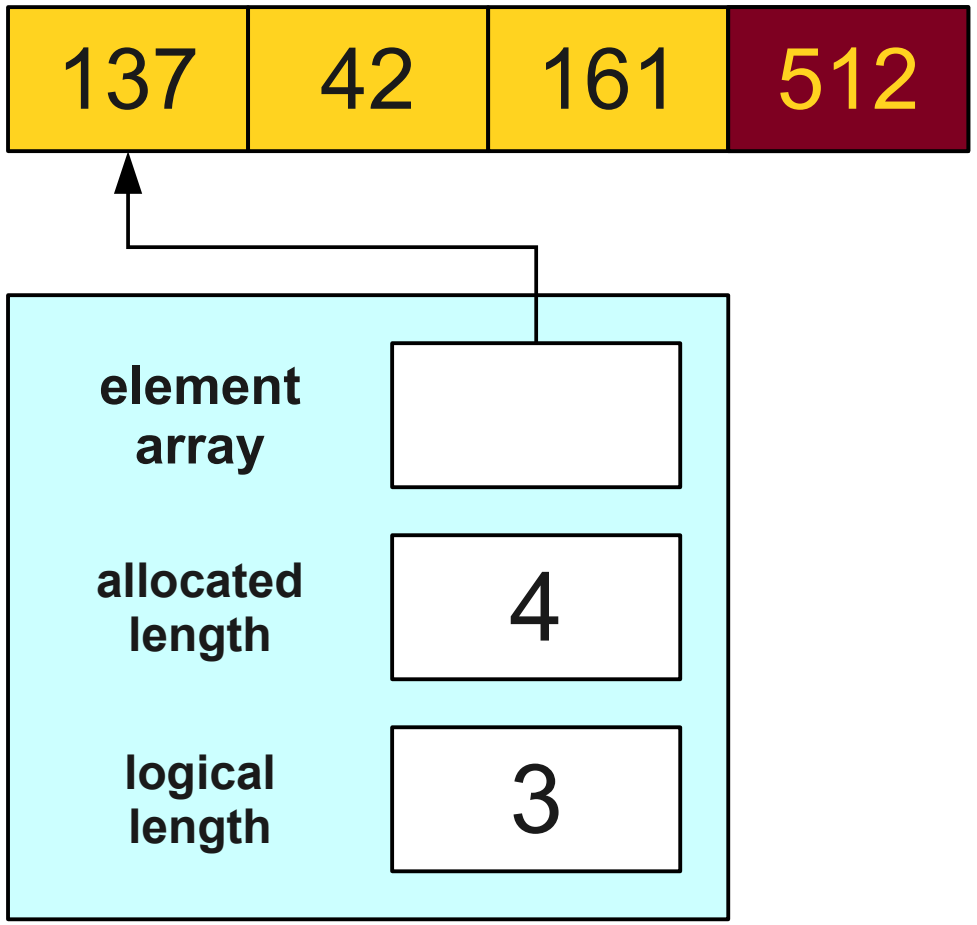

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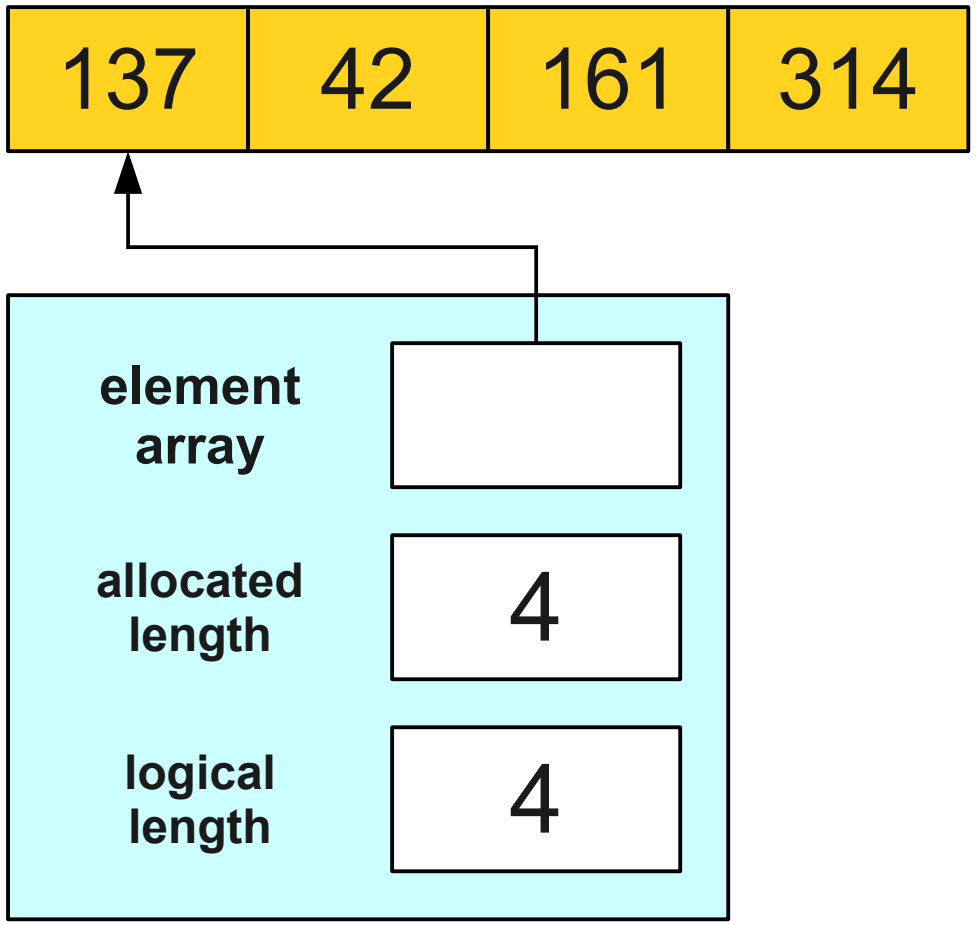

Running out of Space

- Our current implementation very quickly runs out of space to store elements.
- What should we do when this happens?

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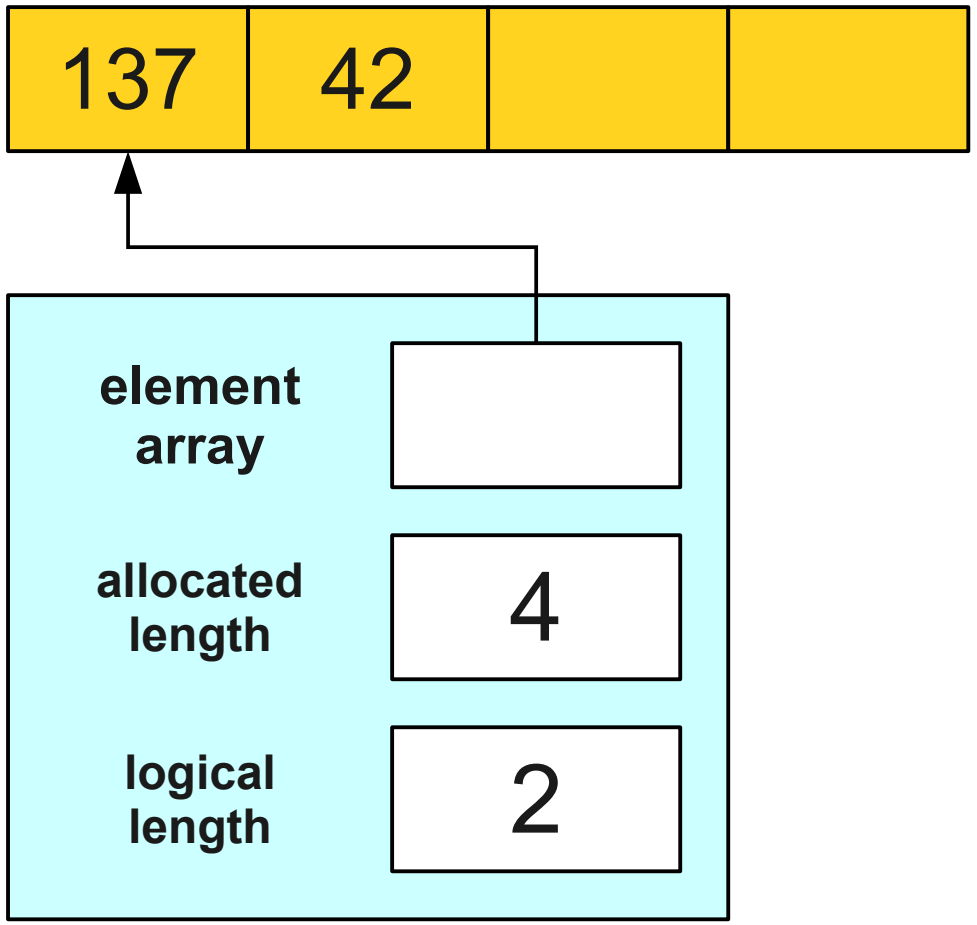

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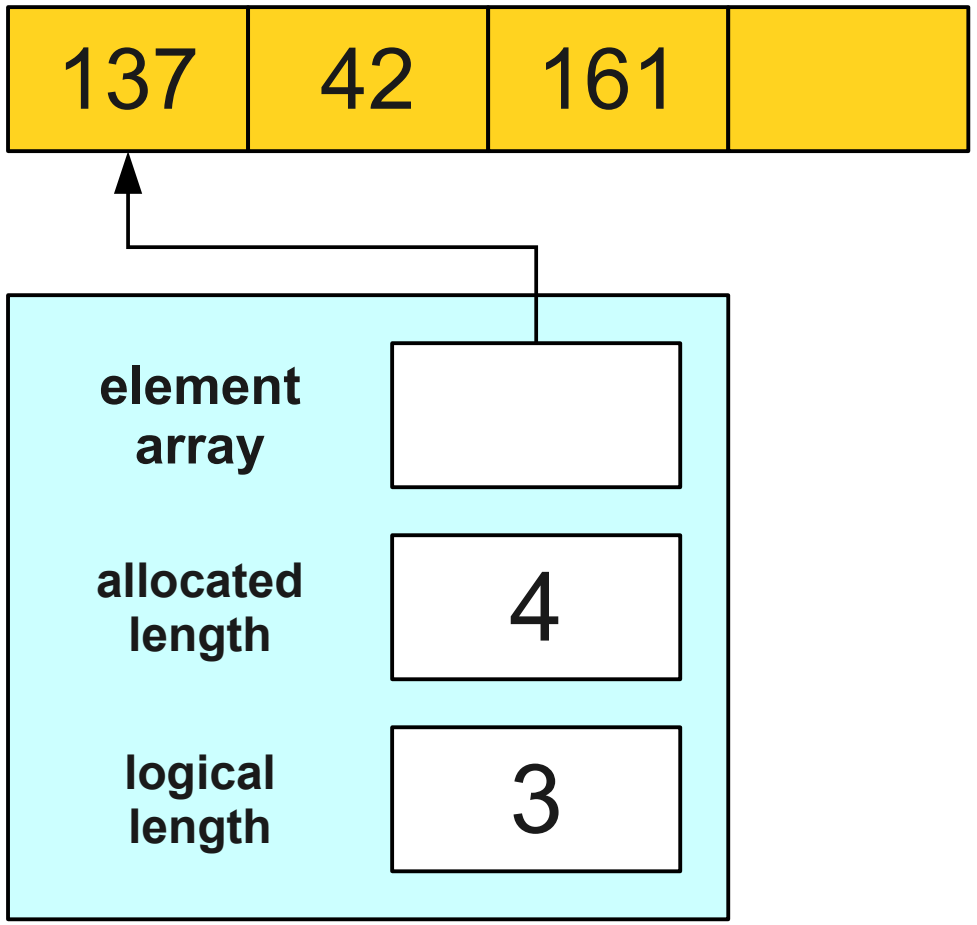

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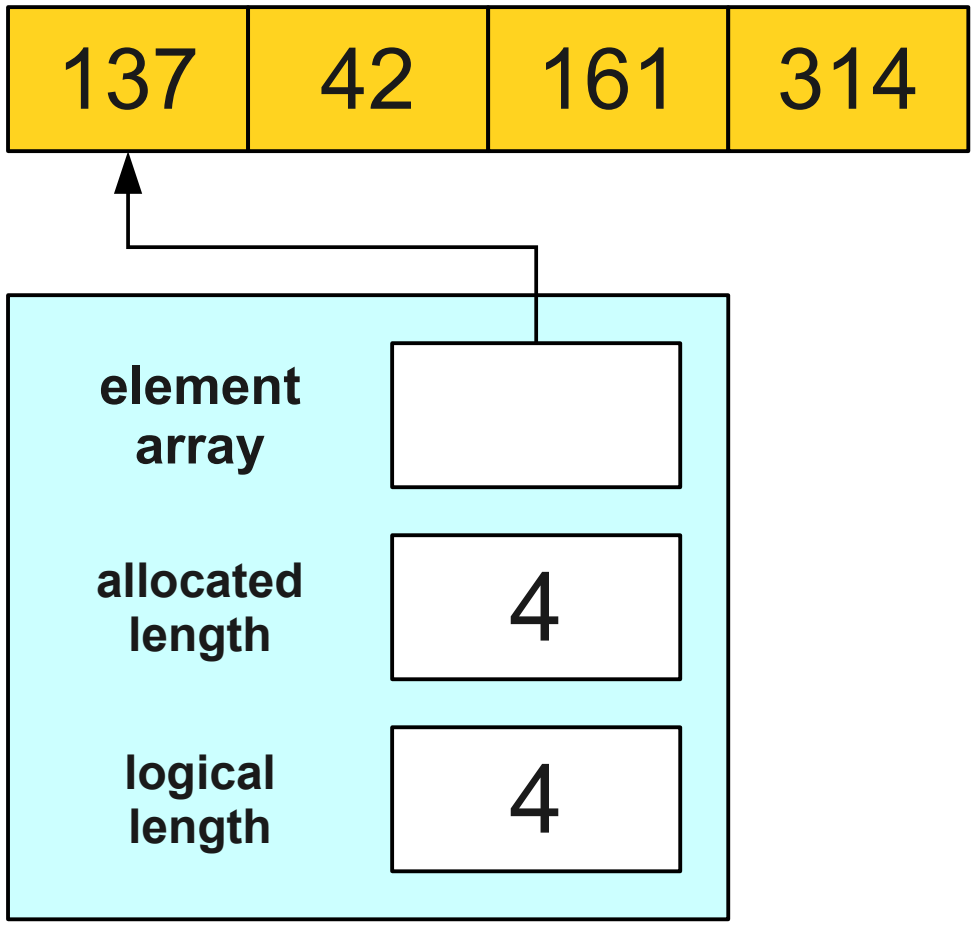

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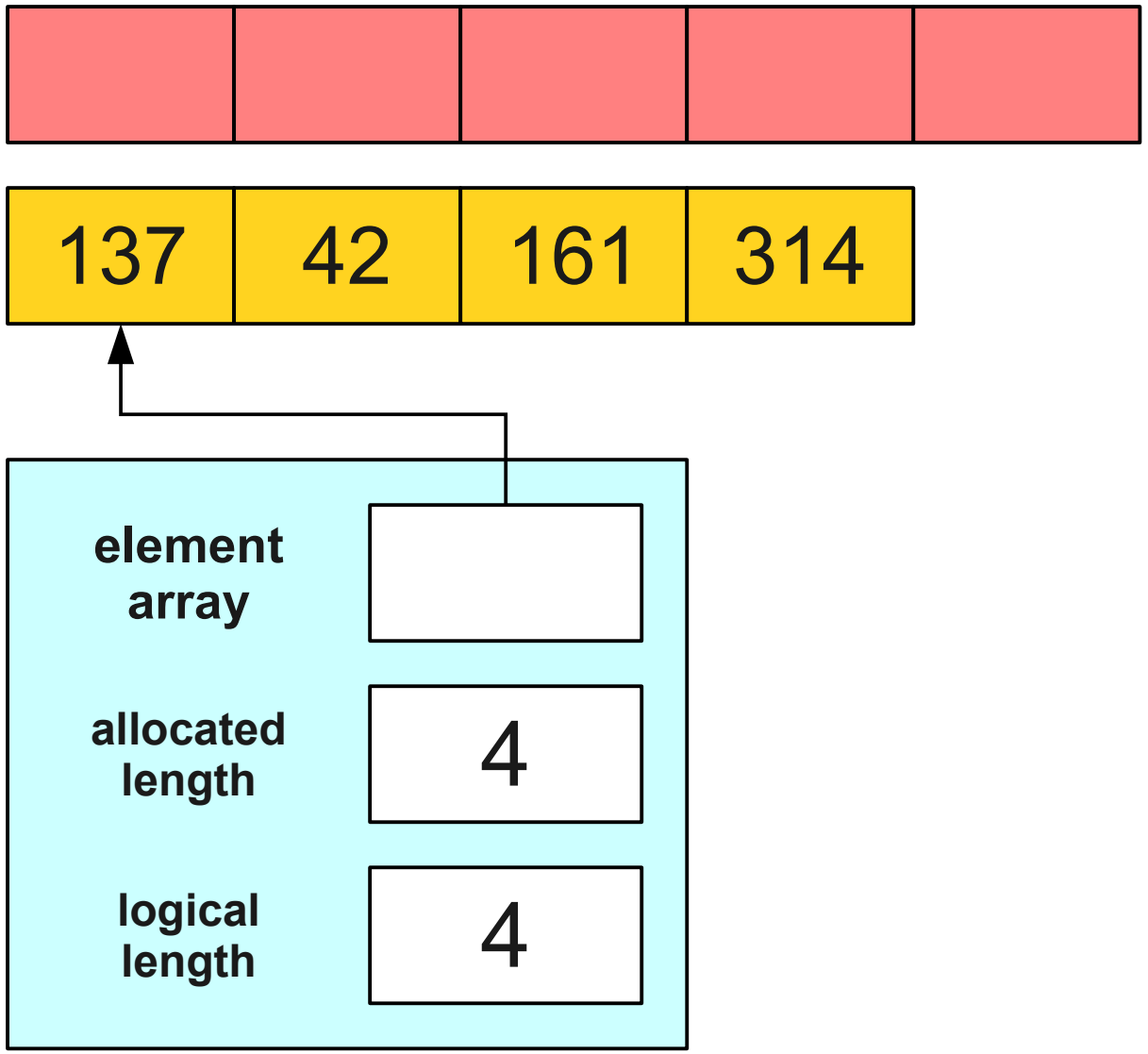

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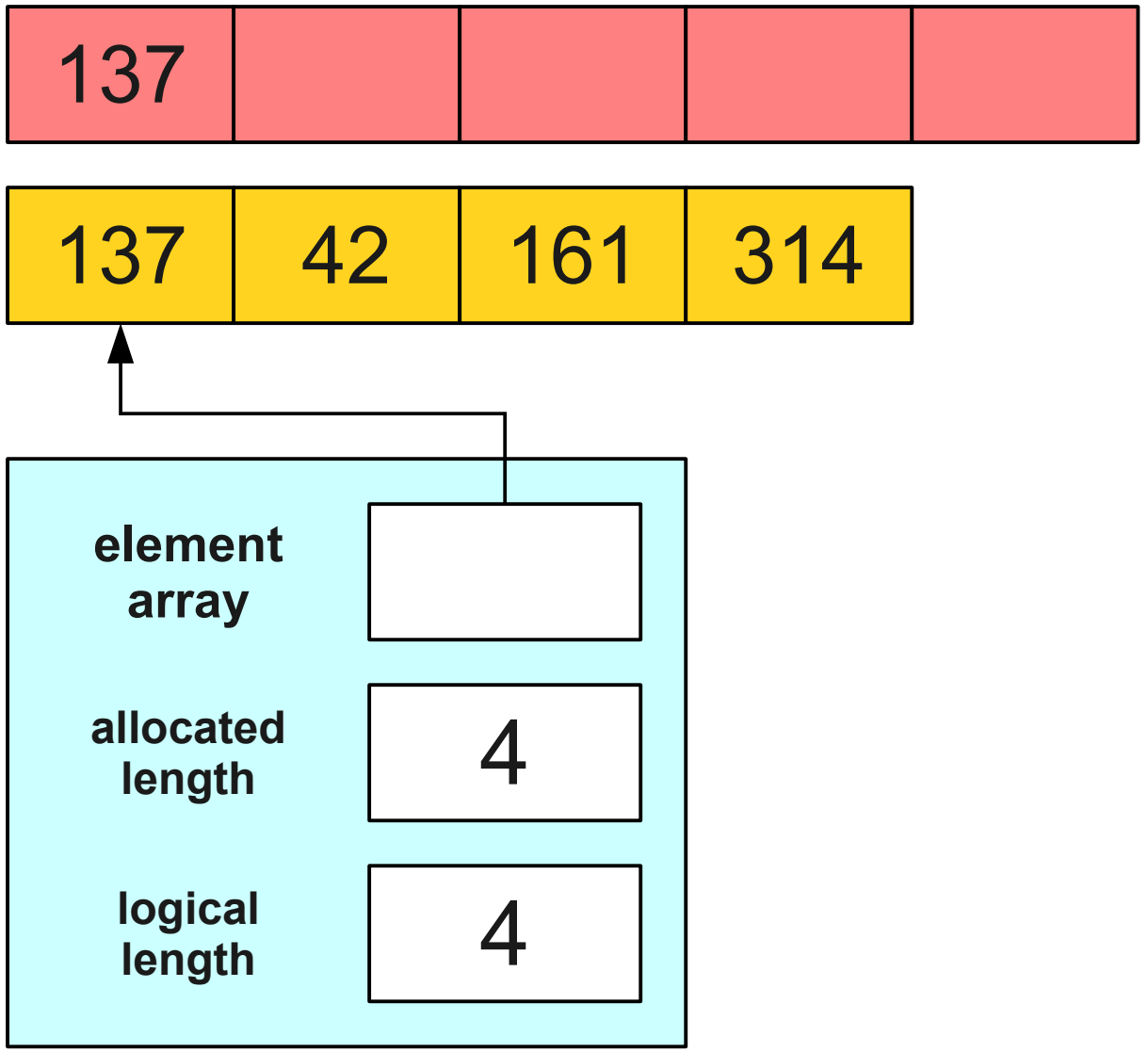

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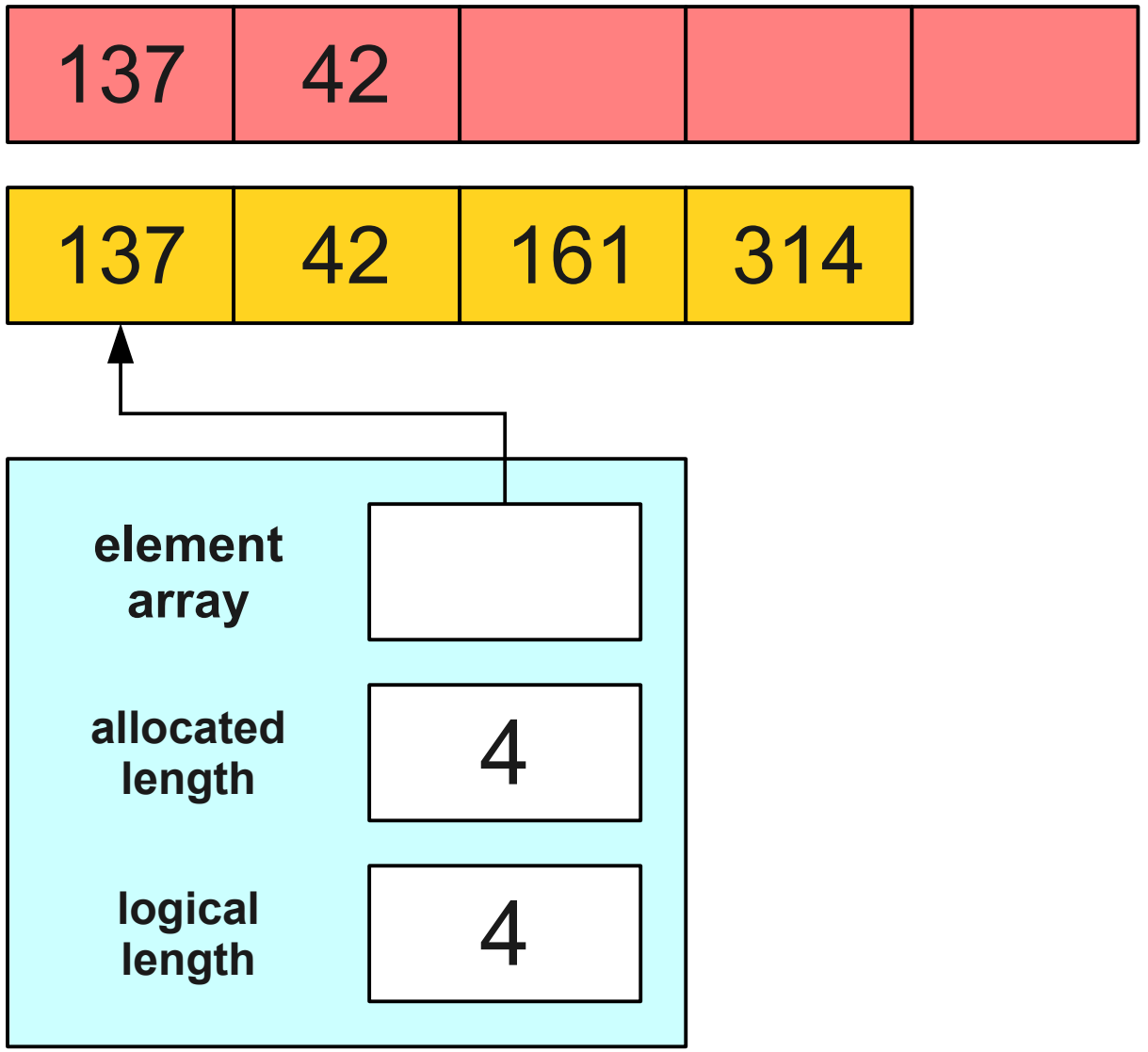

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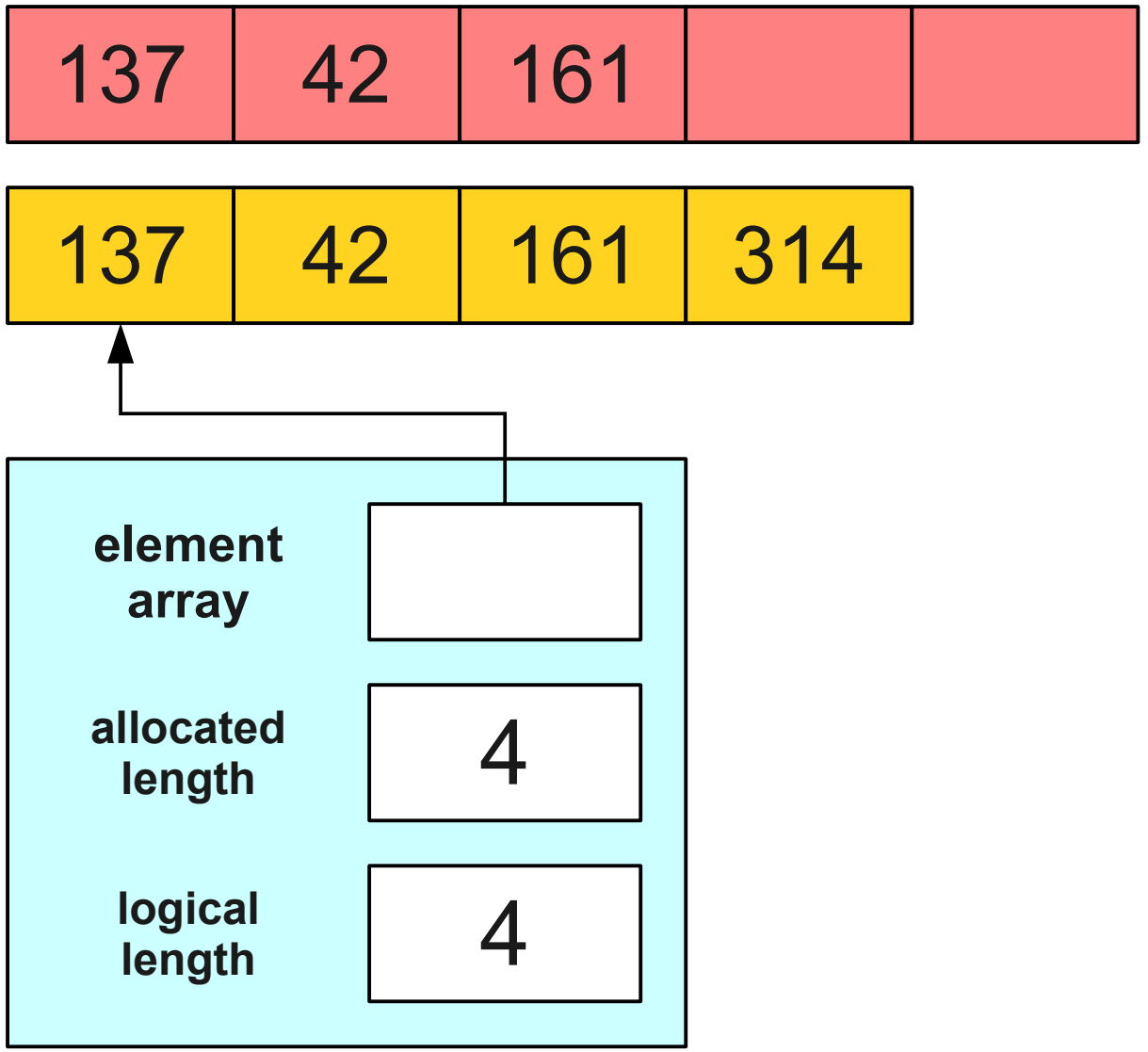

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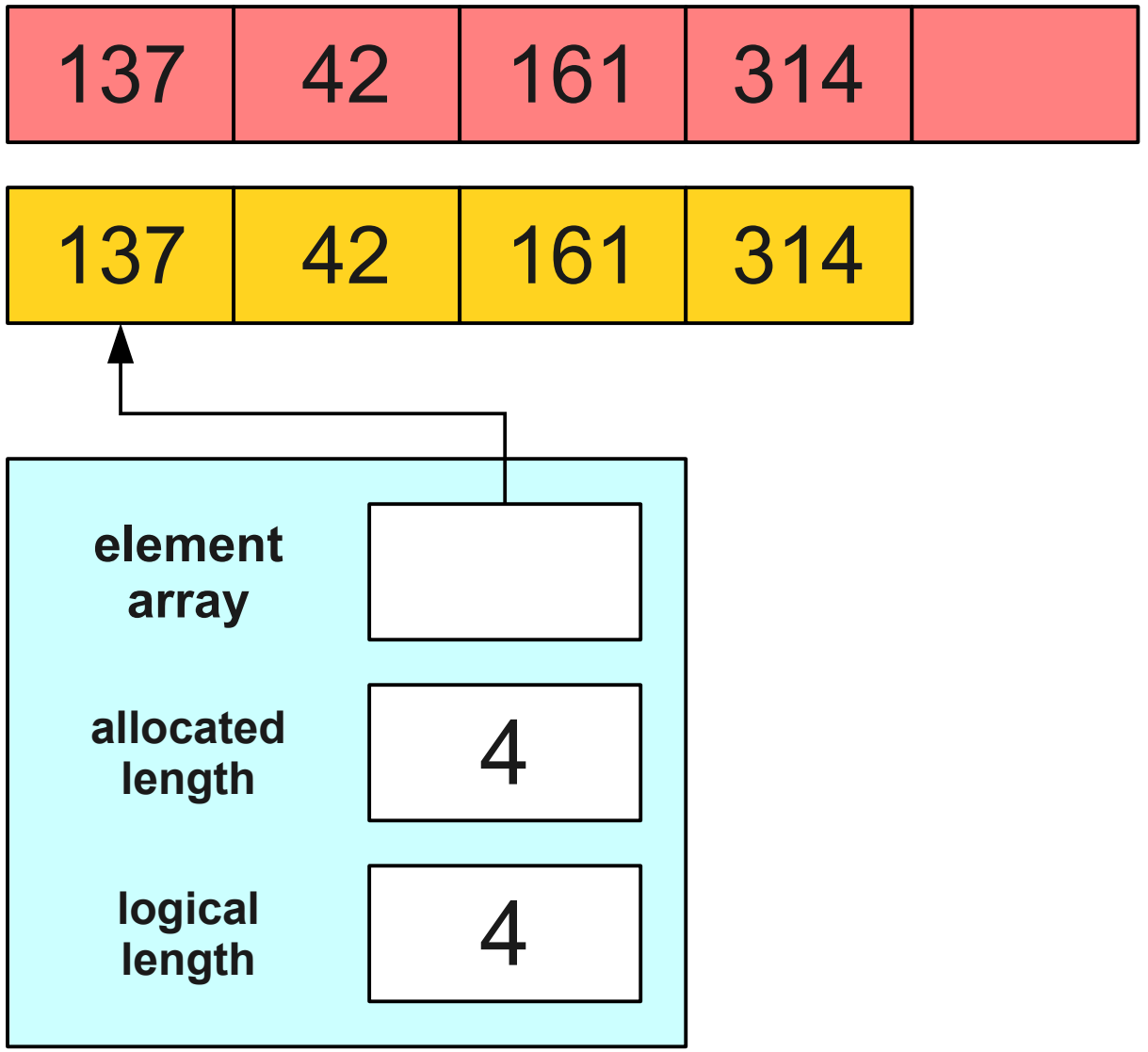

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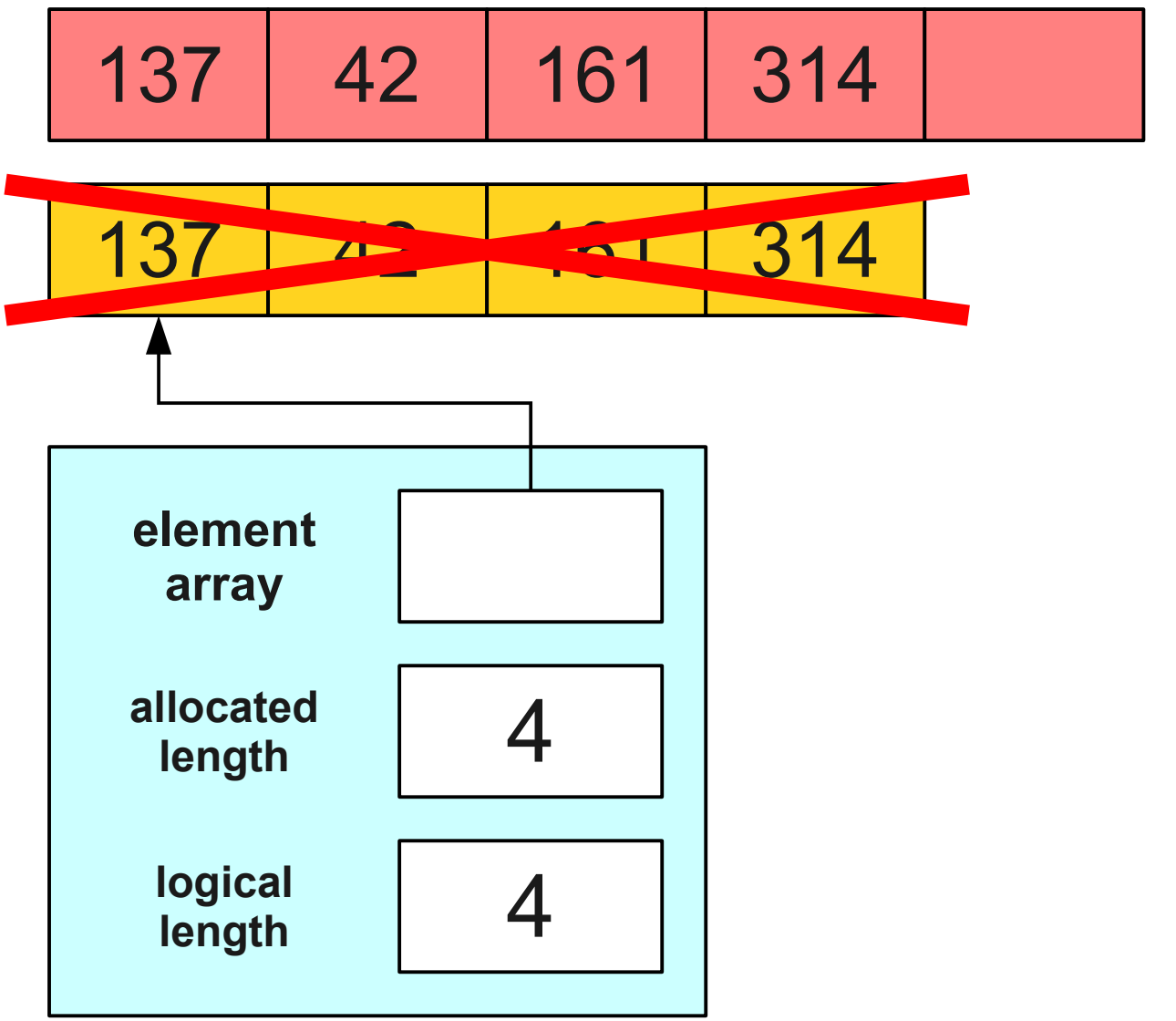

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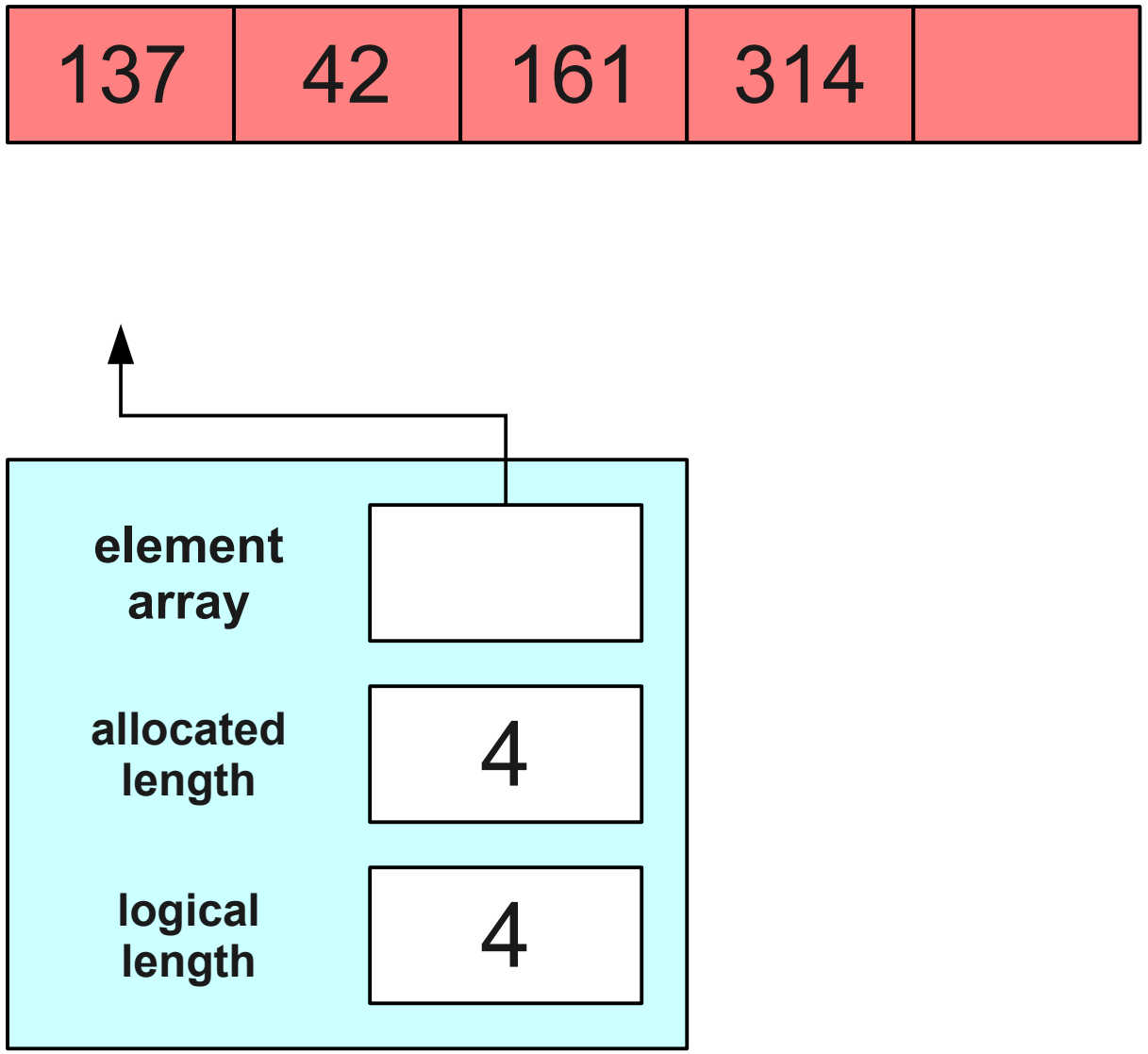

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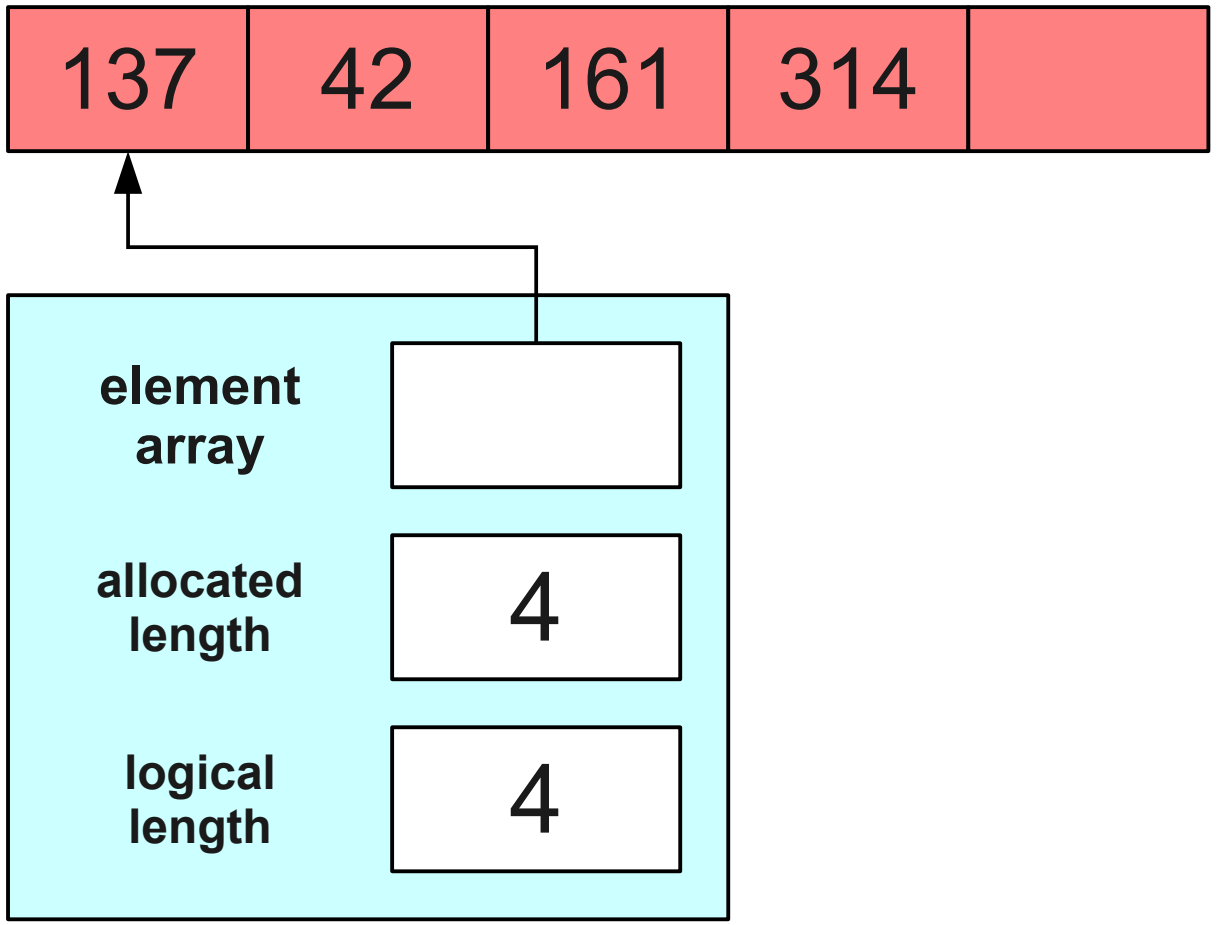

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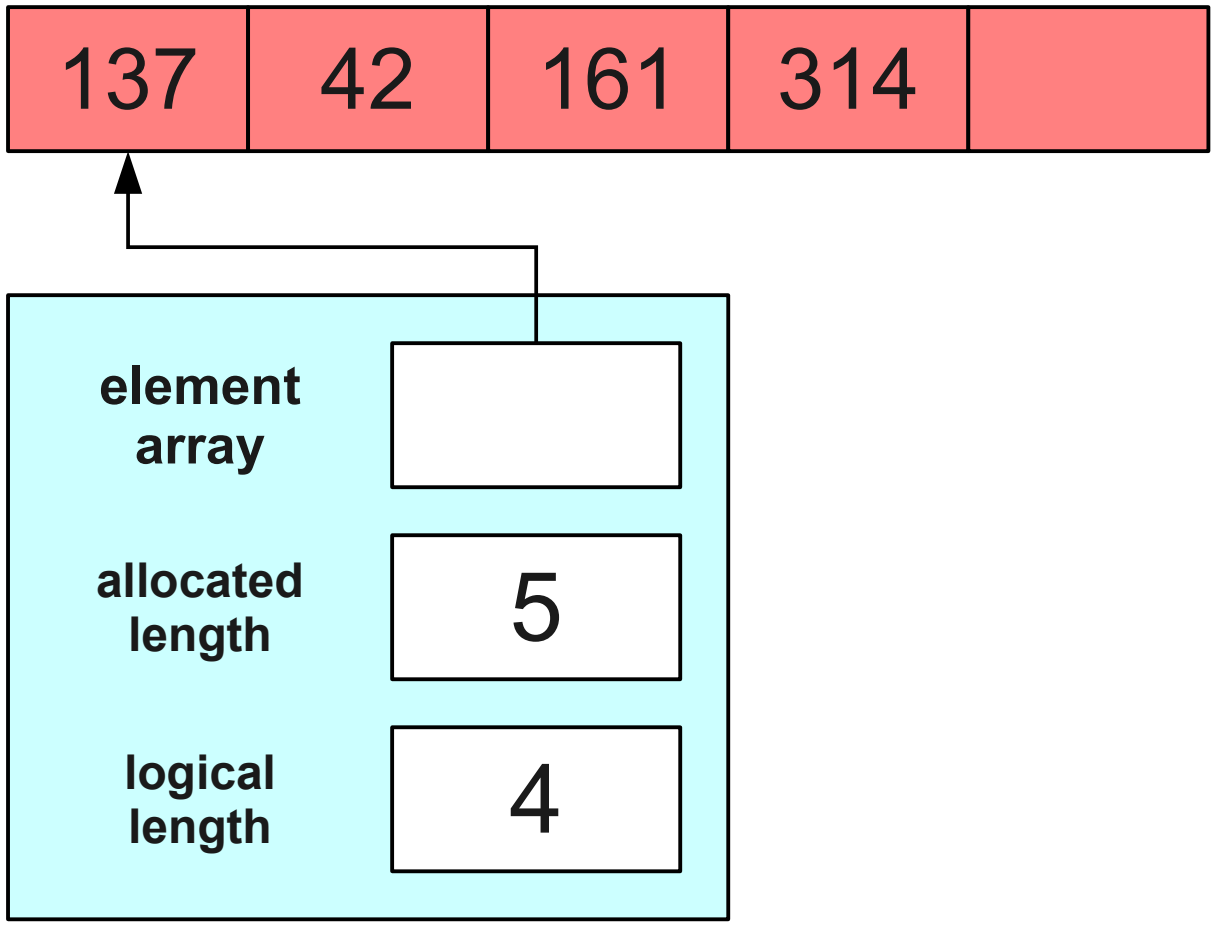

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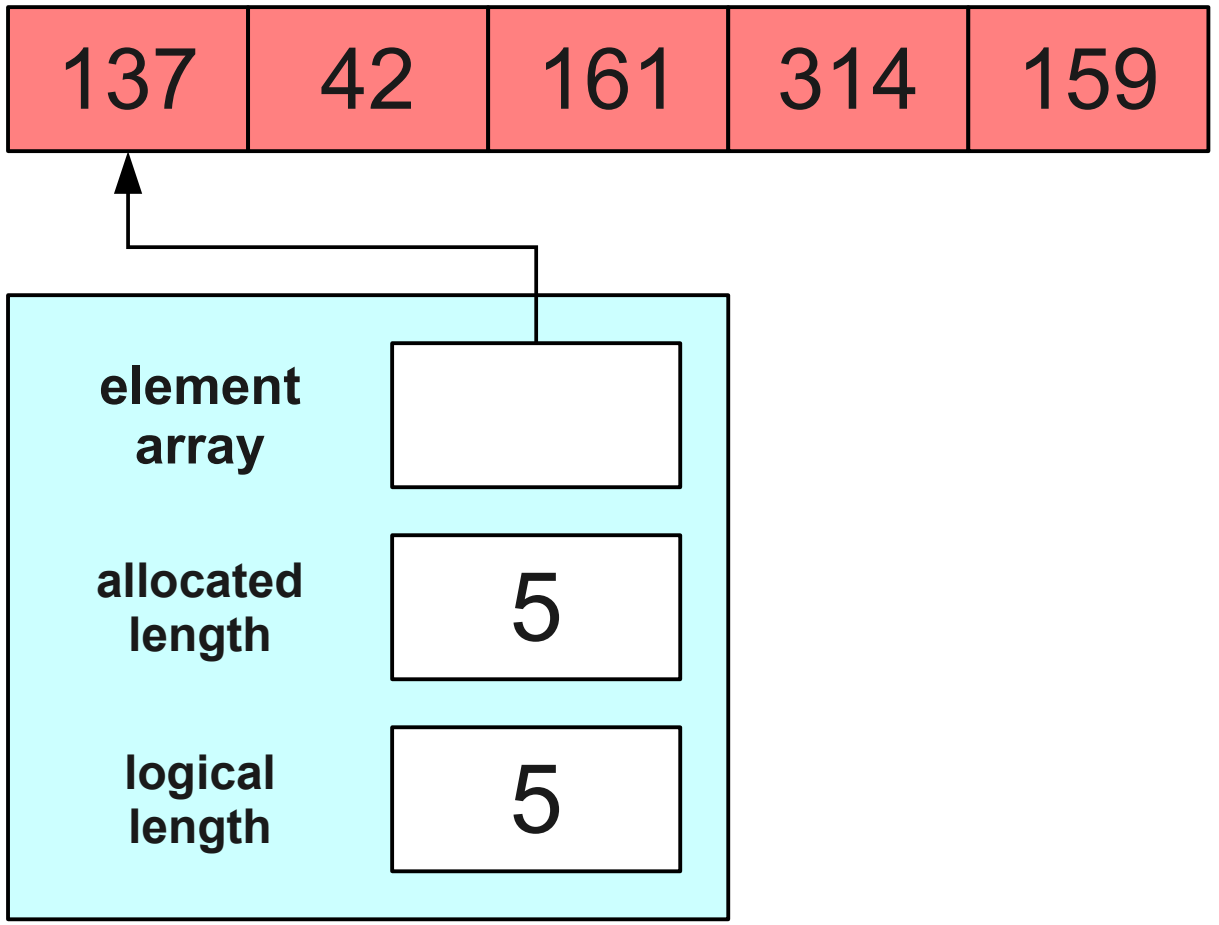

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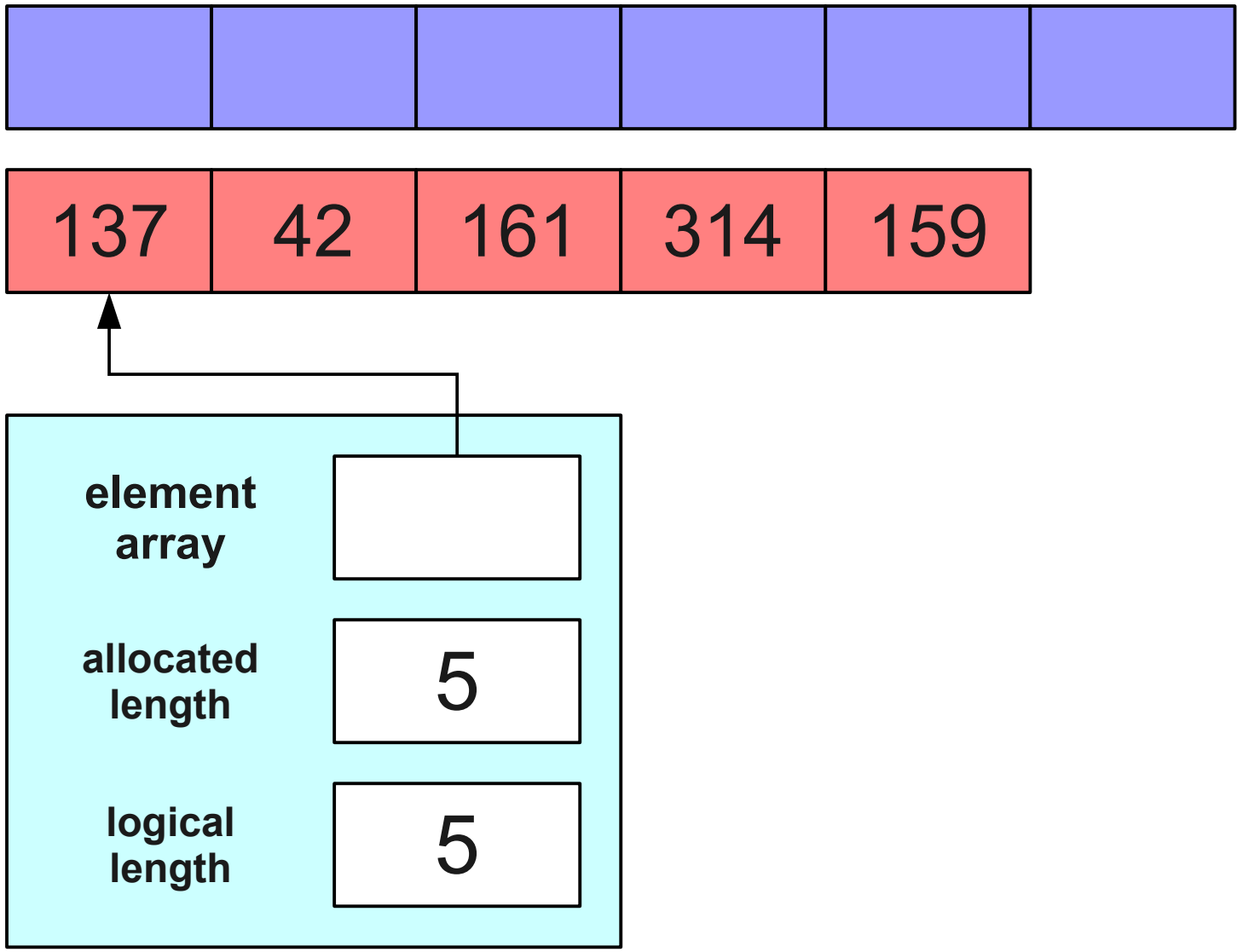

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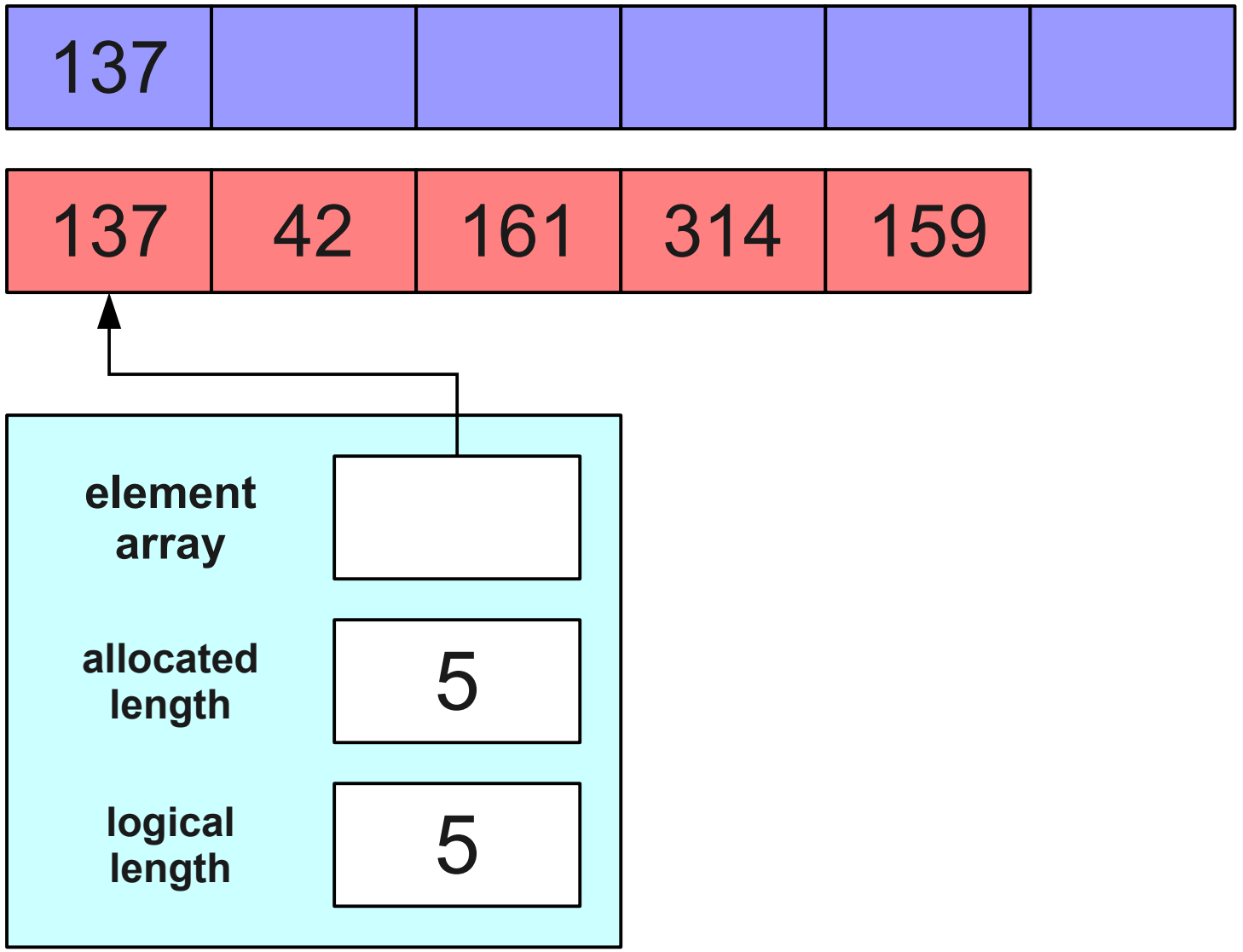

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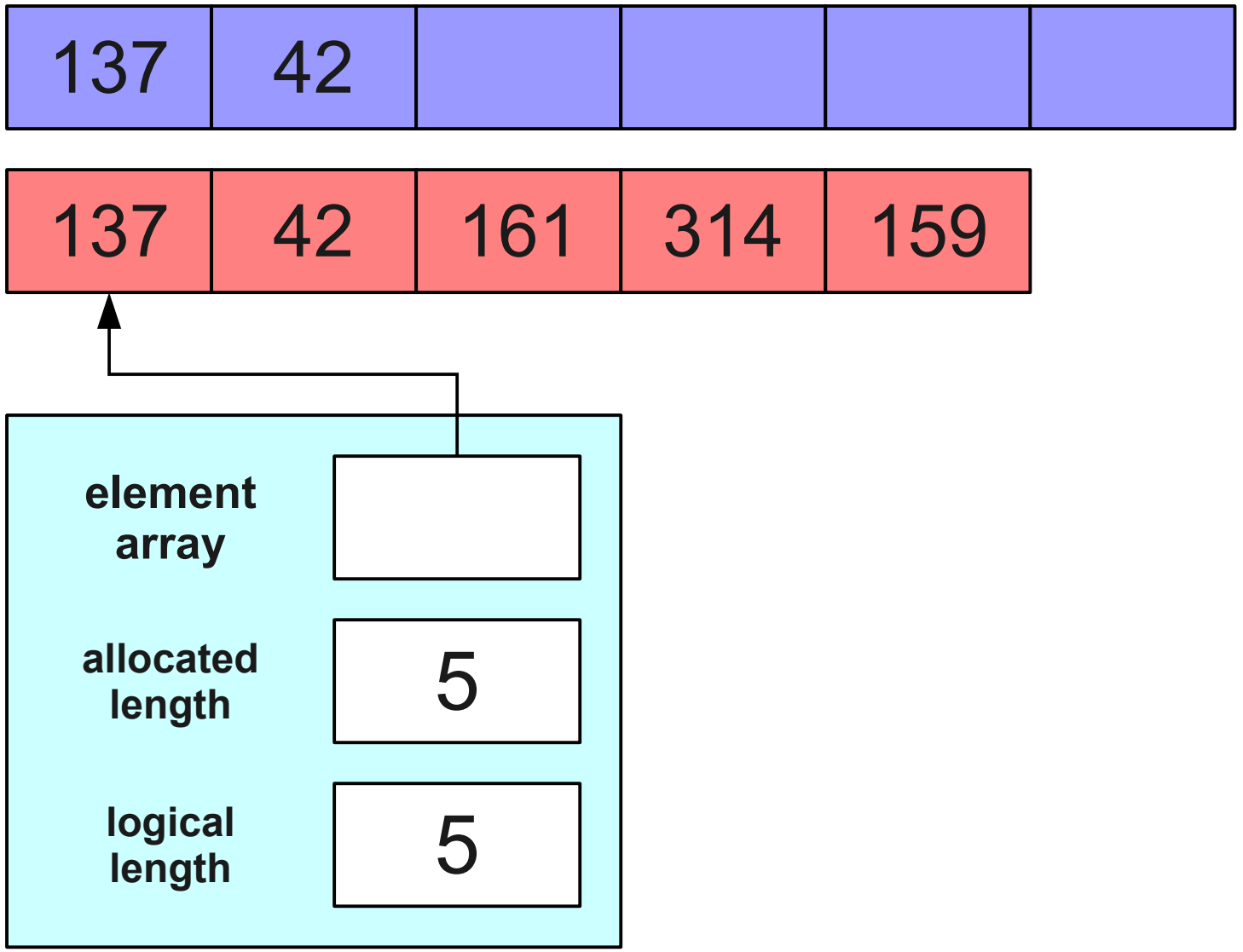

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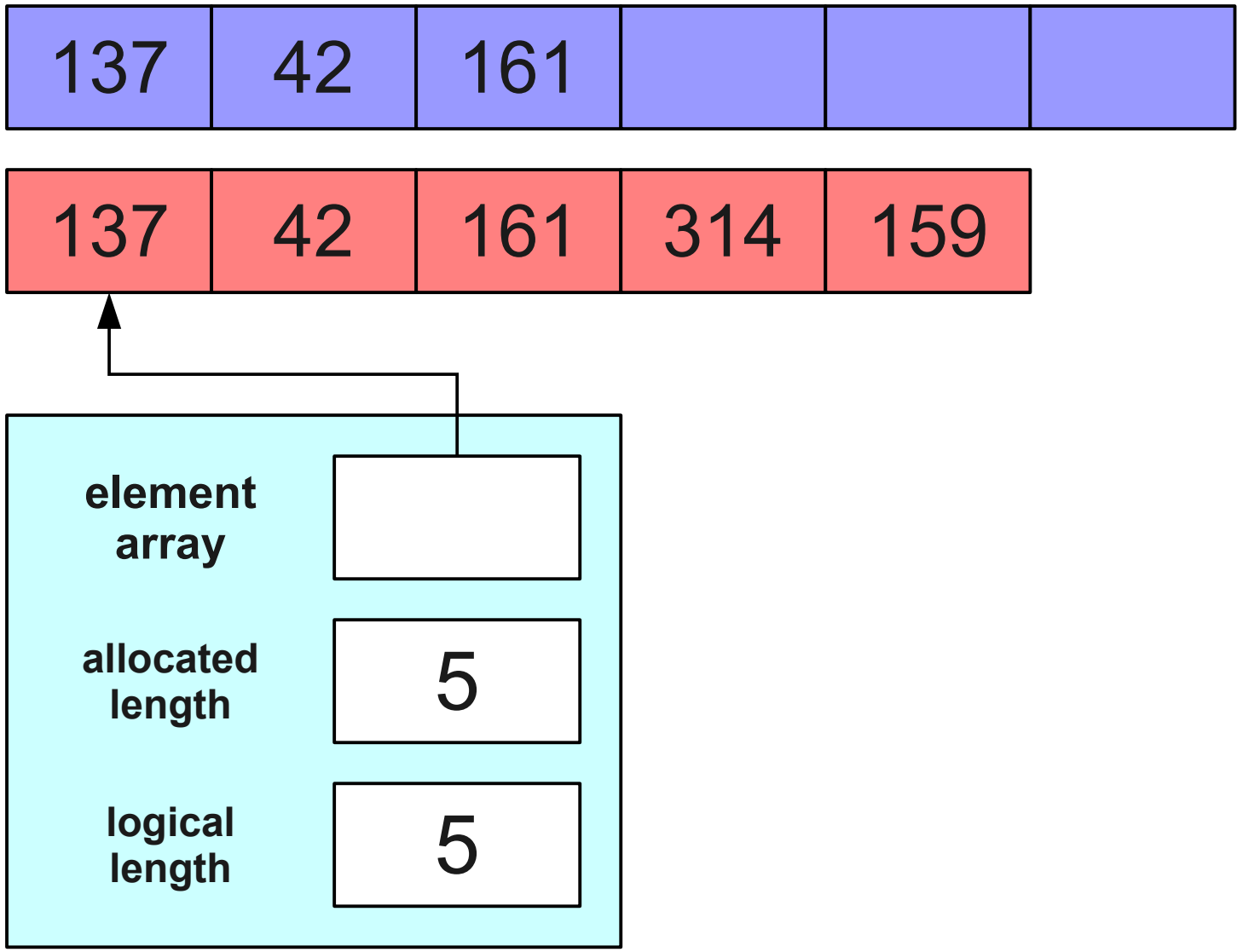

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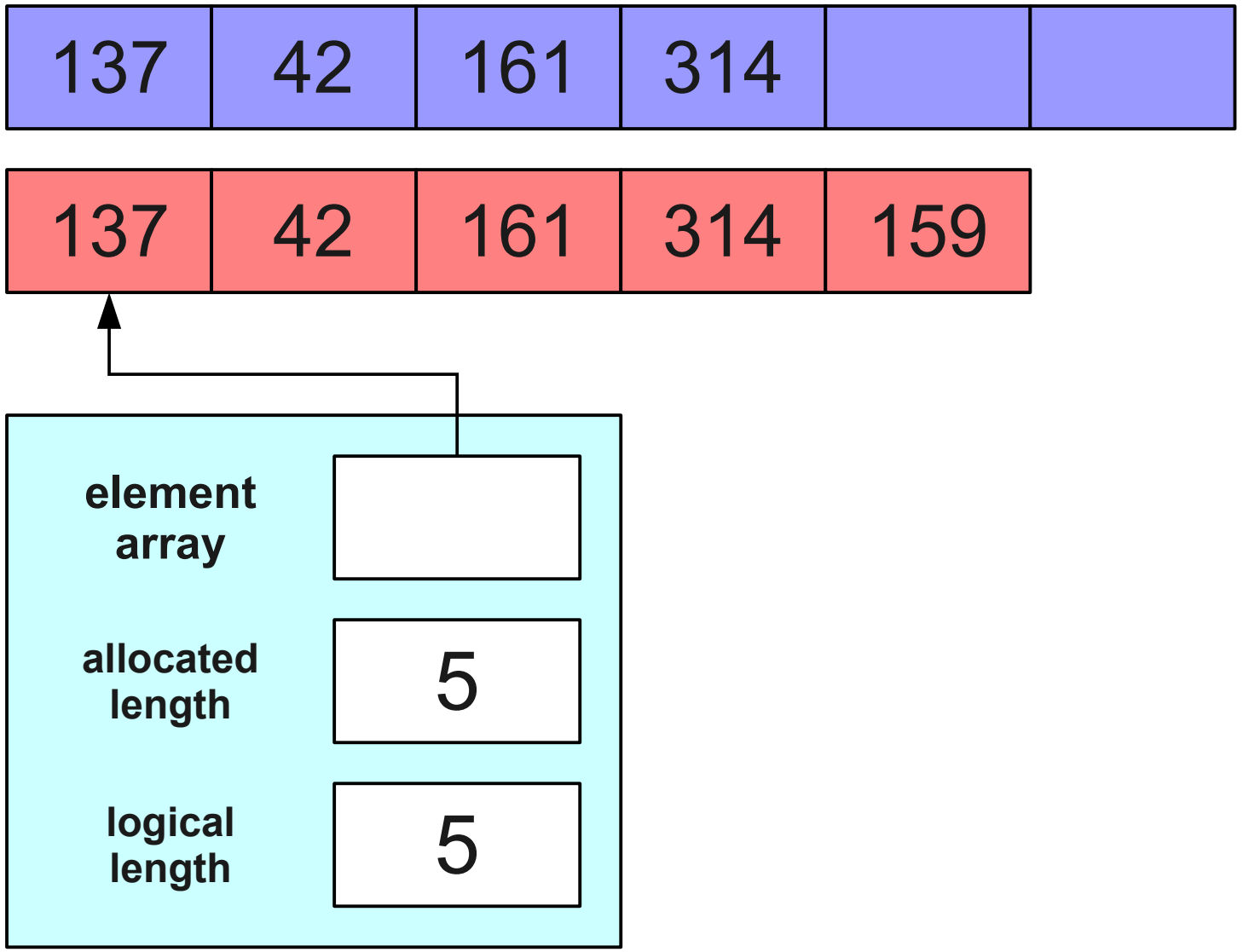

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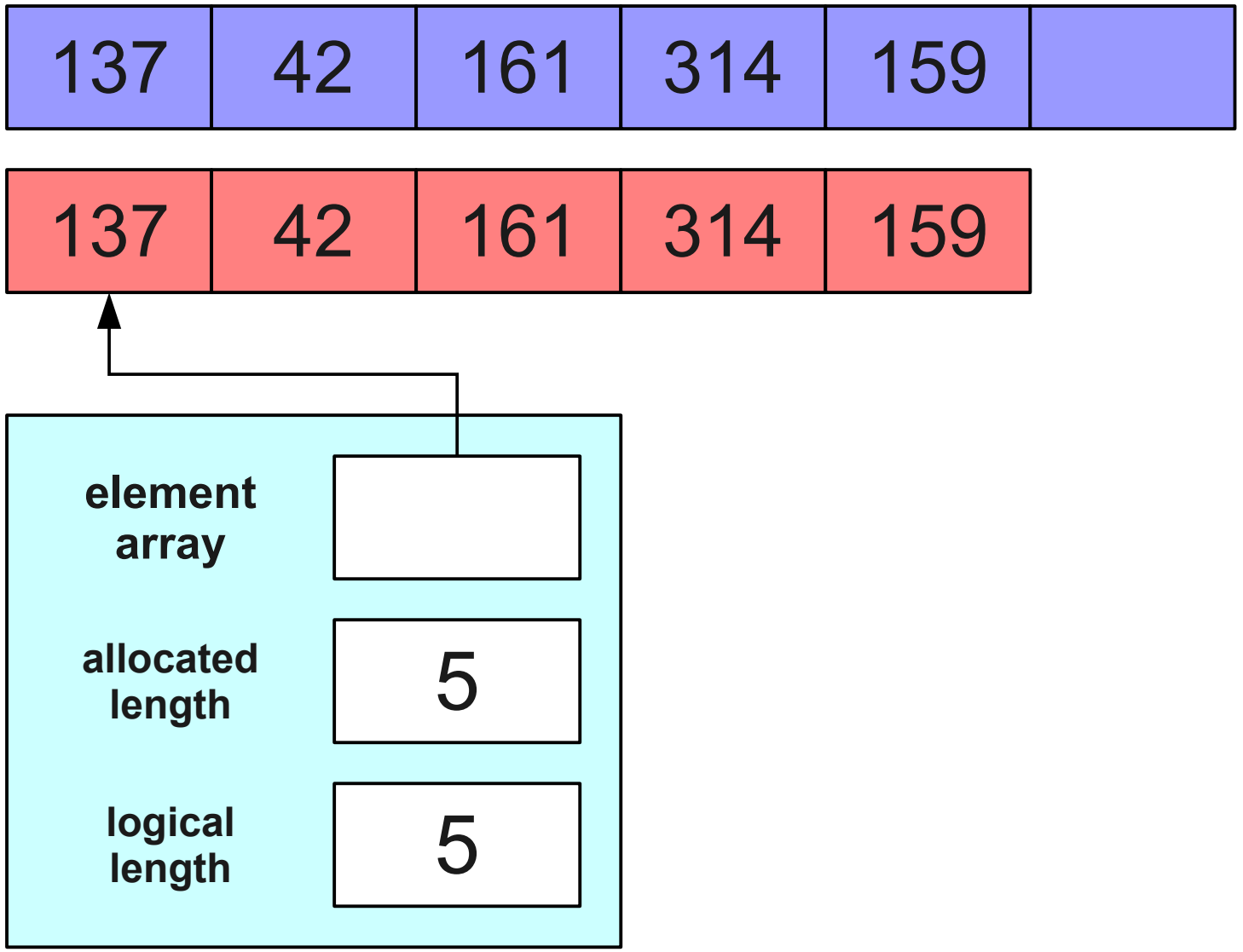

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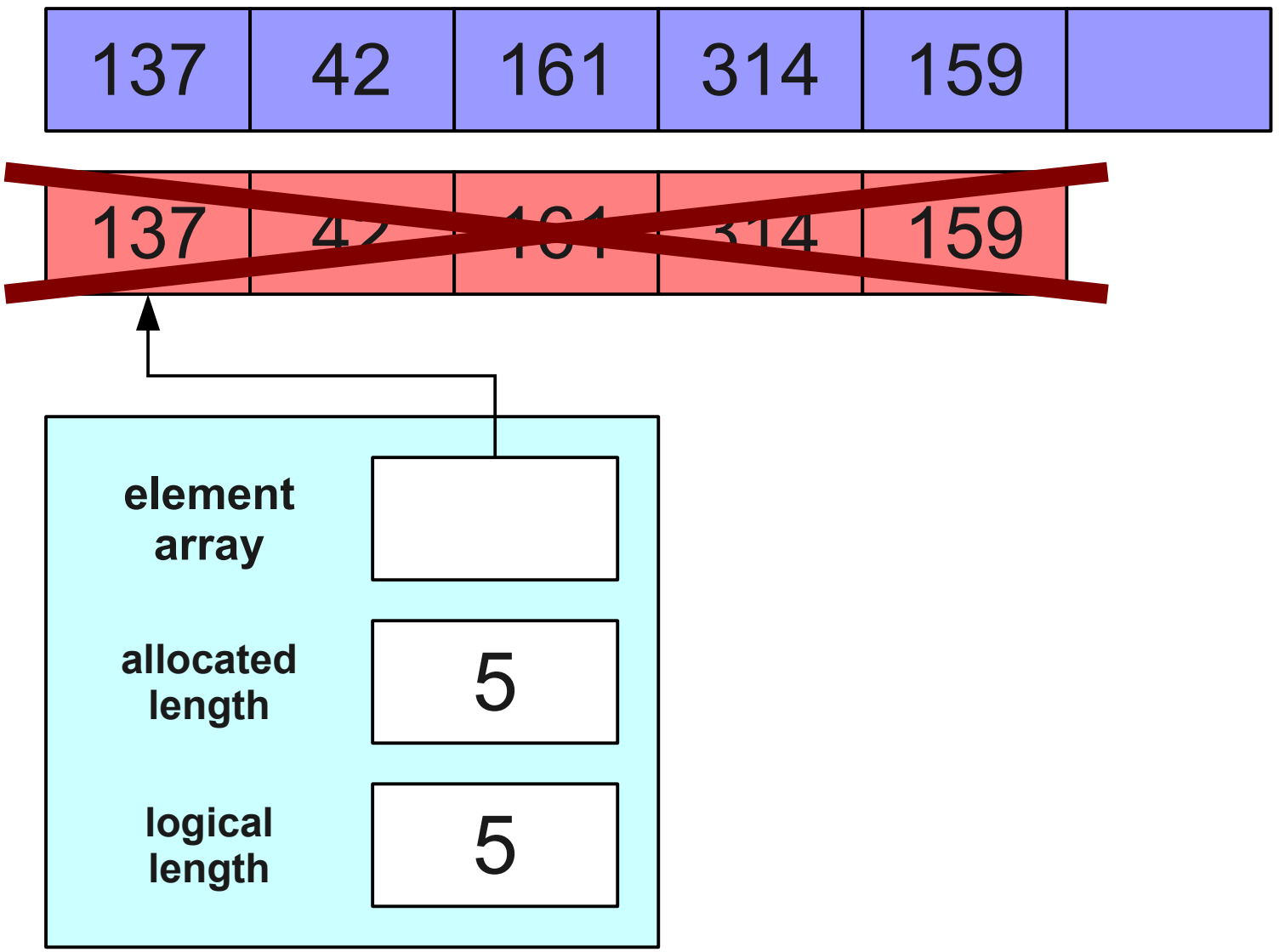

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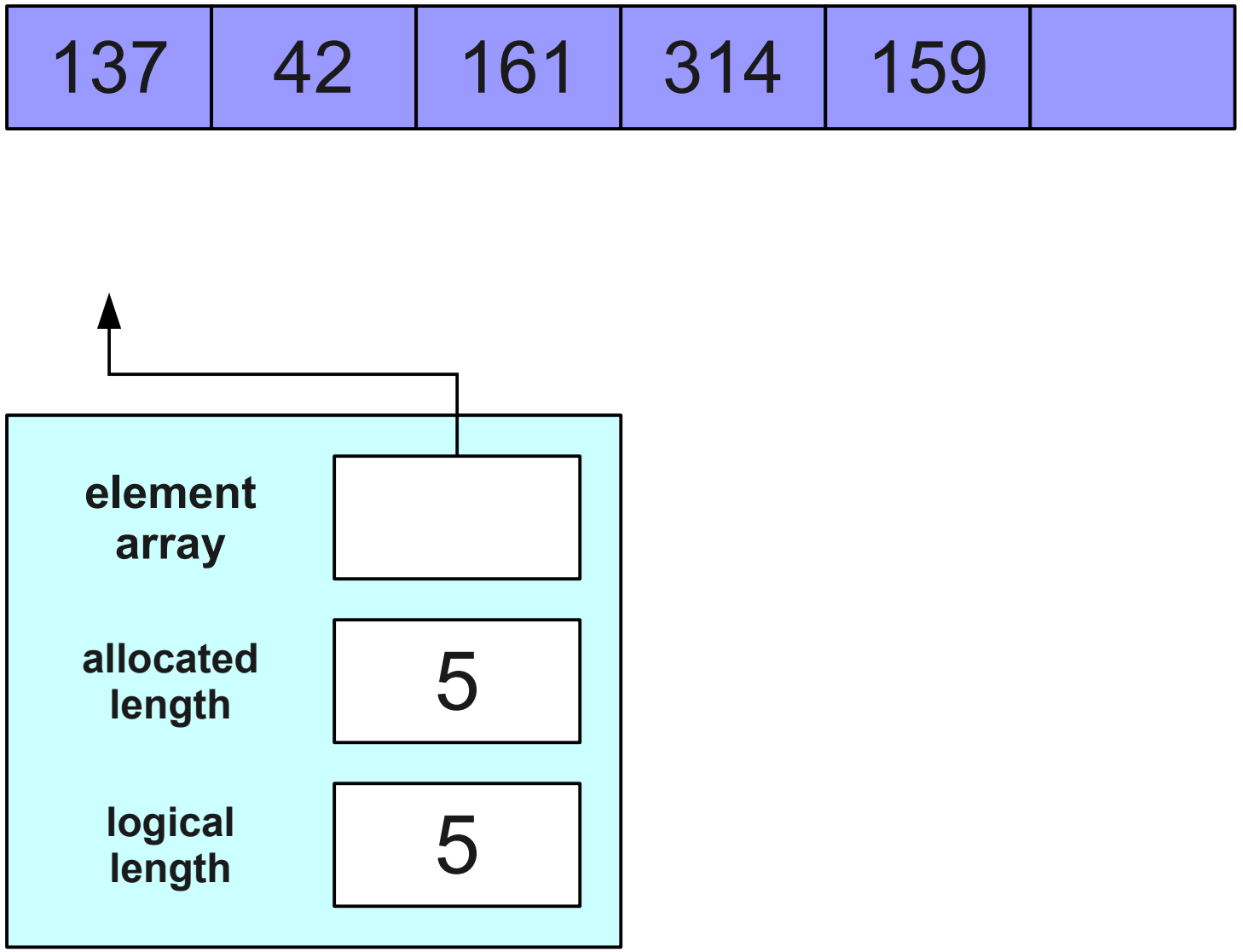

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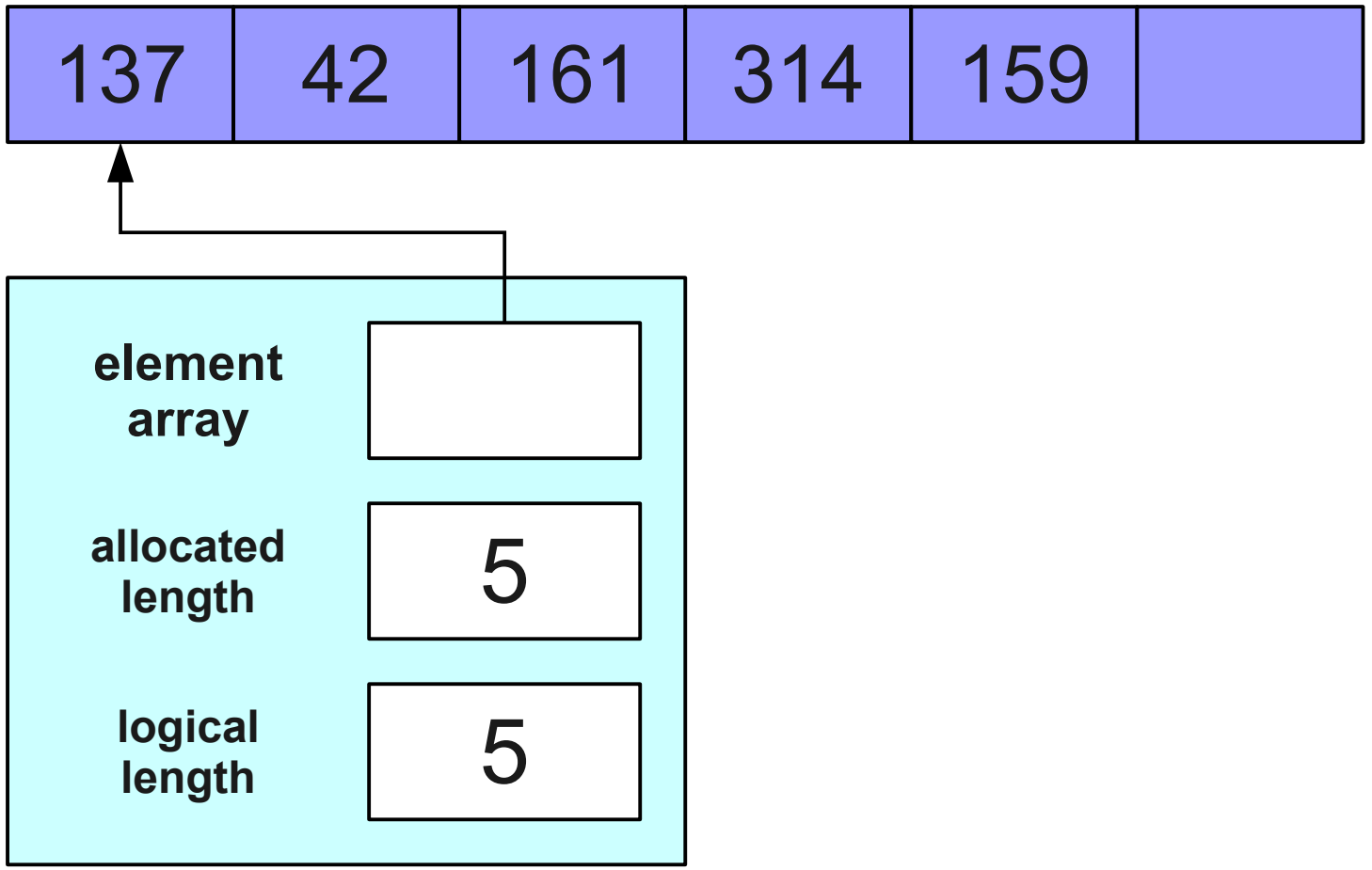

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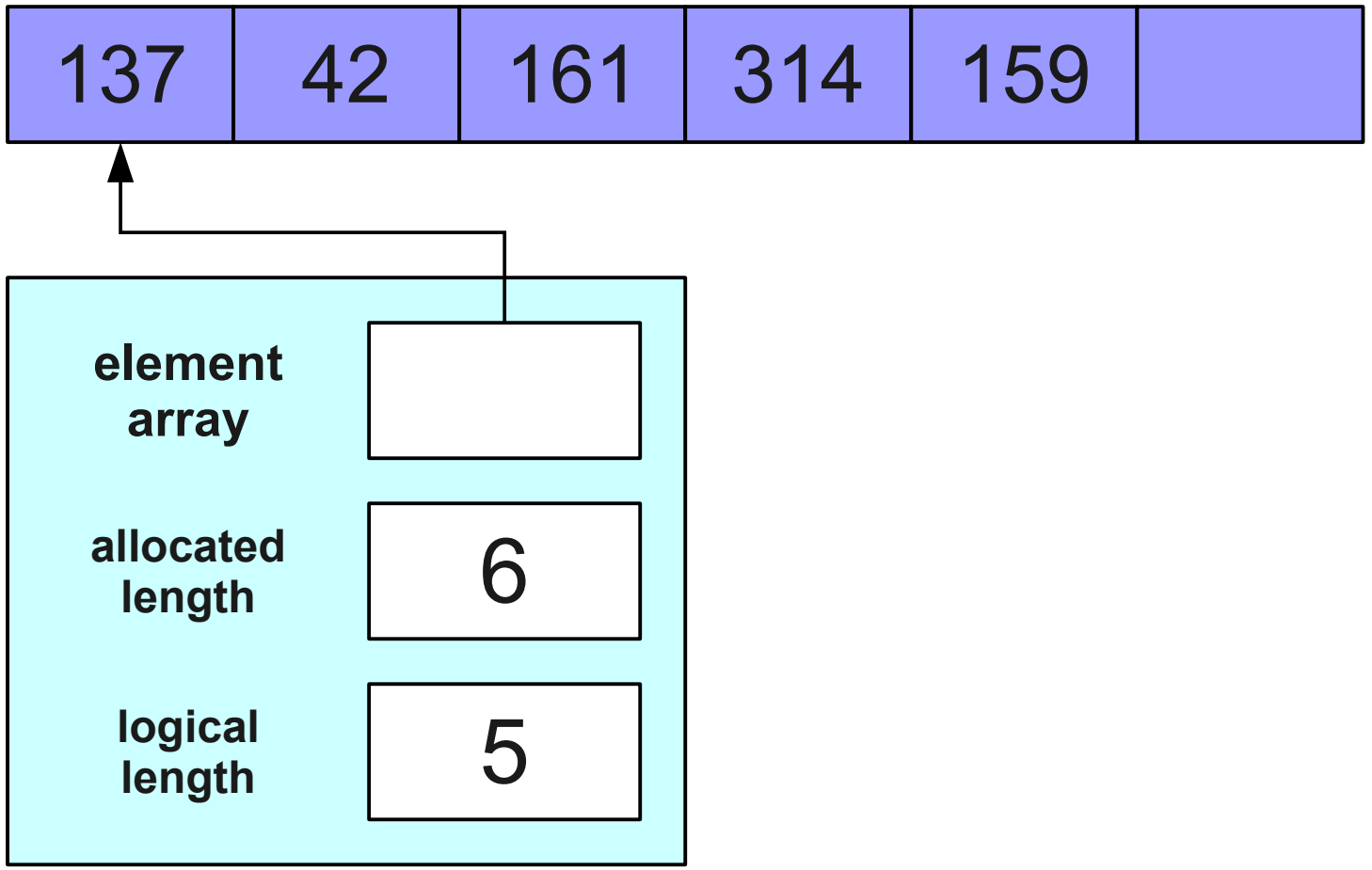

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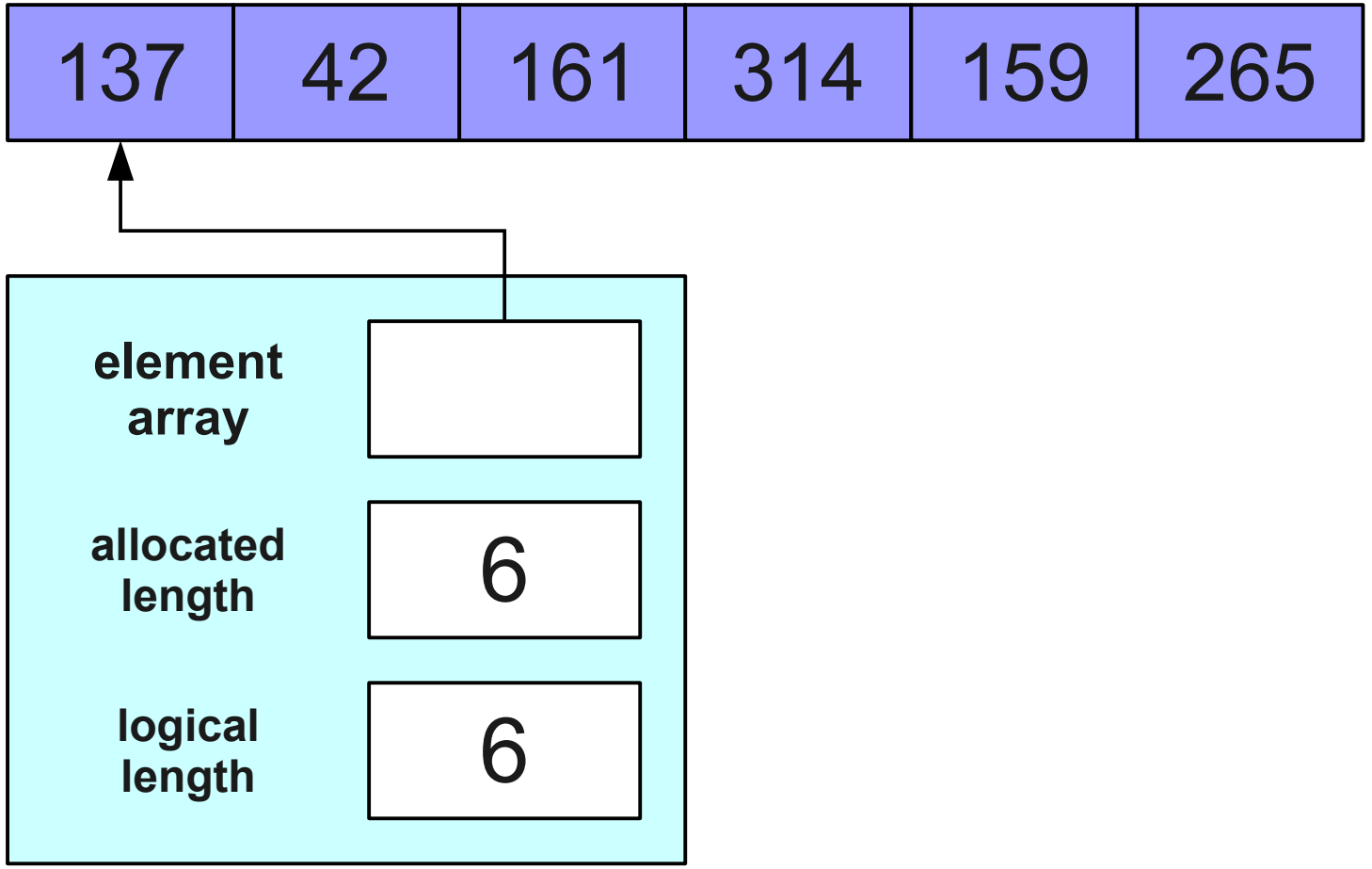

Ready... set... grow!

Analyzing Our Approach

- We now have a working solution, but is it an *efficient* solution?
- Let's analyze the big-O complexity of the five operations.
  - **`size`**:
  - **`isEmpty`**:
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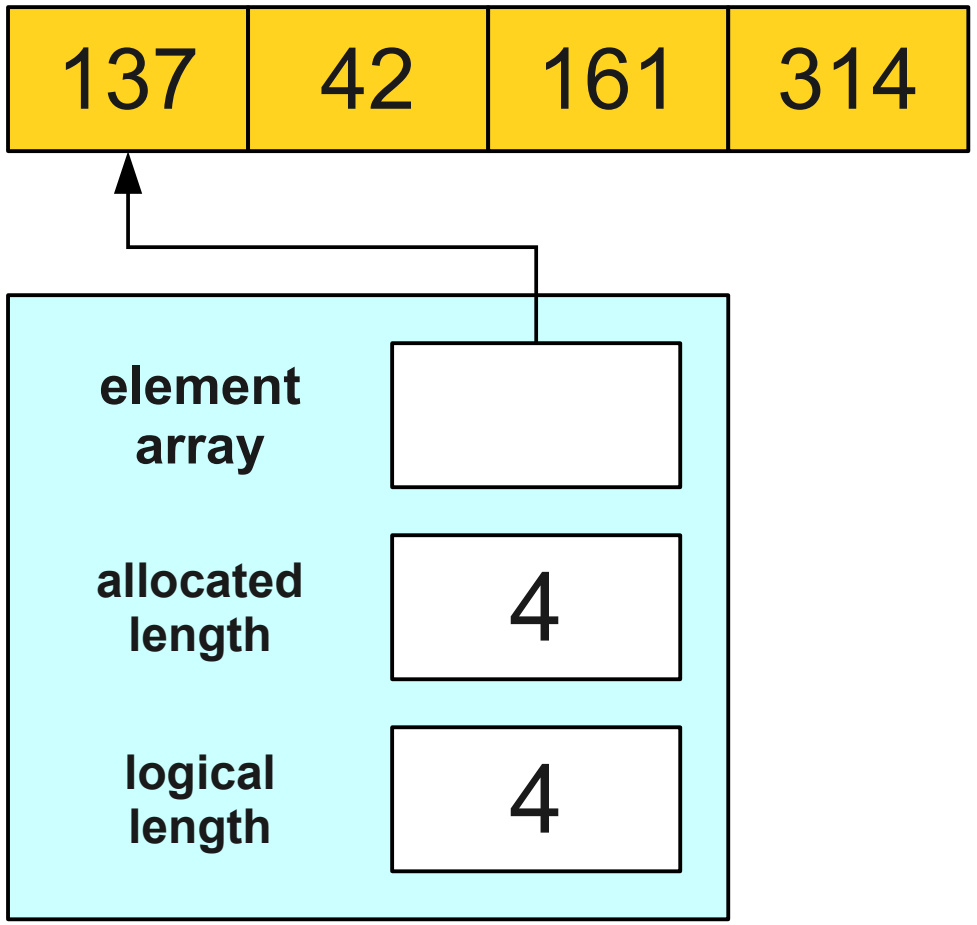
Validating Our Model

Speeding up the Stack

A Better Idea



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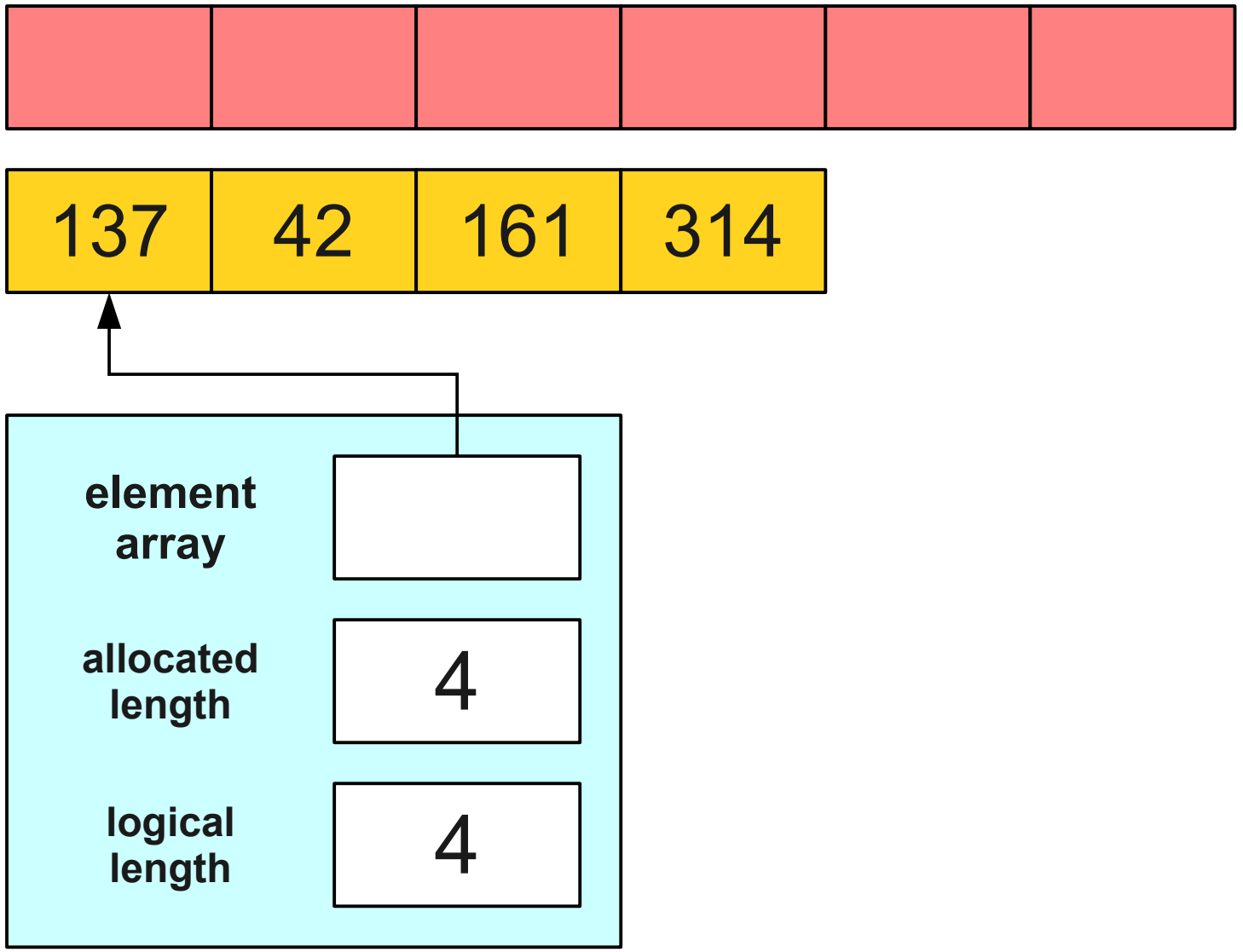

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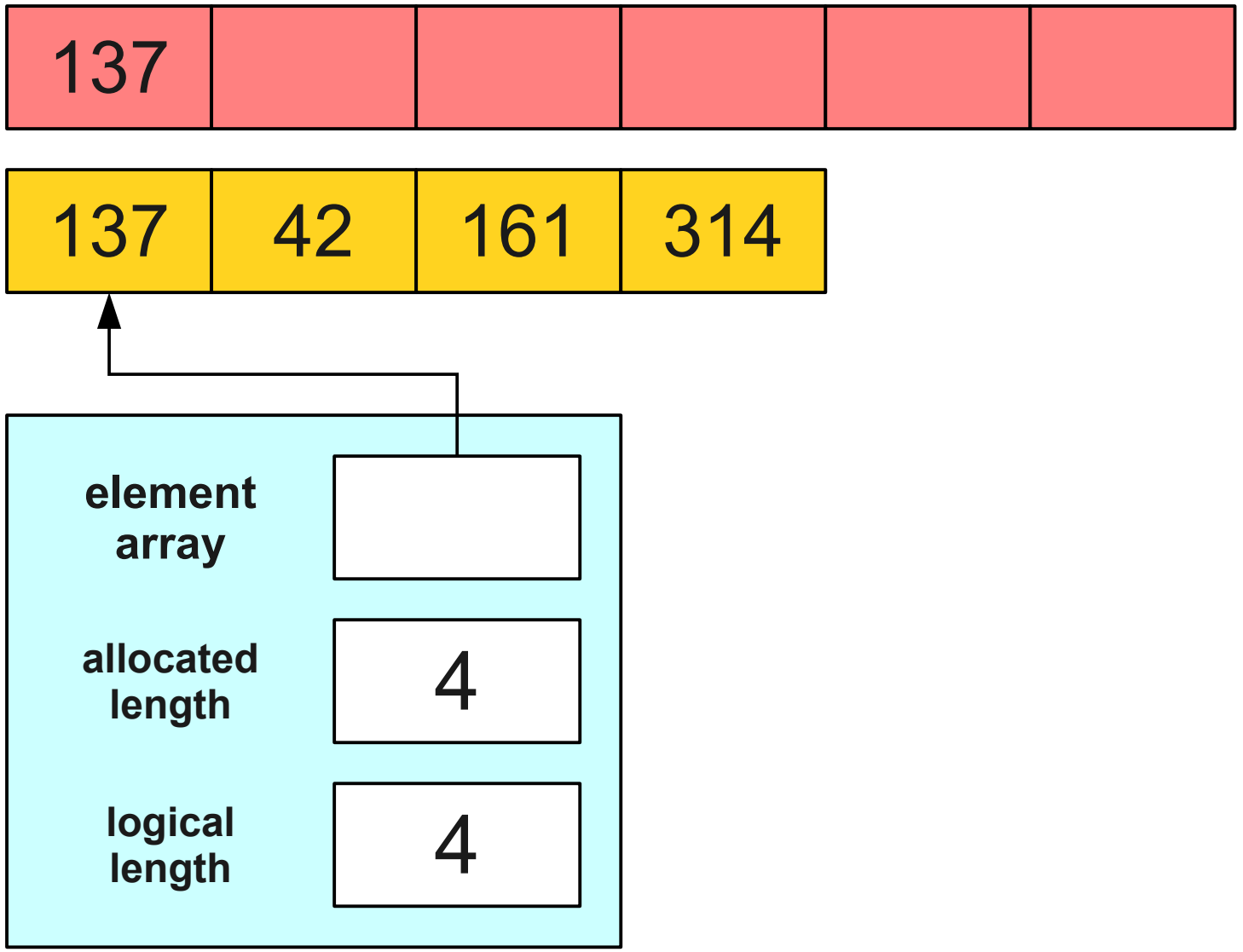

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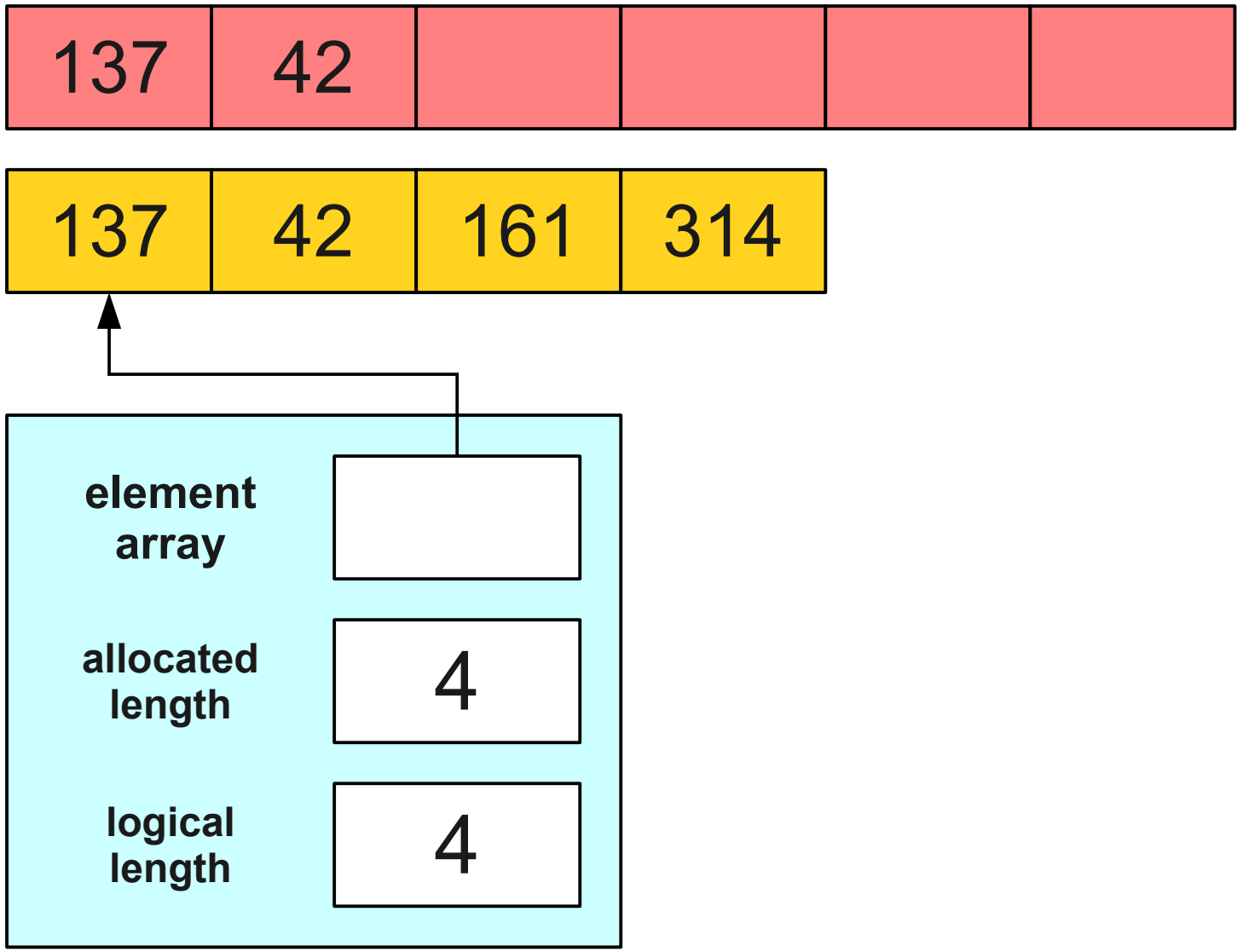

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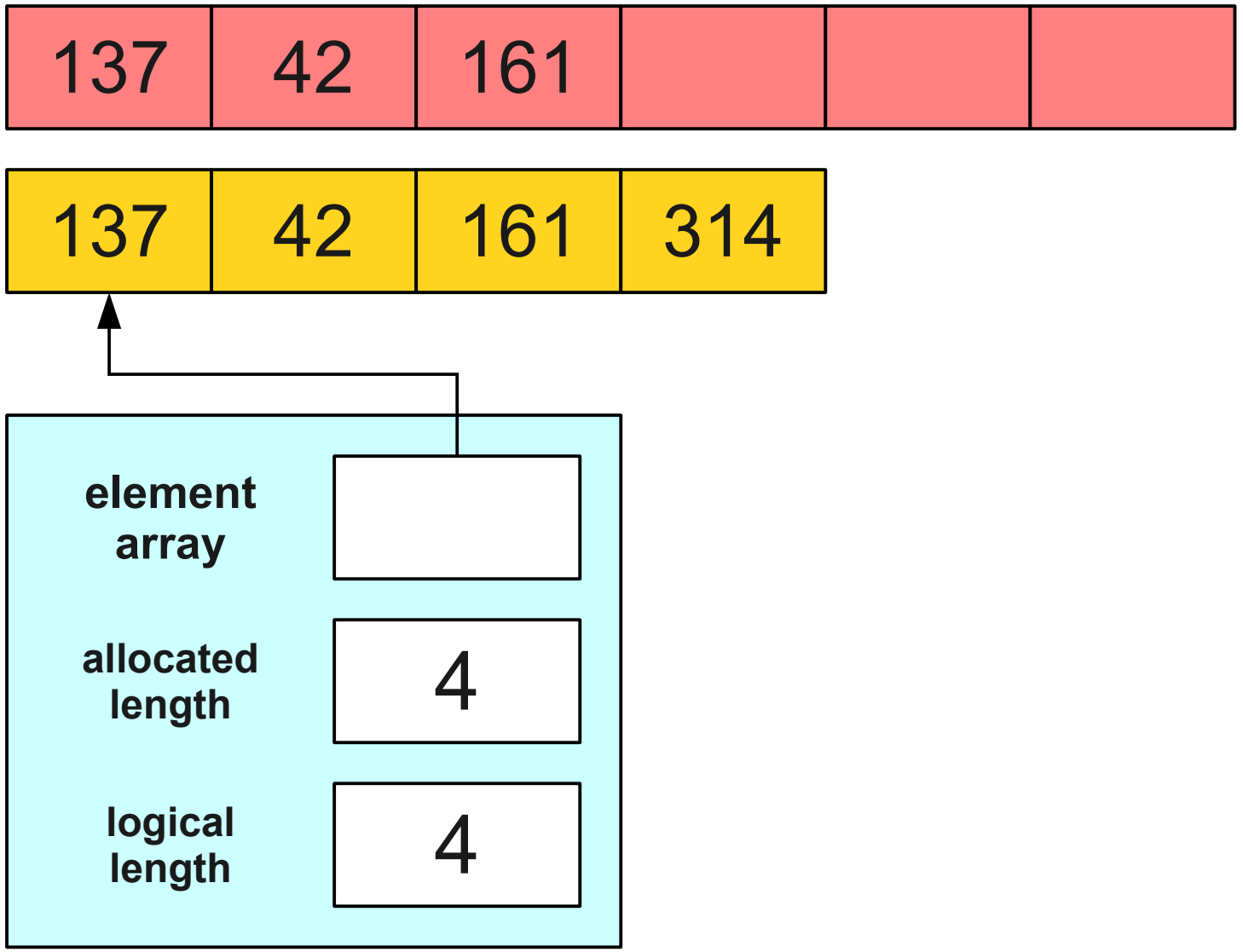

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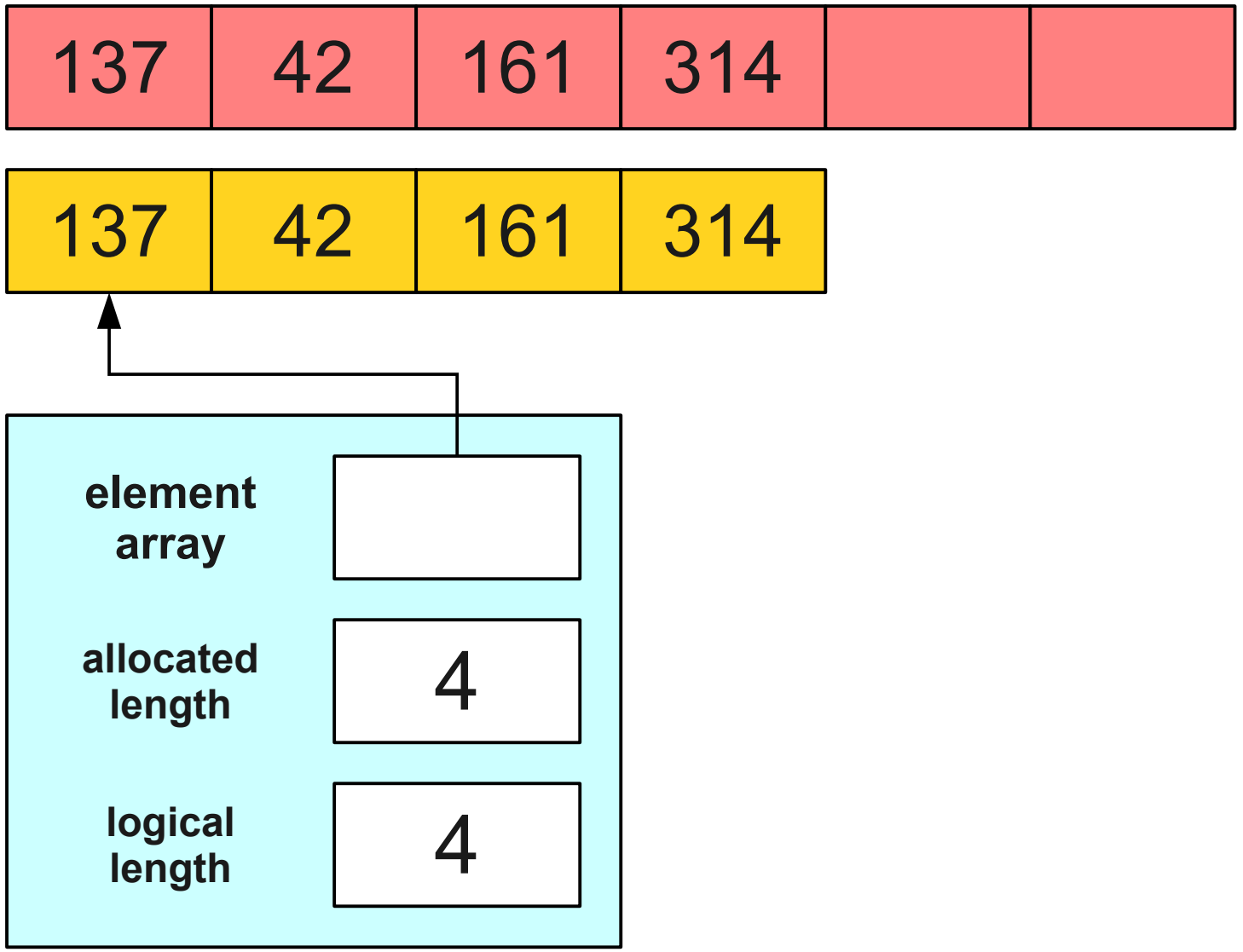

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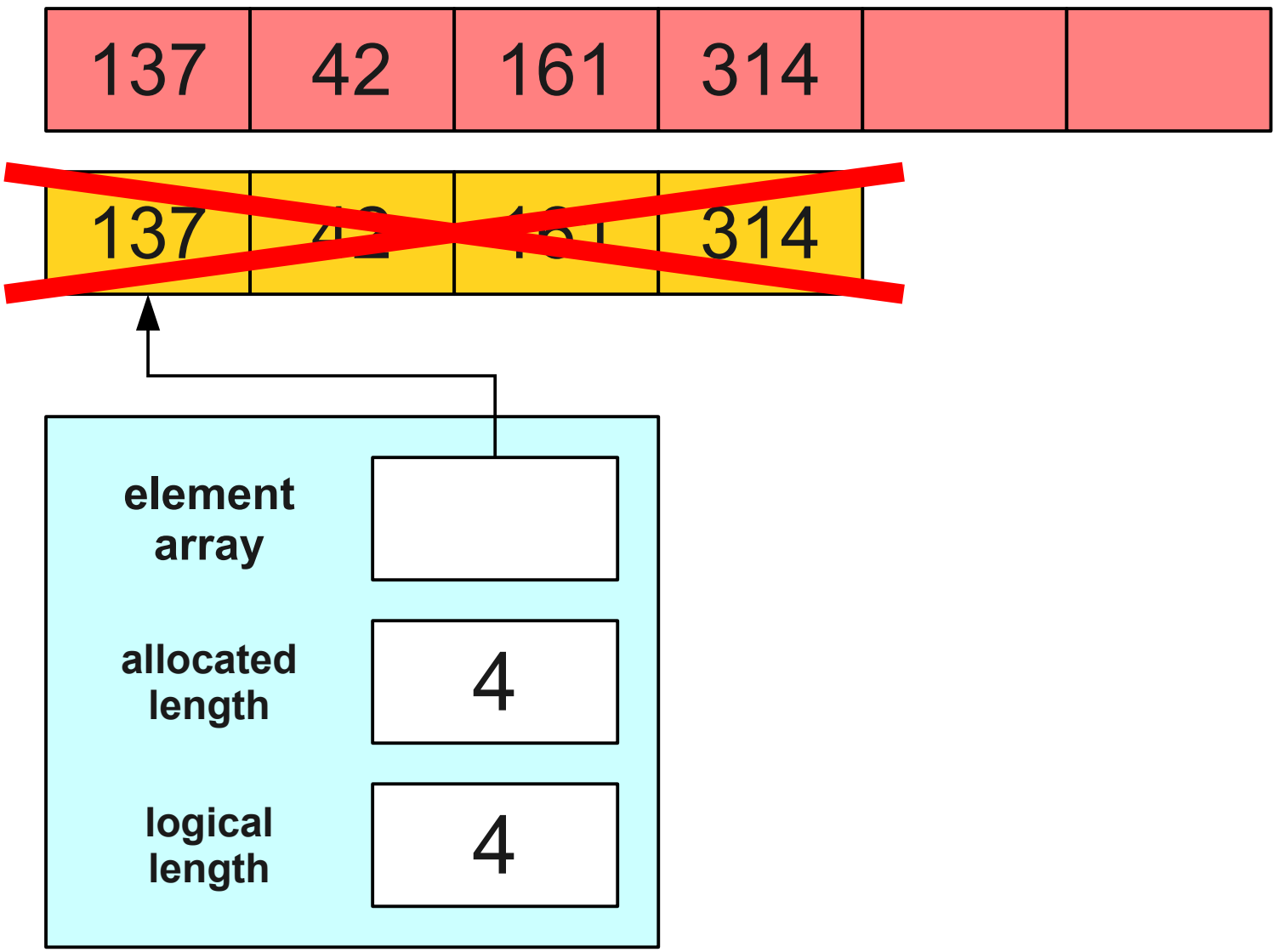

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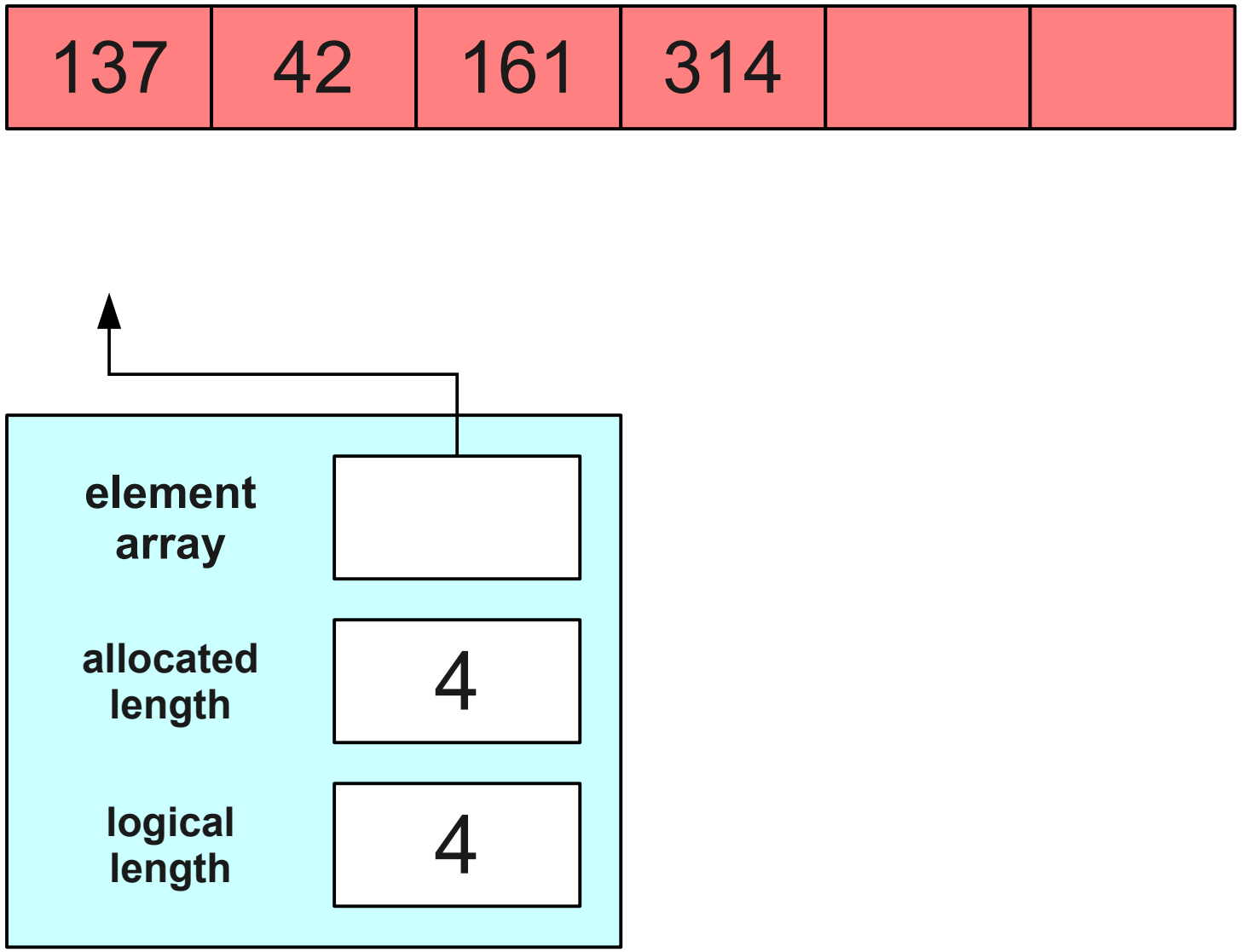

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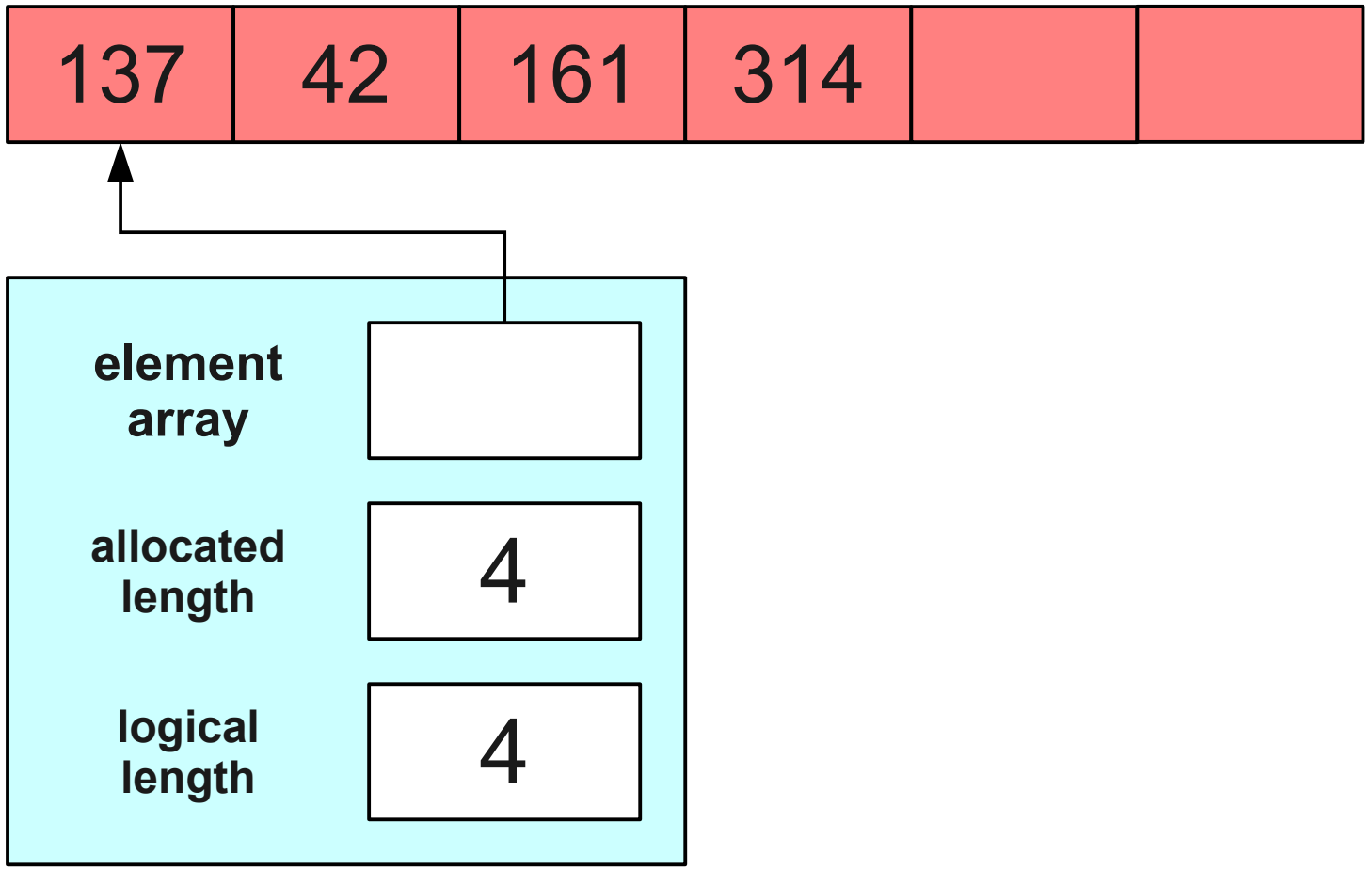

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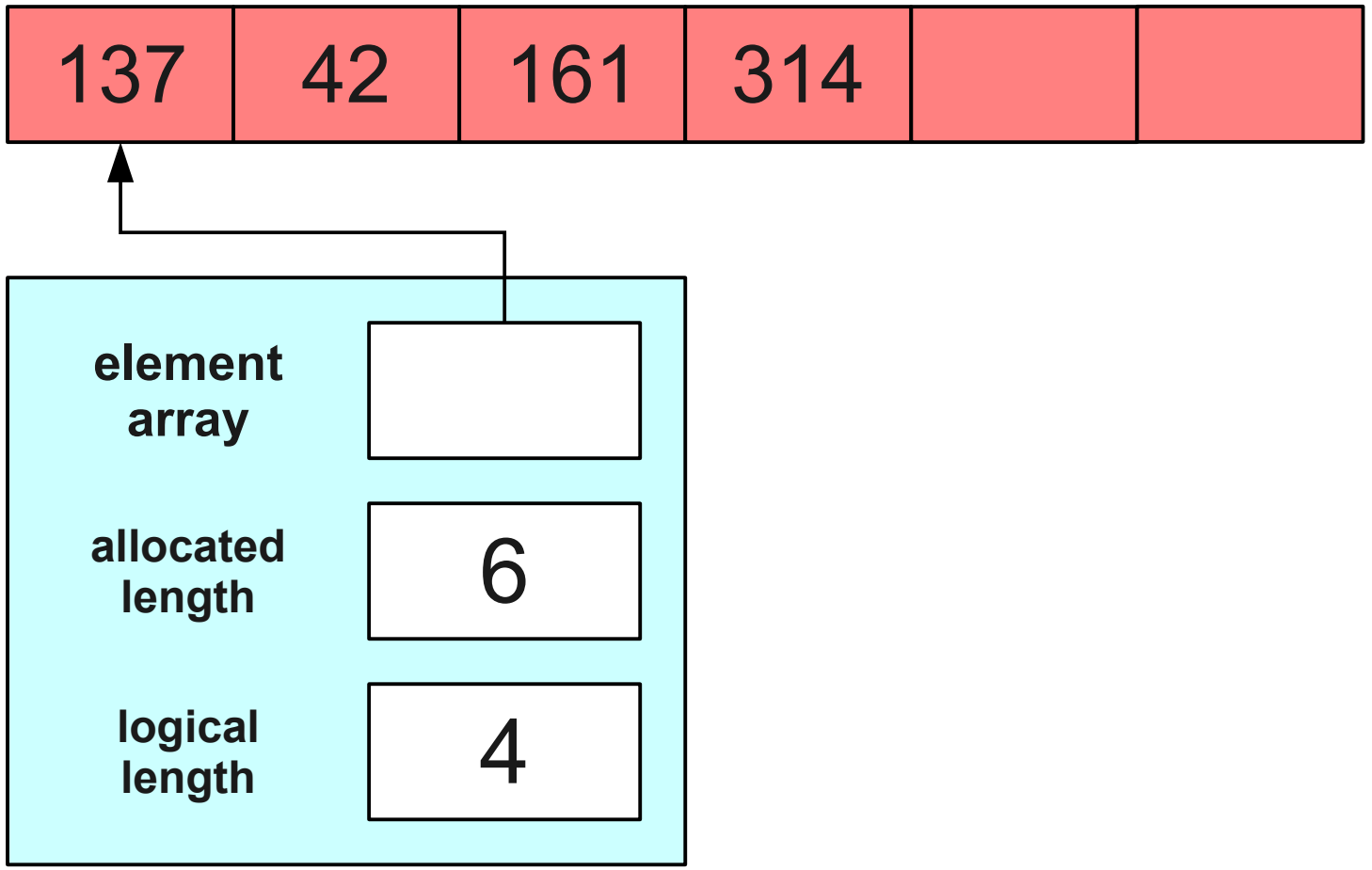

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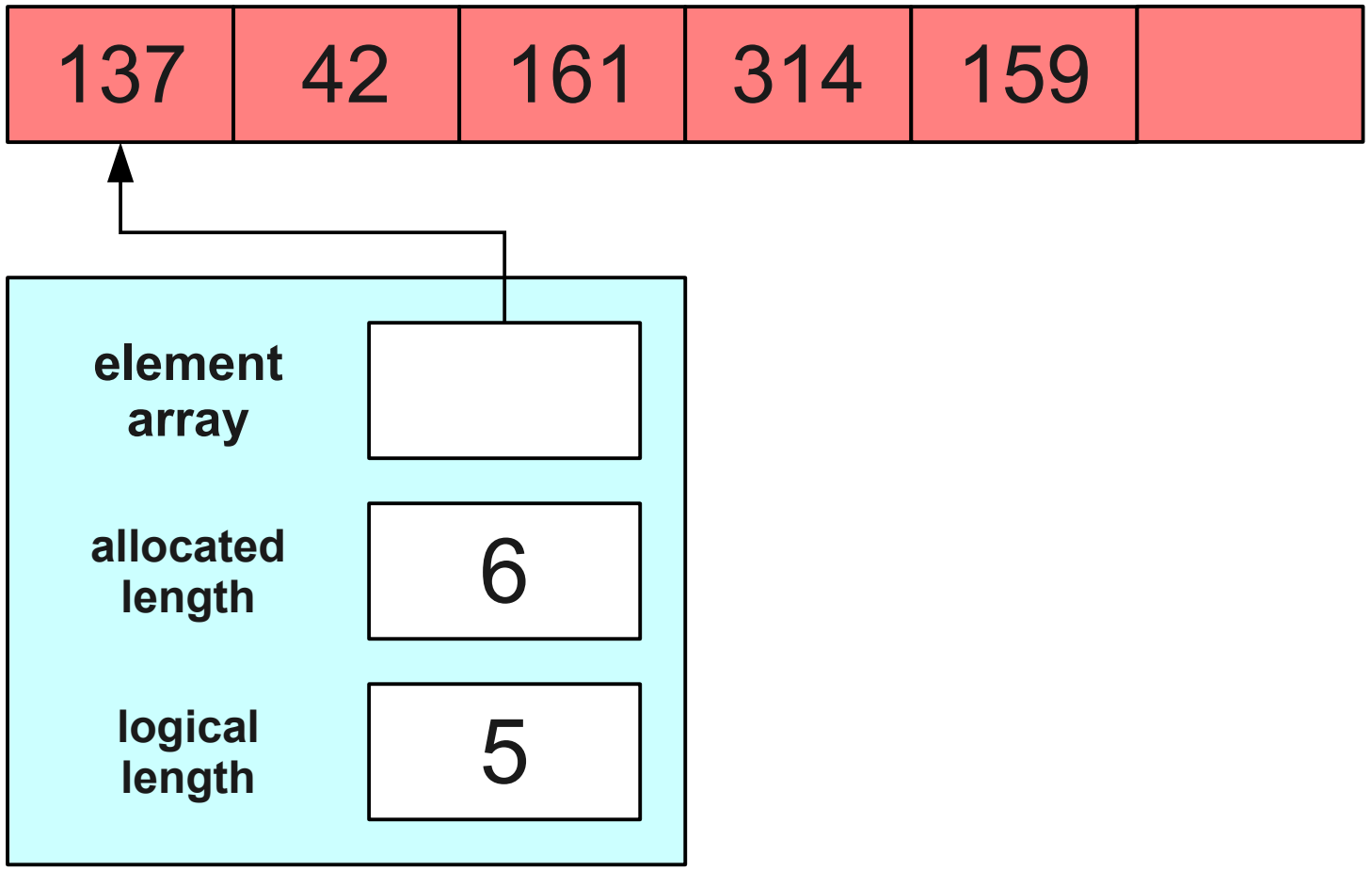

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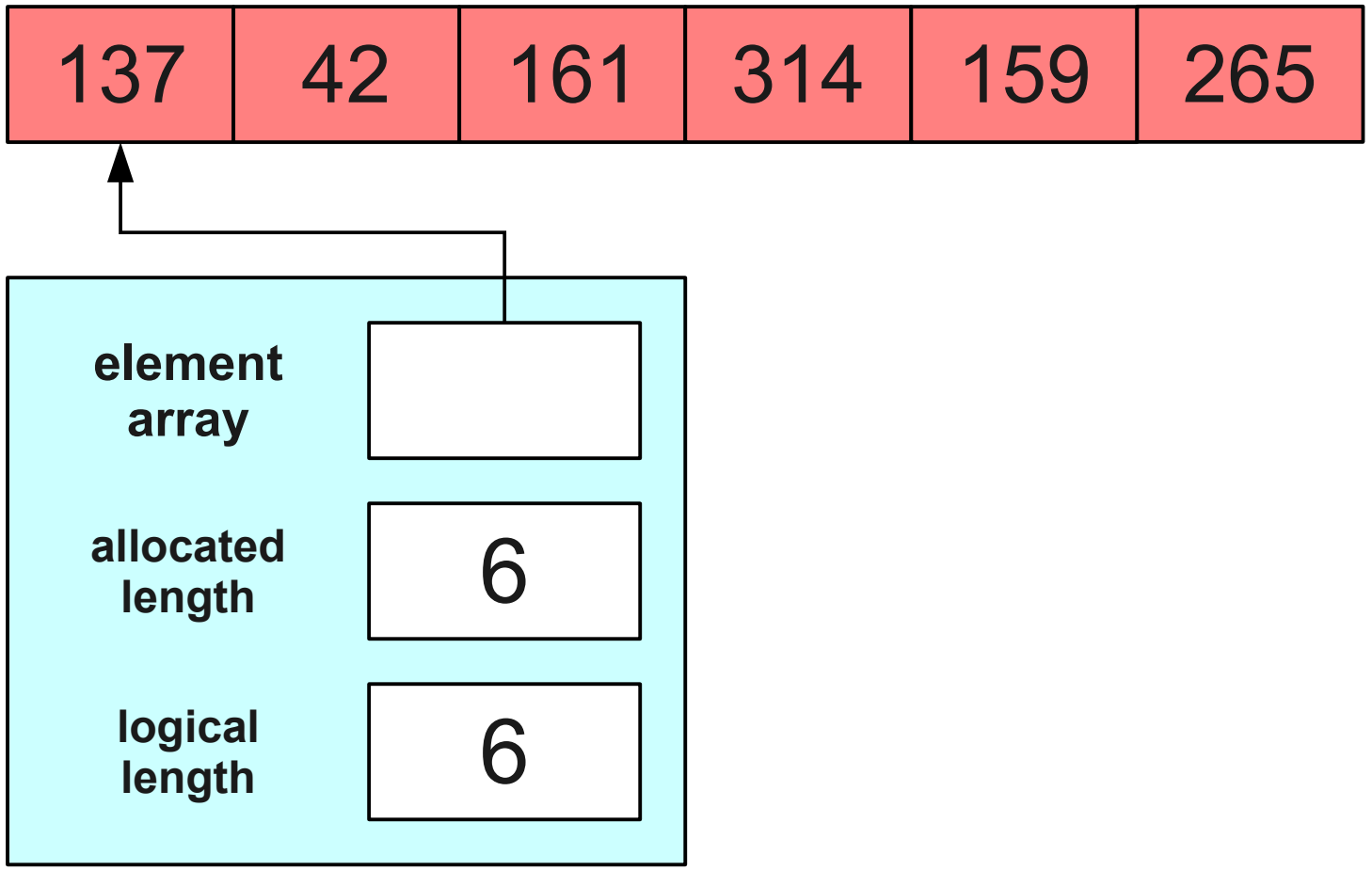
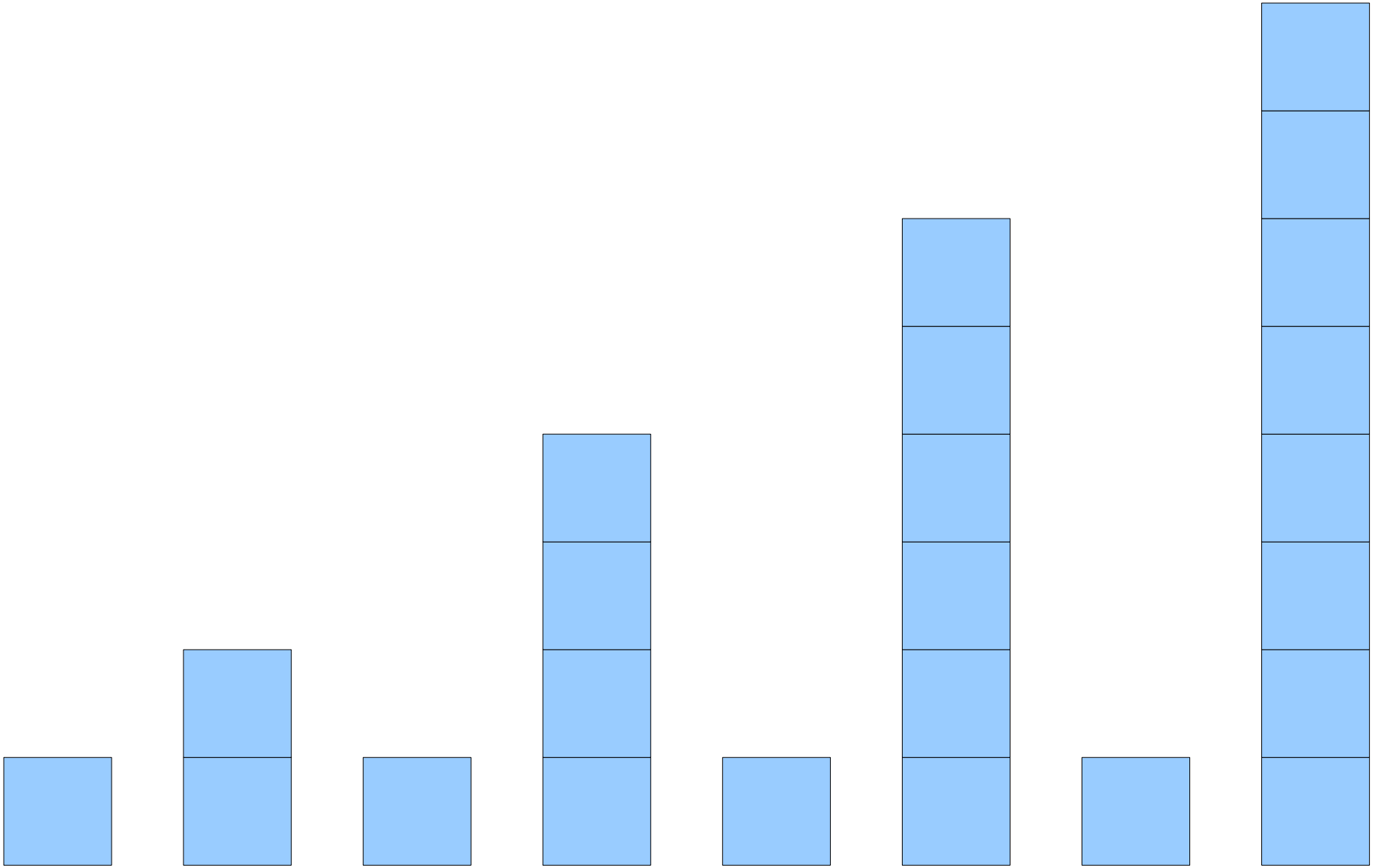


What Just Happened?

- Half of our pushes are now “easy” pushes, and half of our pushes are now “hard” pushes.
- Hard pushes still take time O(n).
- Easy pushes only take time O(1).
- Worst-case is still O(n).
- What about the average case?

Analyzing the Work



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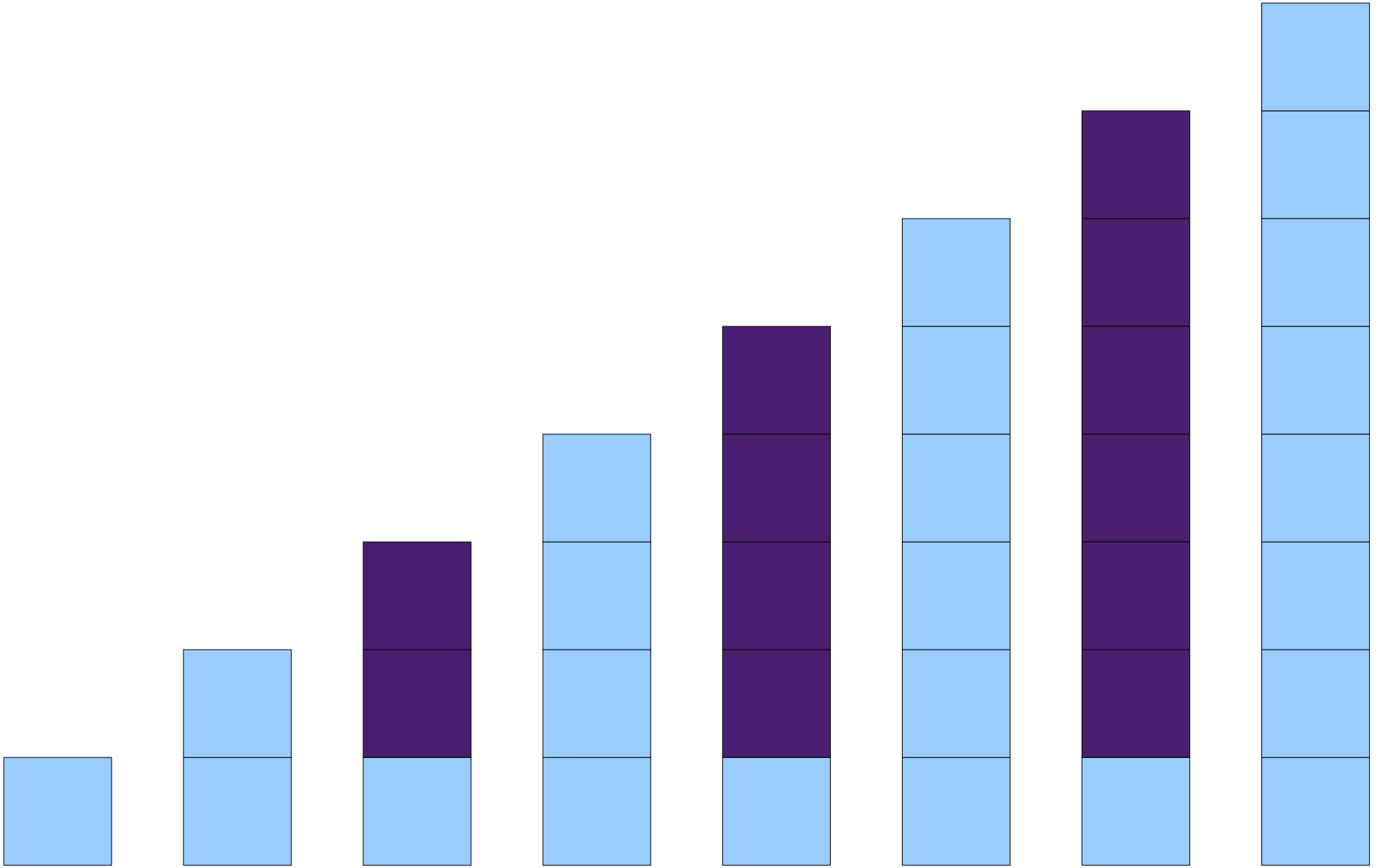
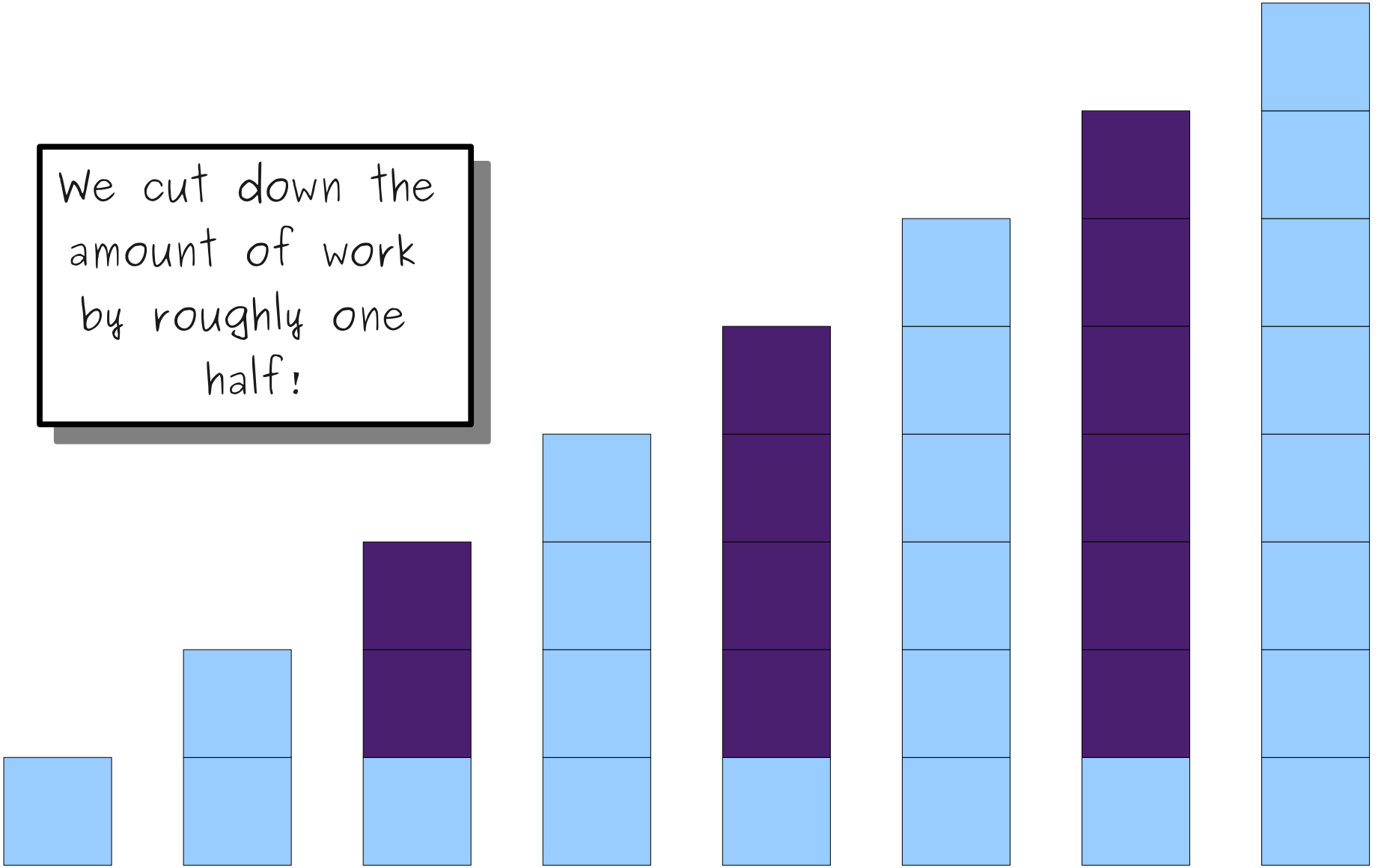


Analyzing the Work

We cut down the amount of work by roughly one half!



A Different Analysis

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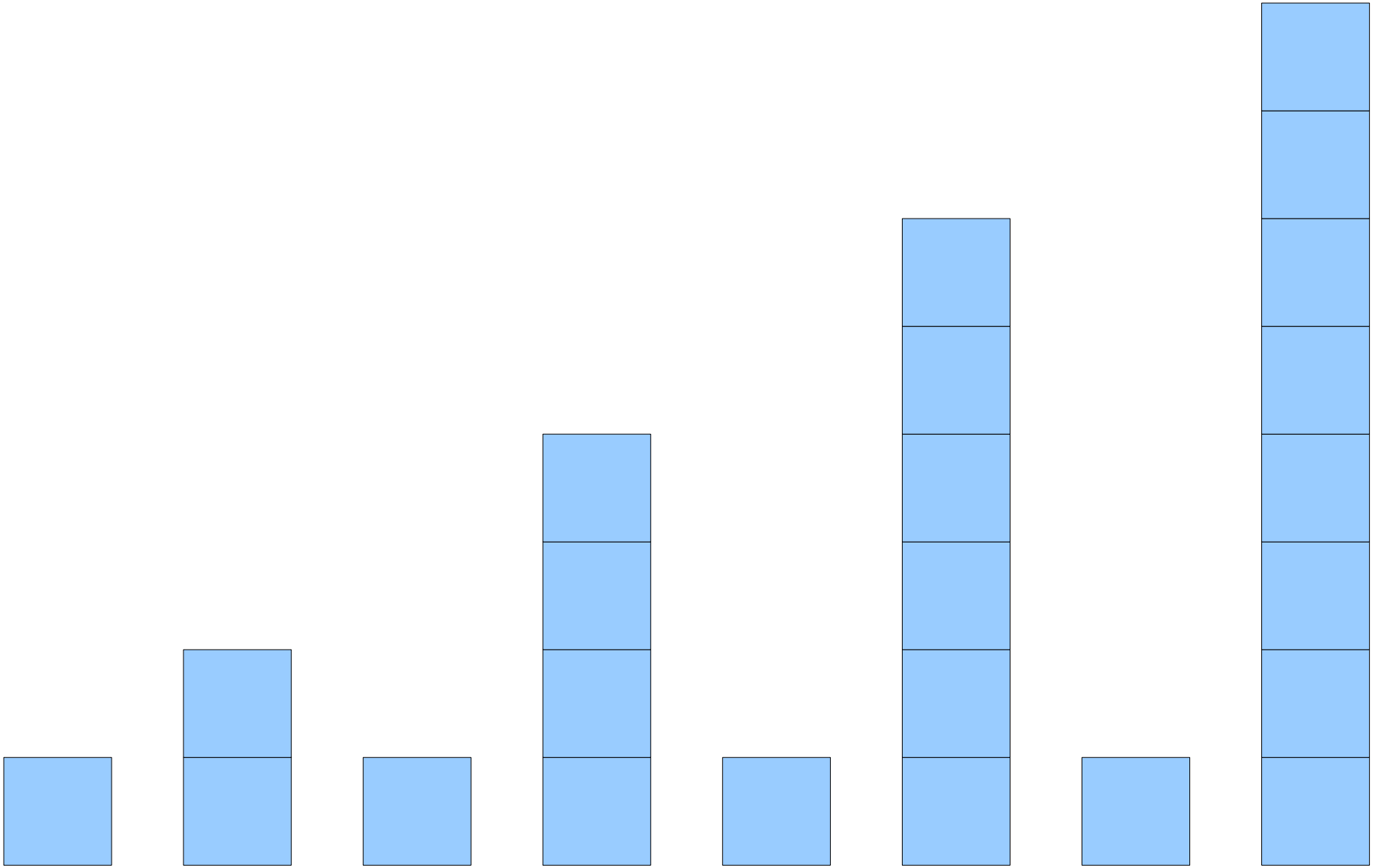
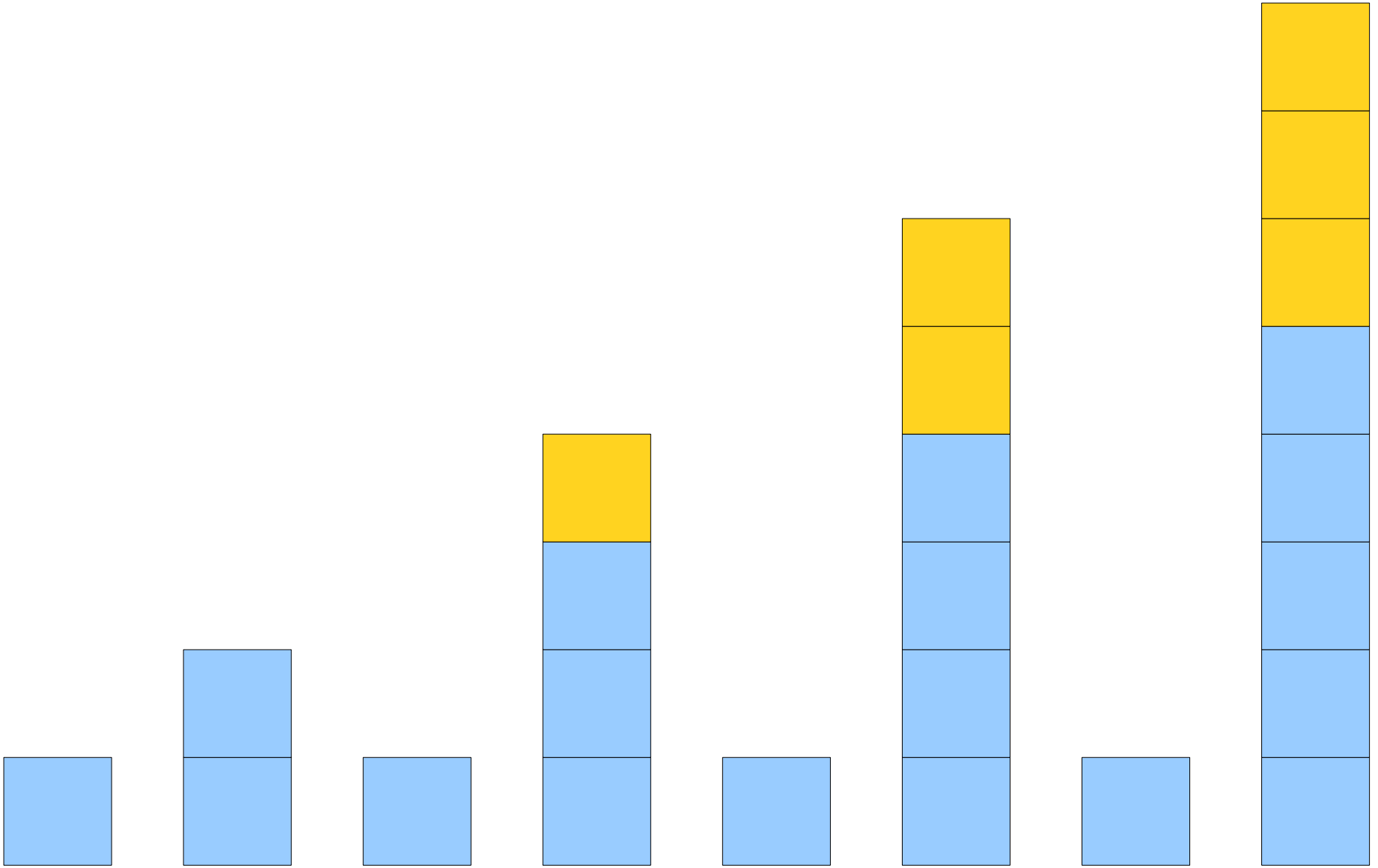


A Different Analysis



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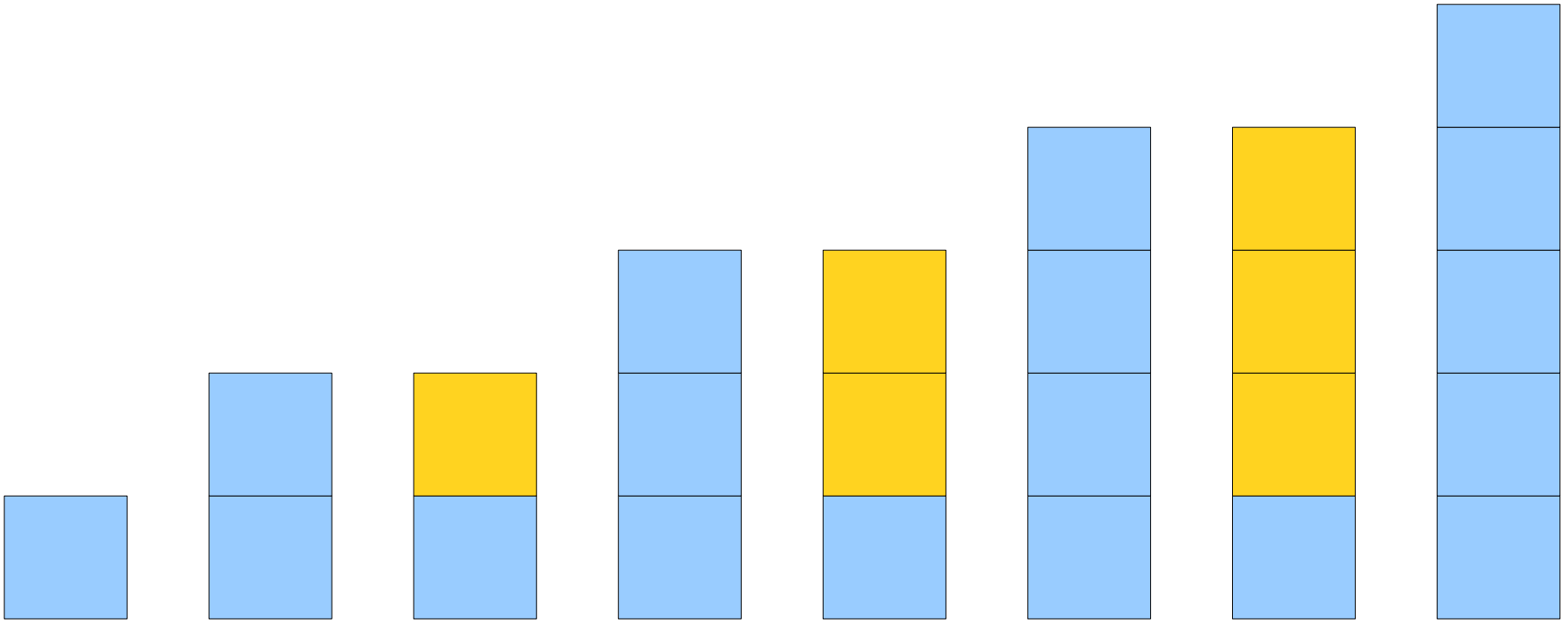

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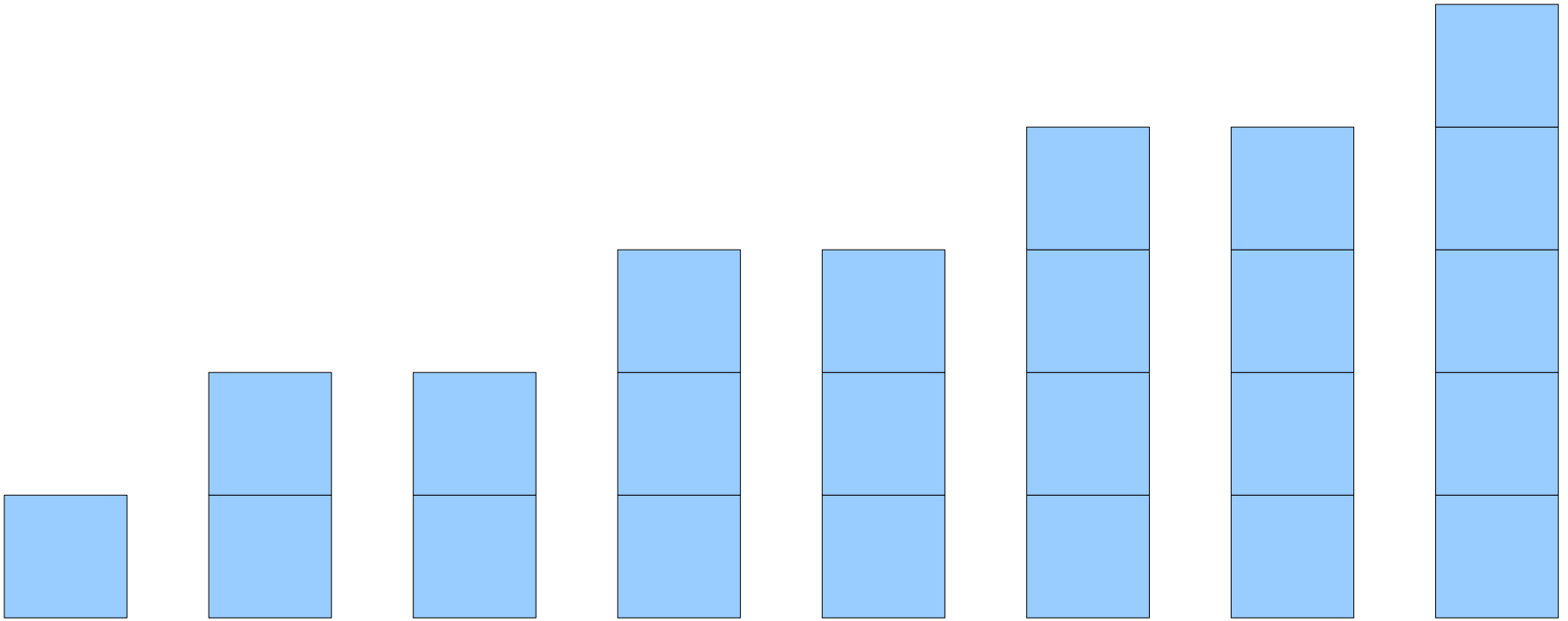

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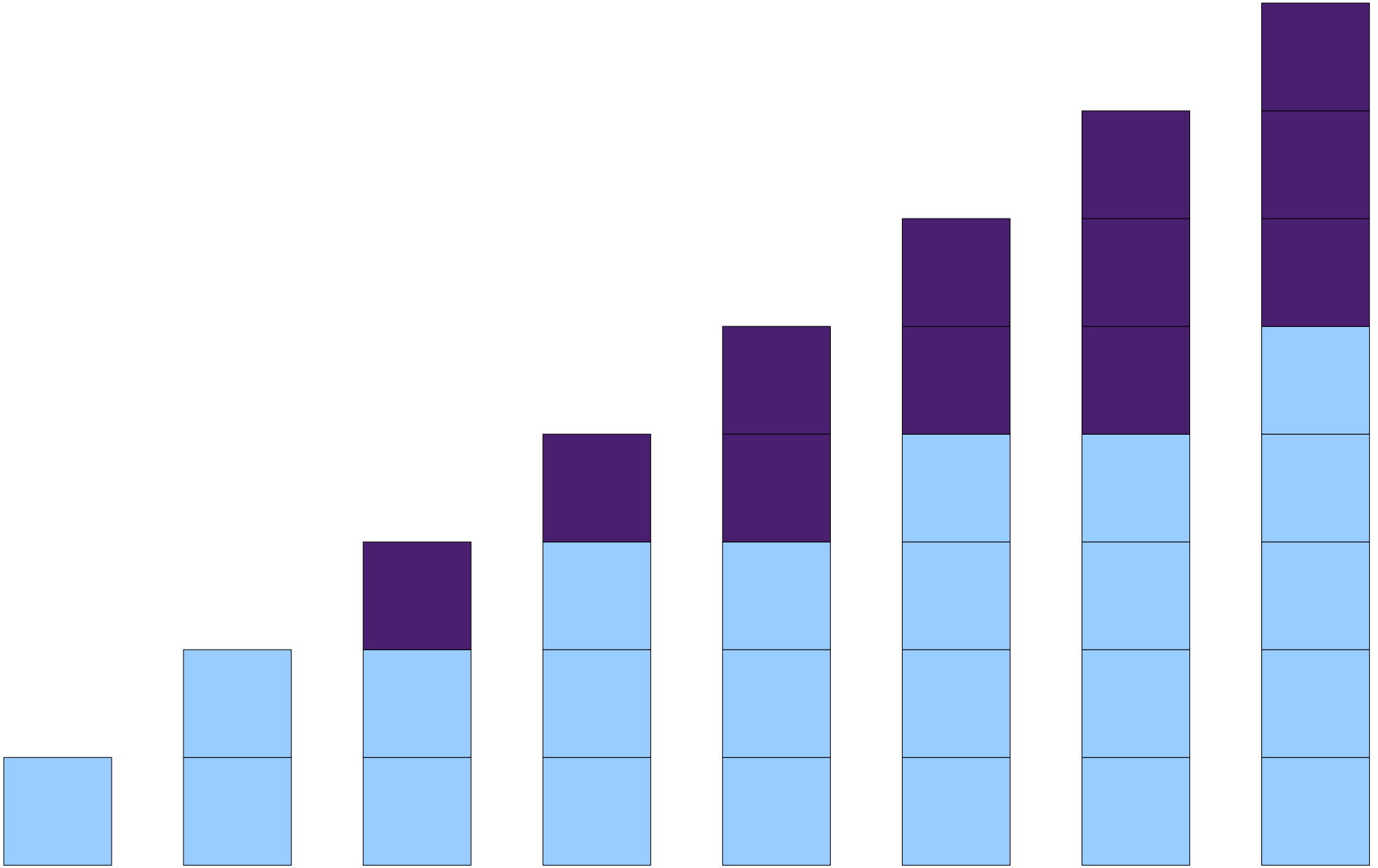

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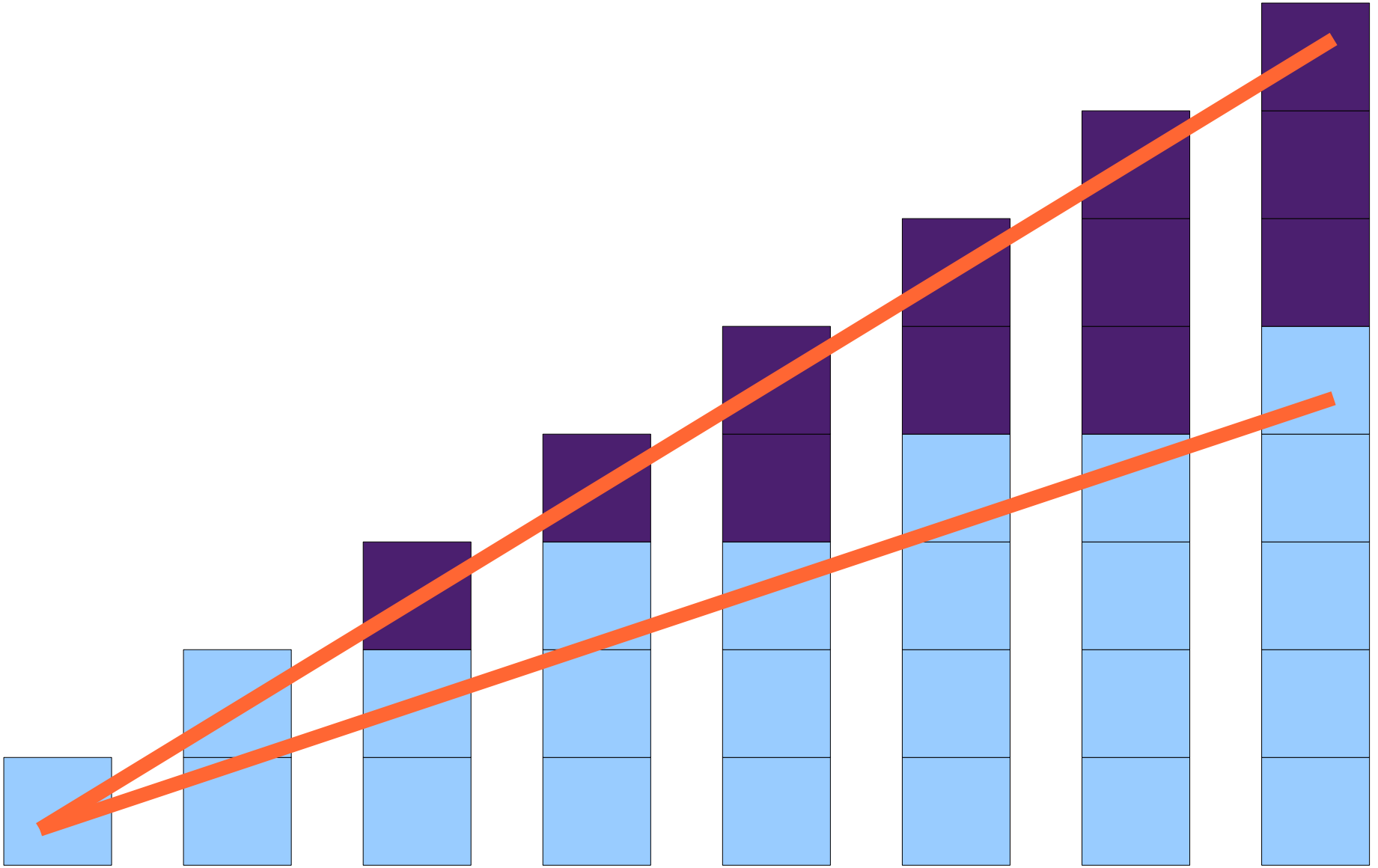

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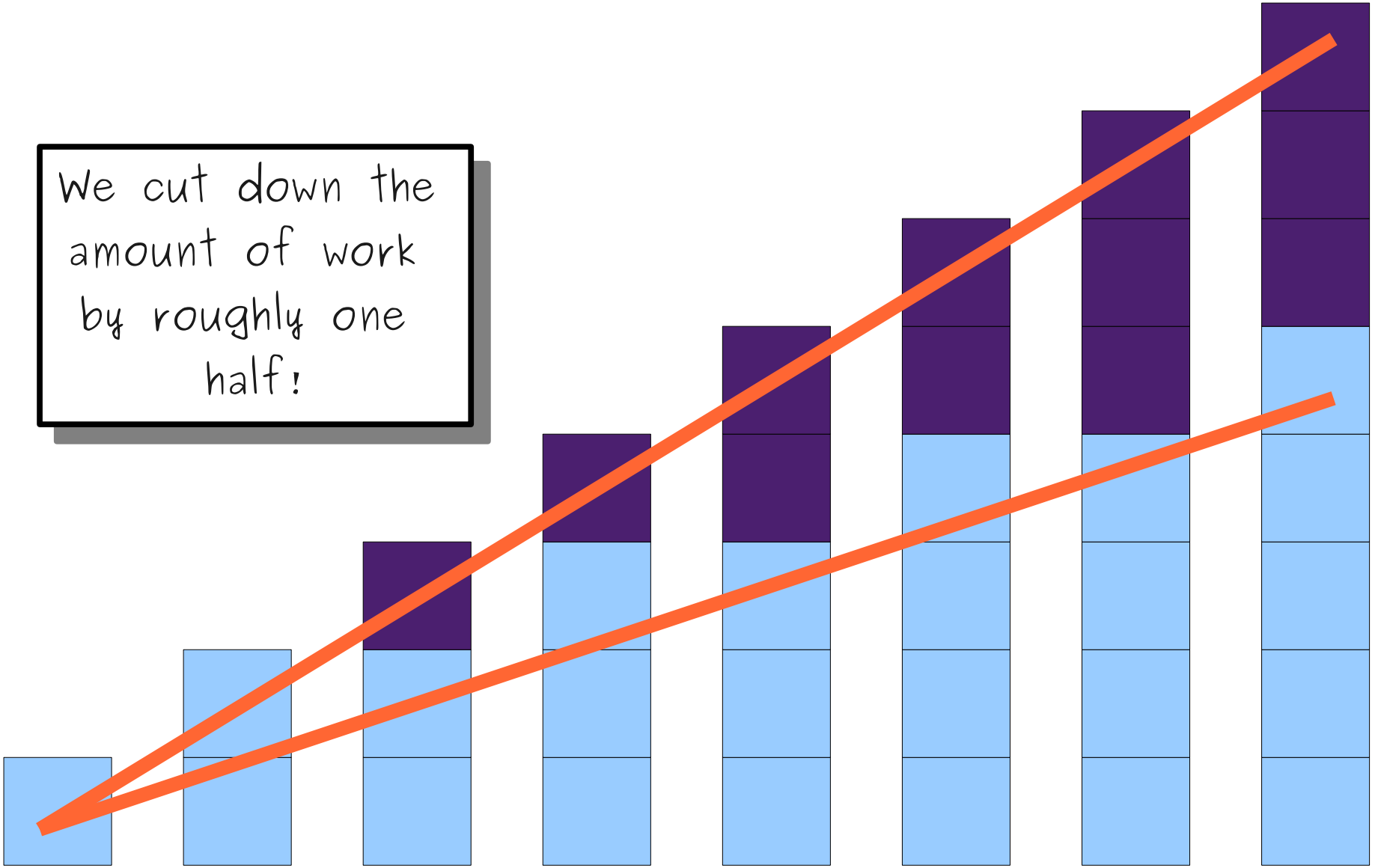
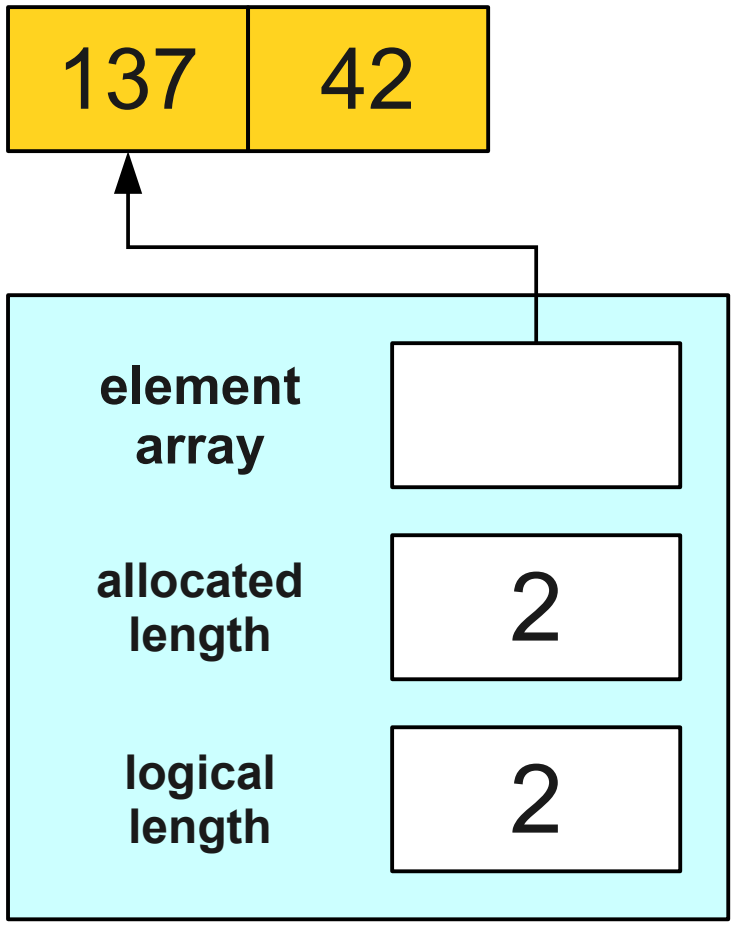


How does it stack up?

A Much Better Idea



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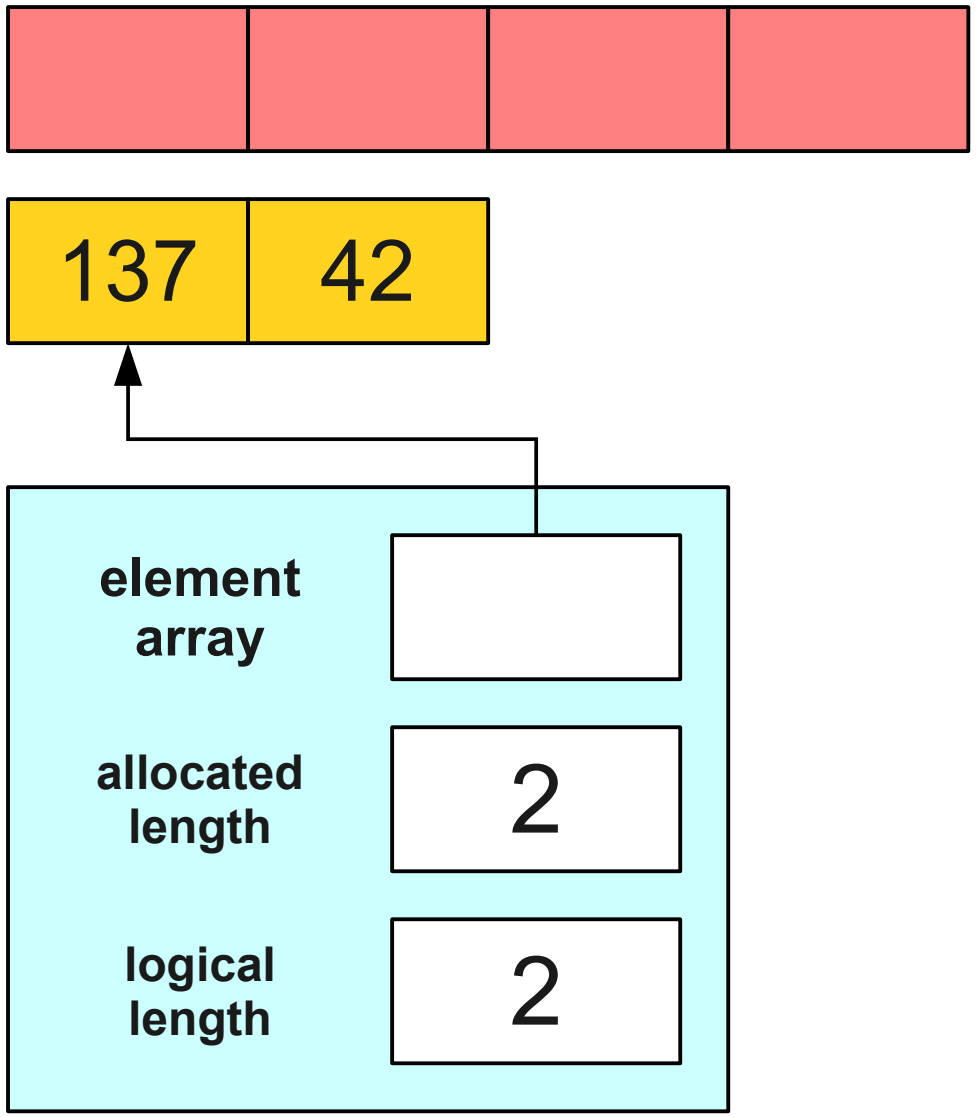

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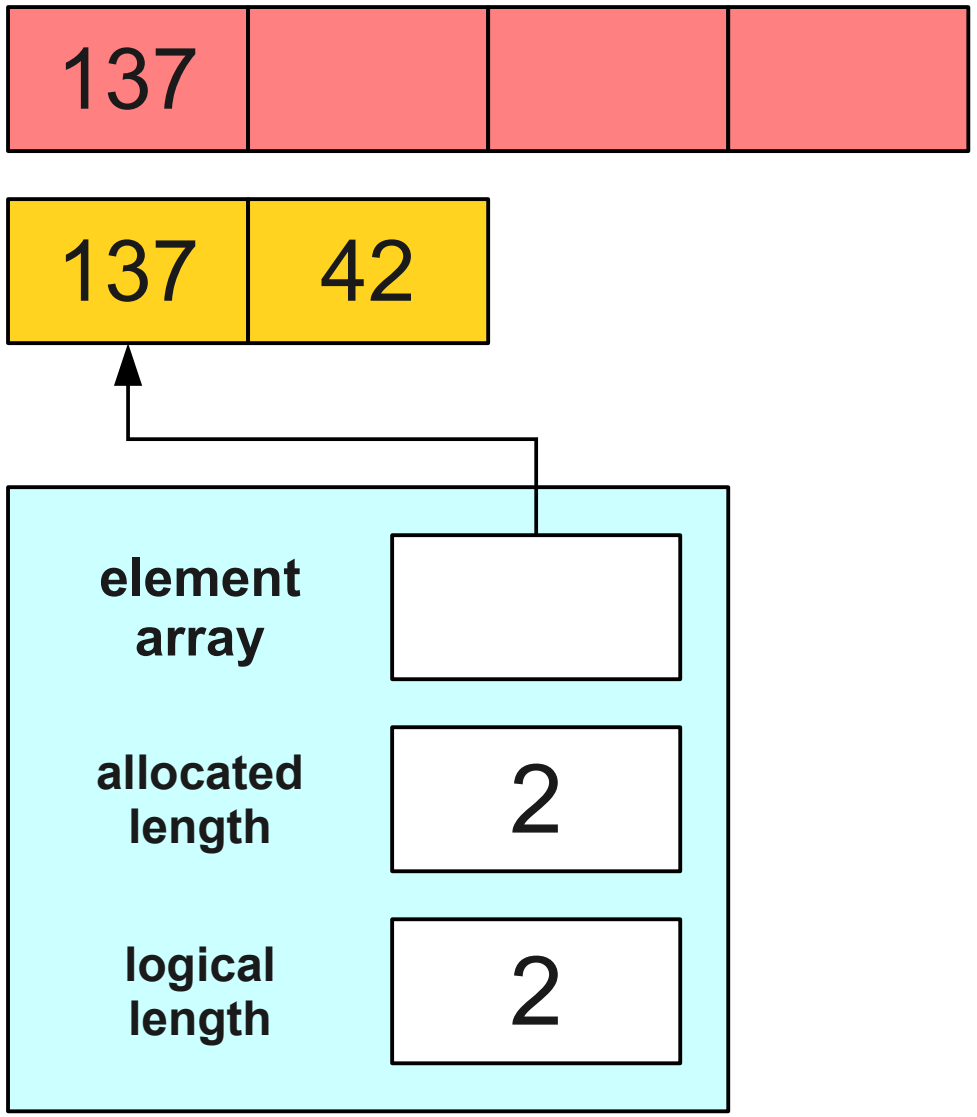

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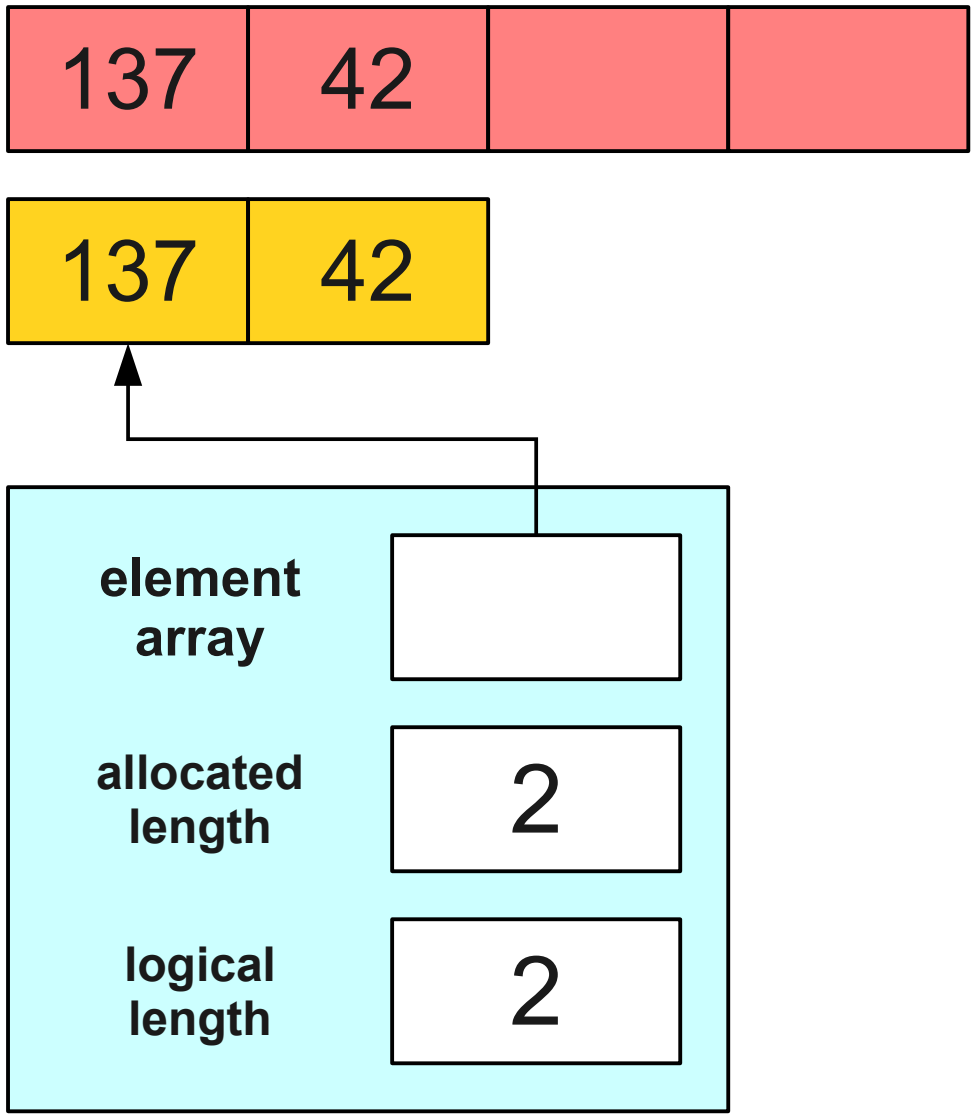

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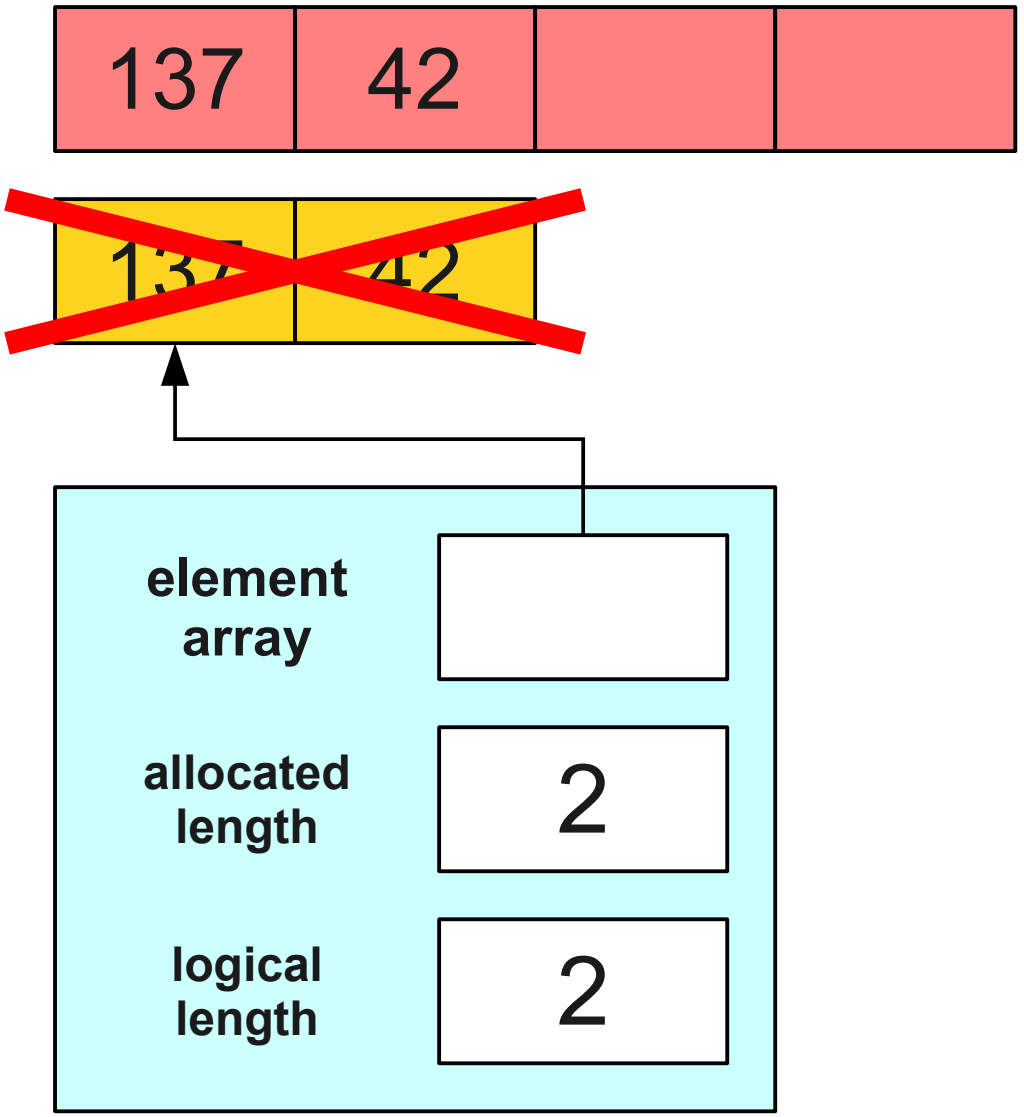

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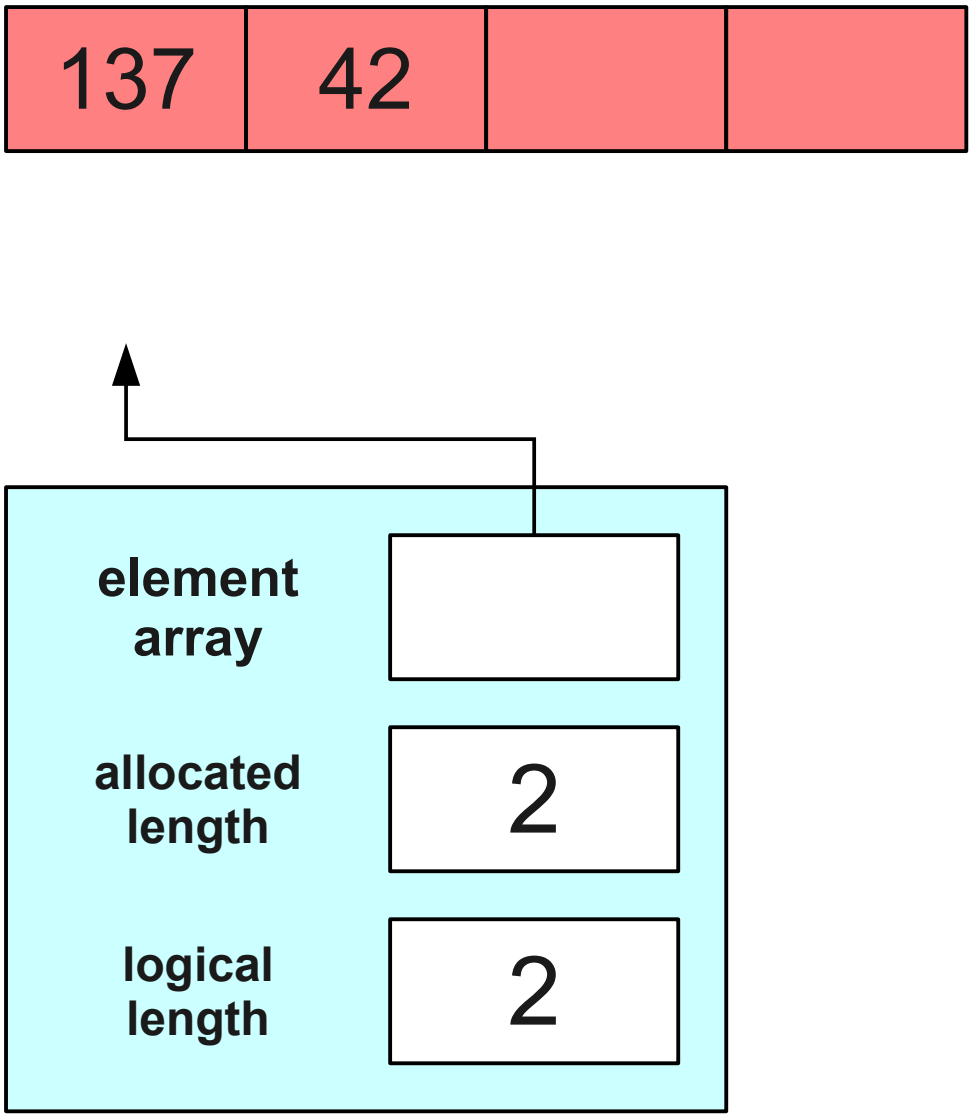

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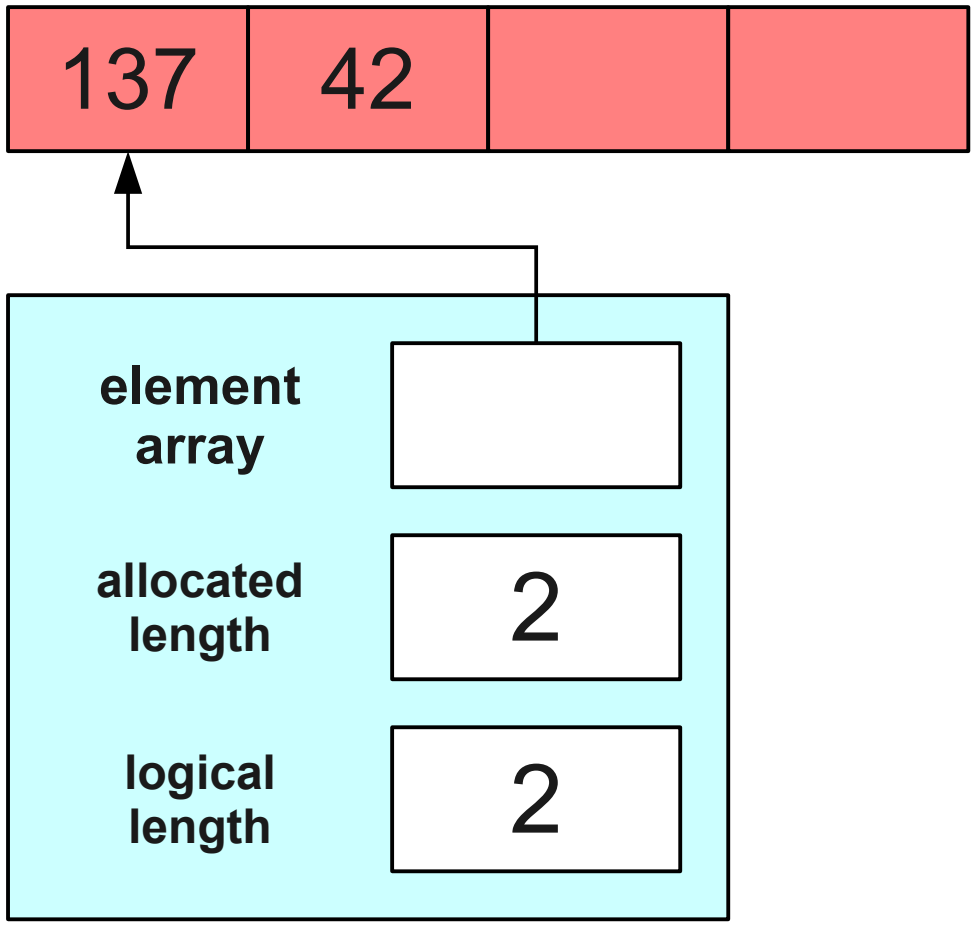

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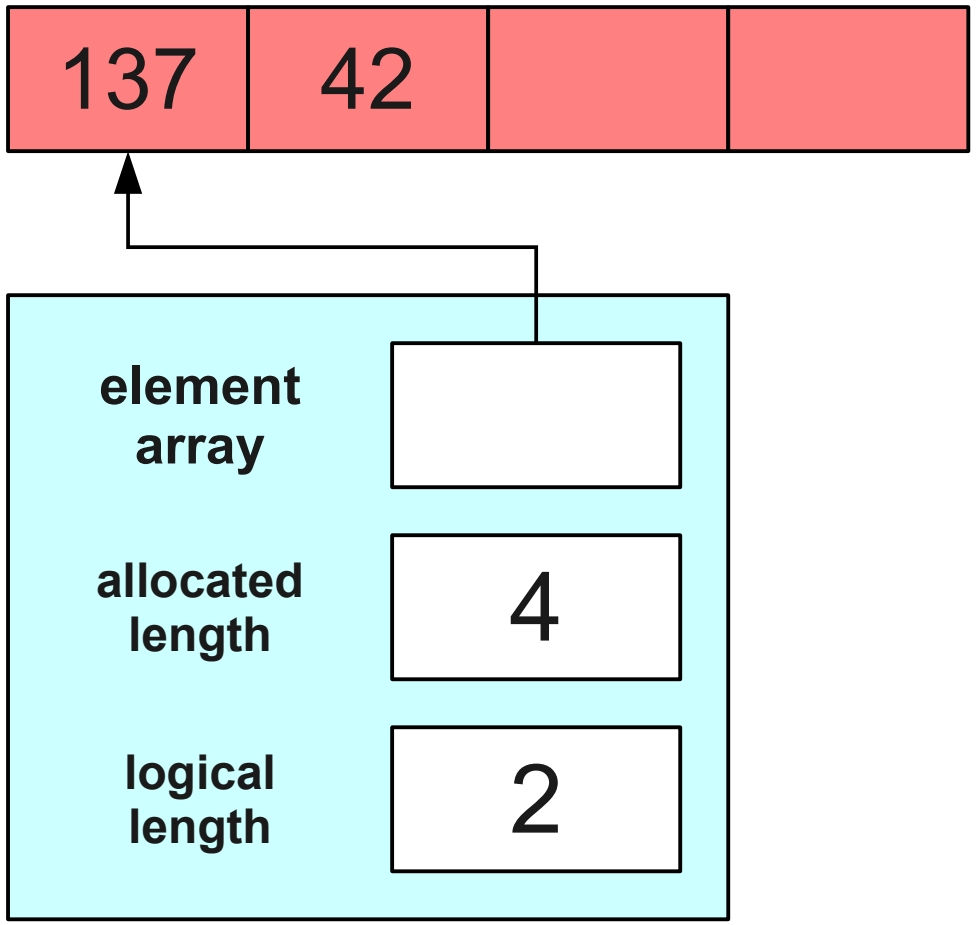

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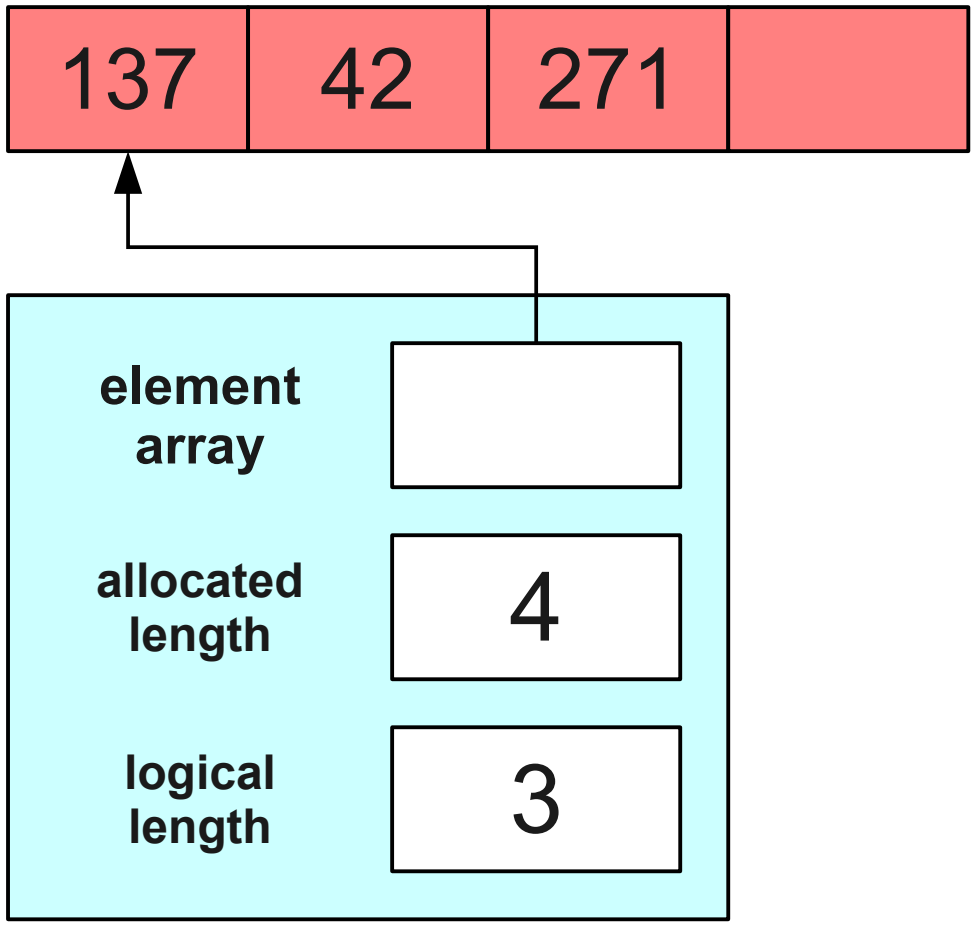

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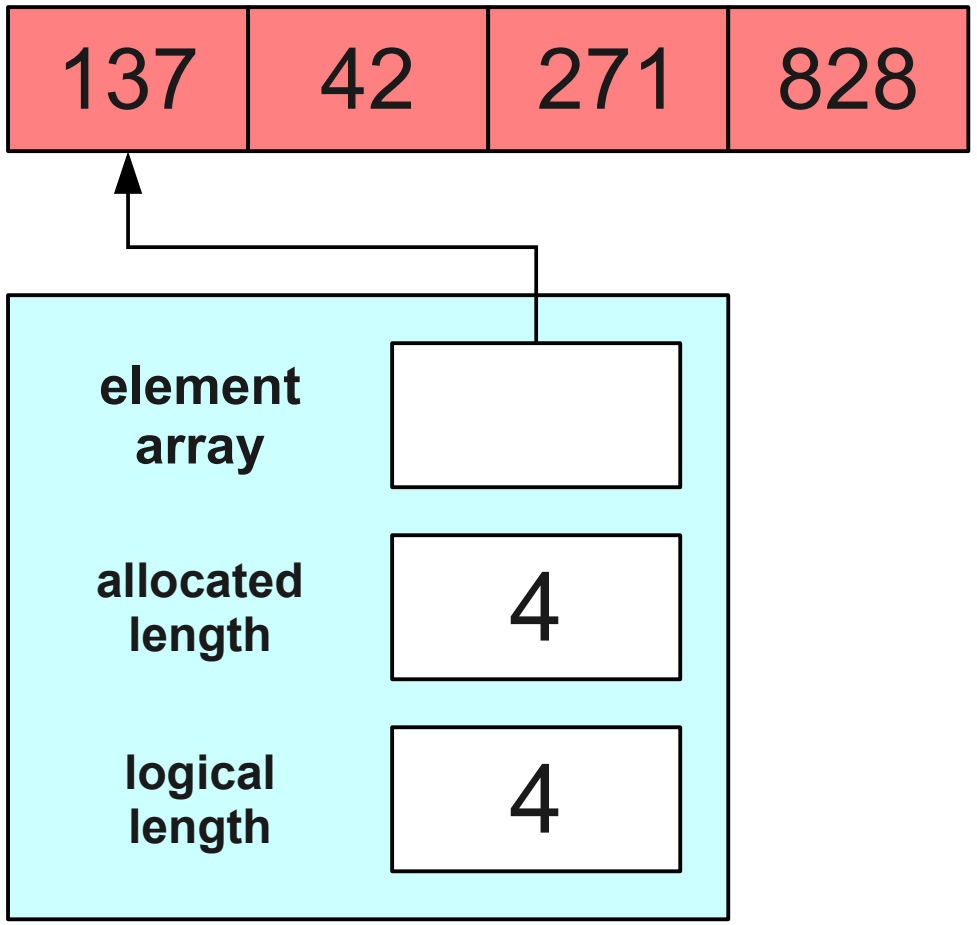

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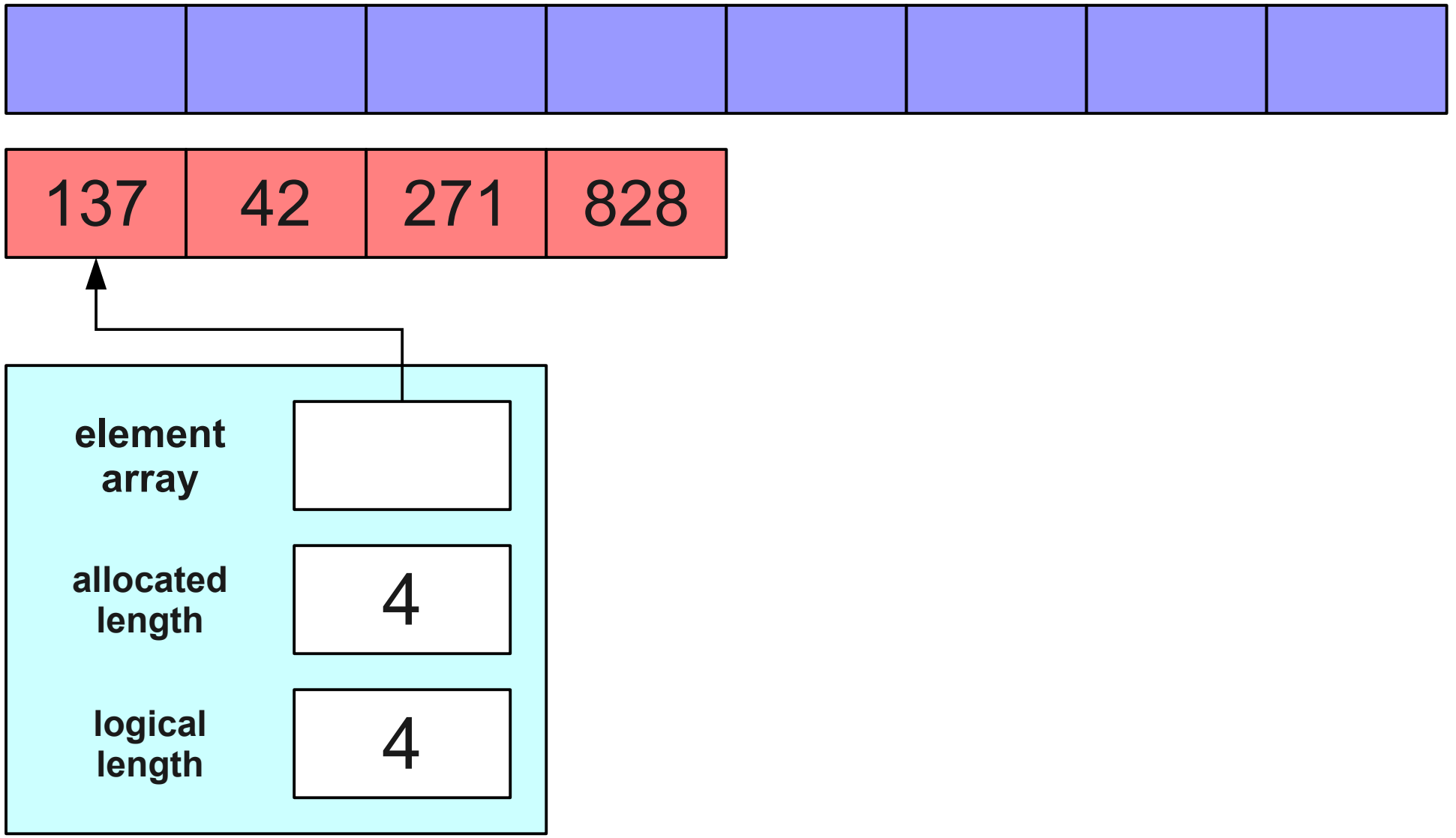

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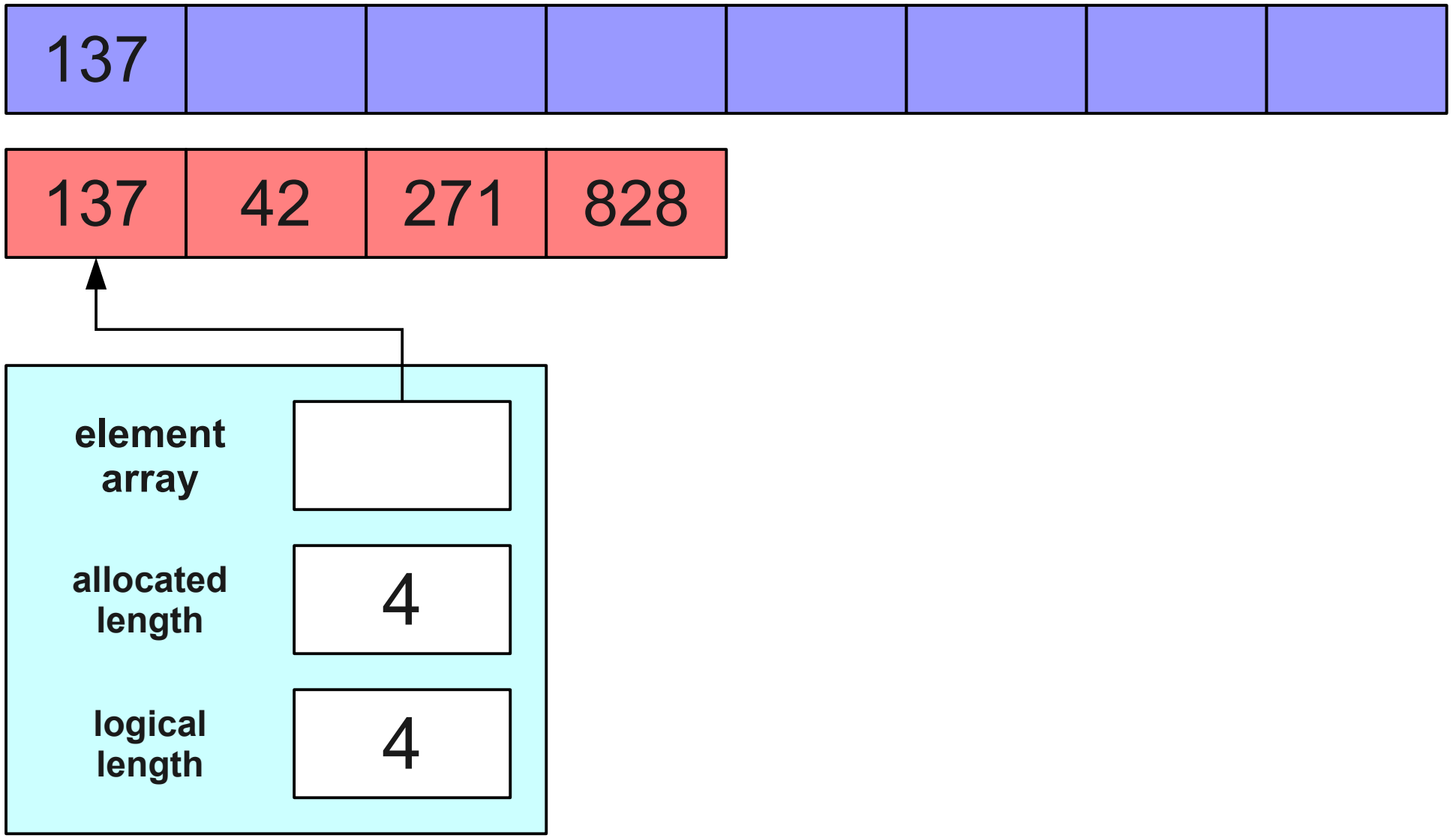

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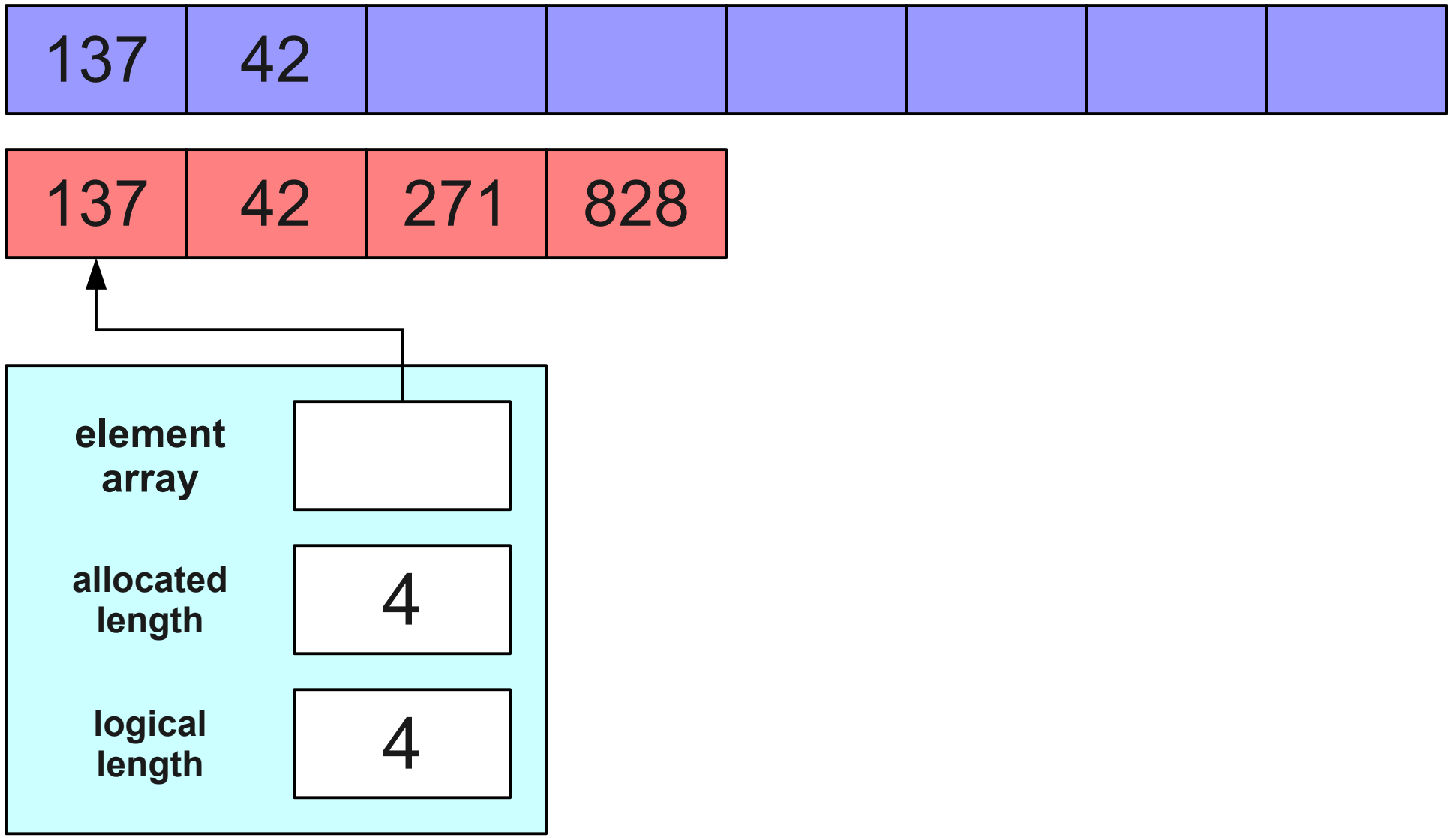

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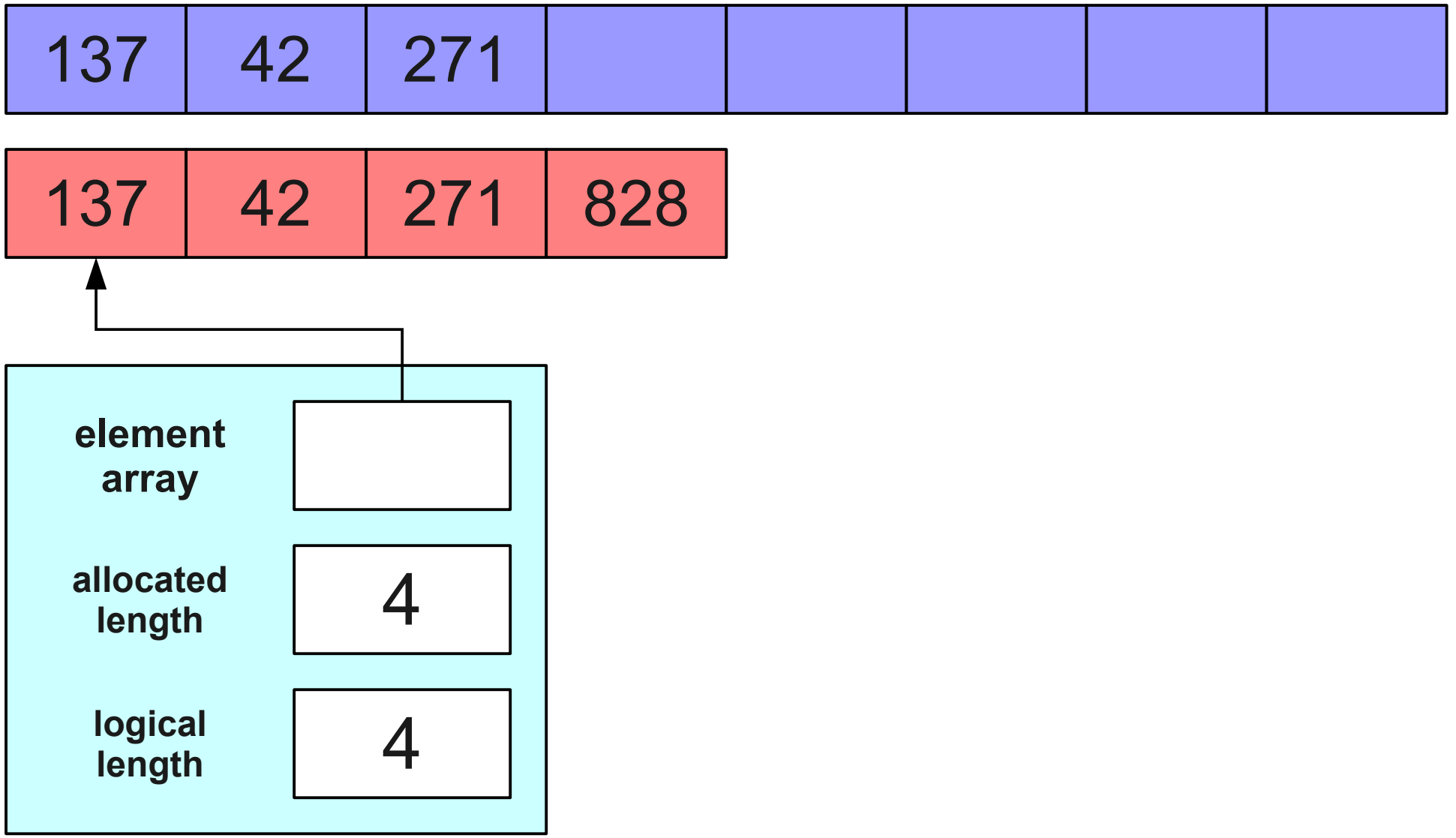

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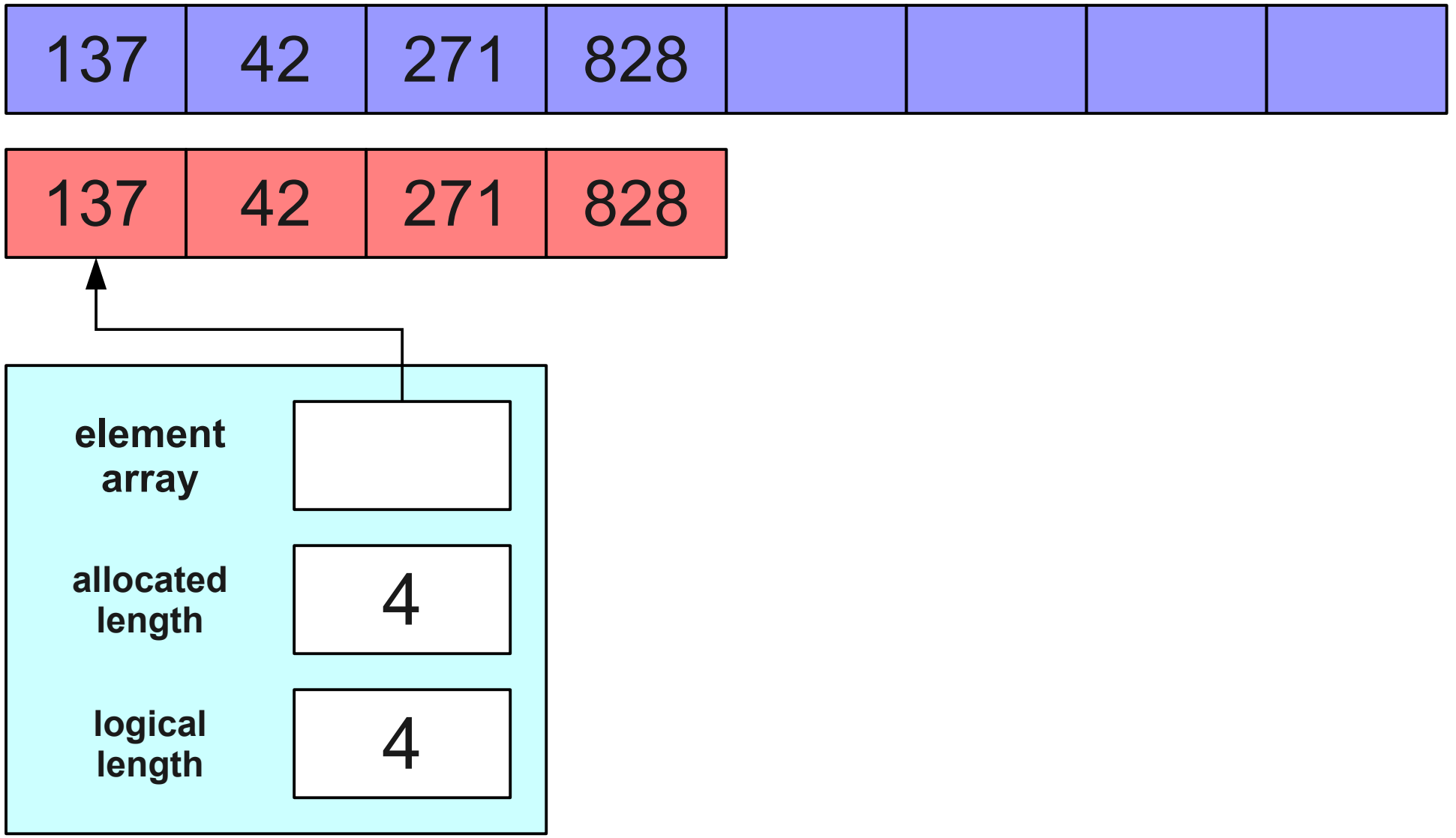

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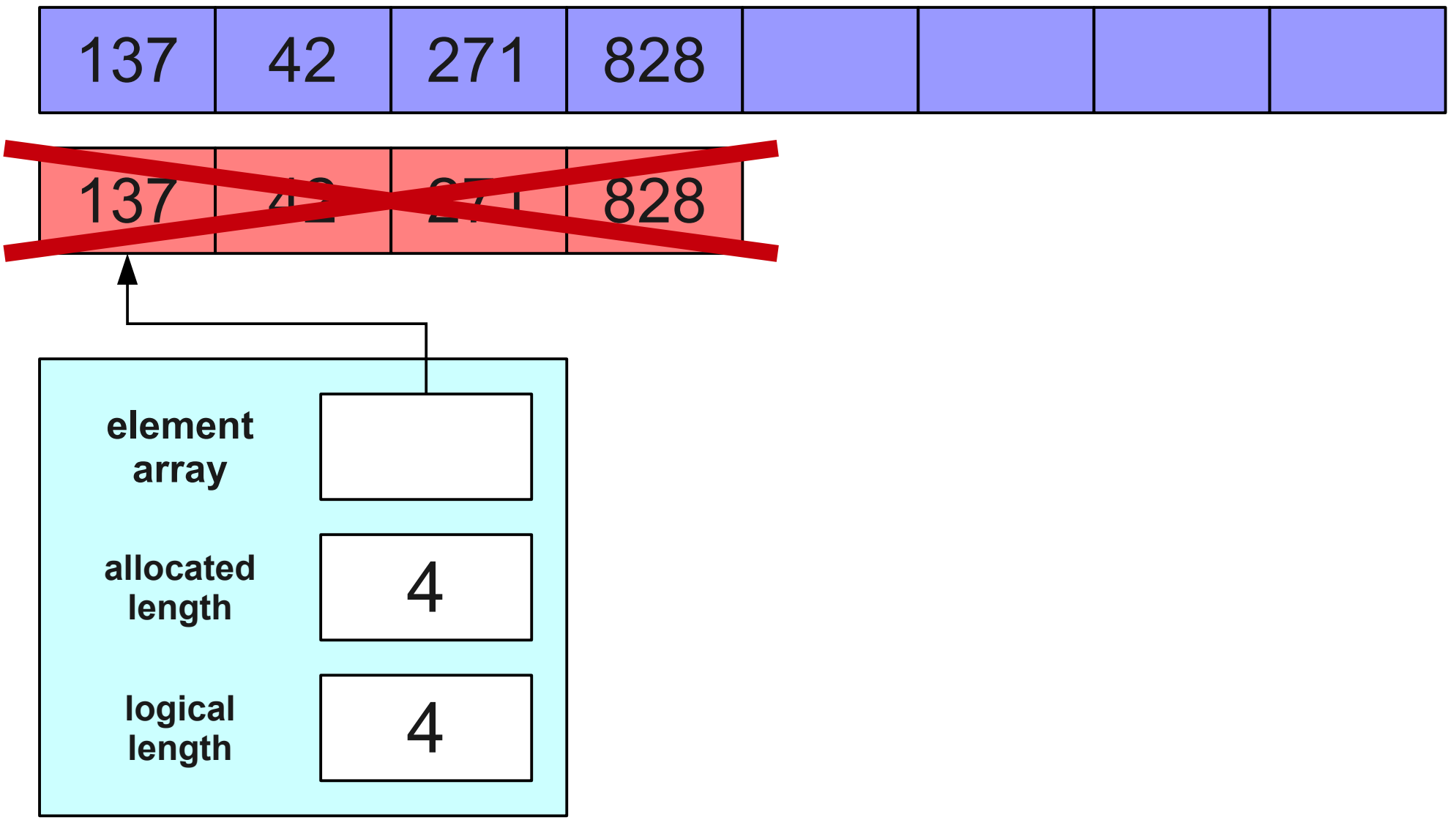

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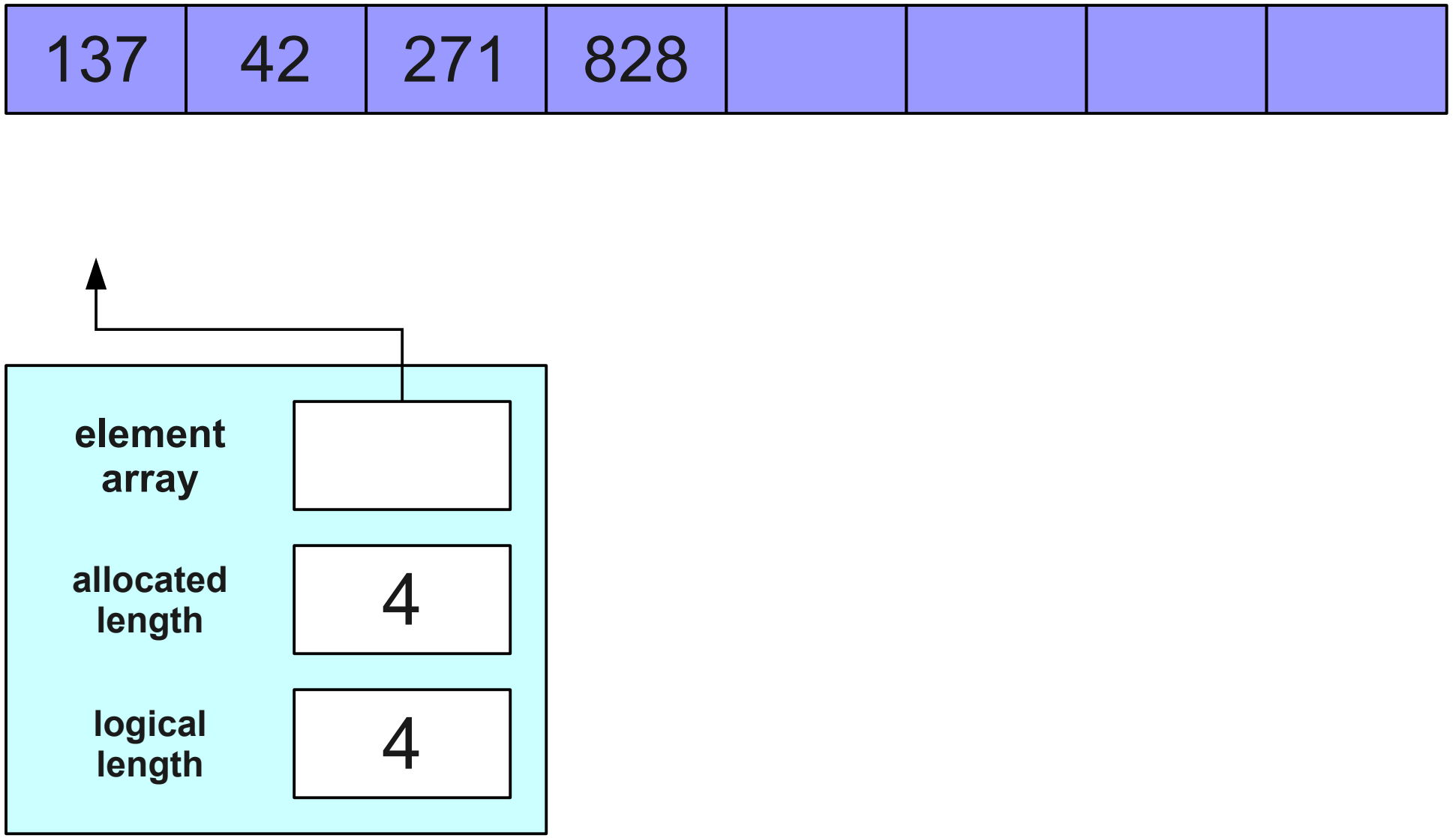

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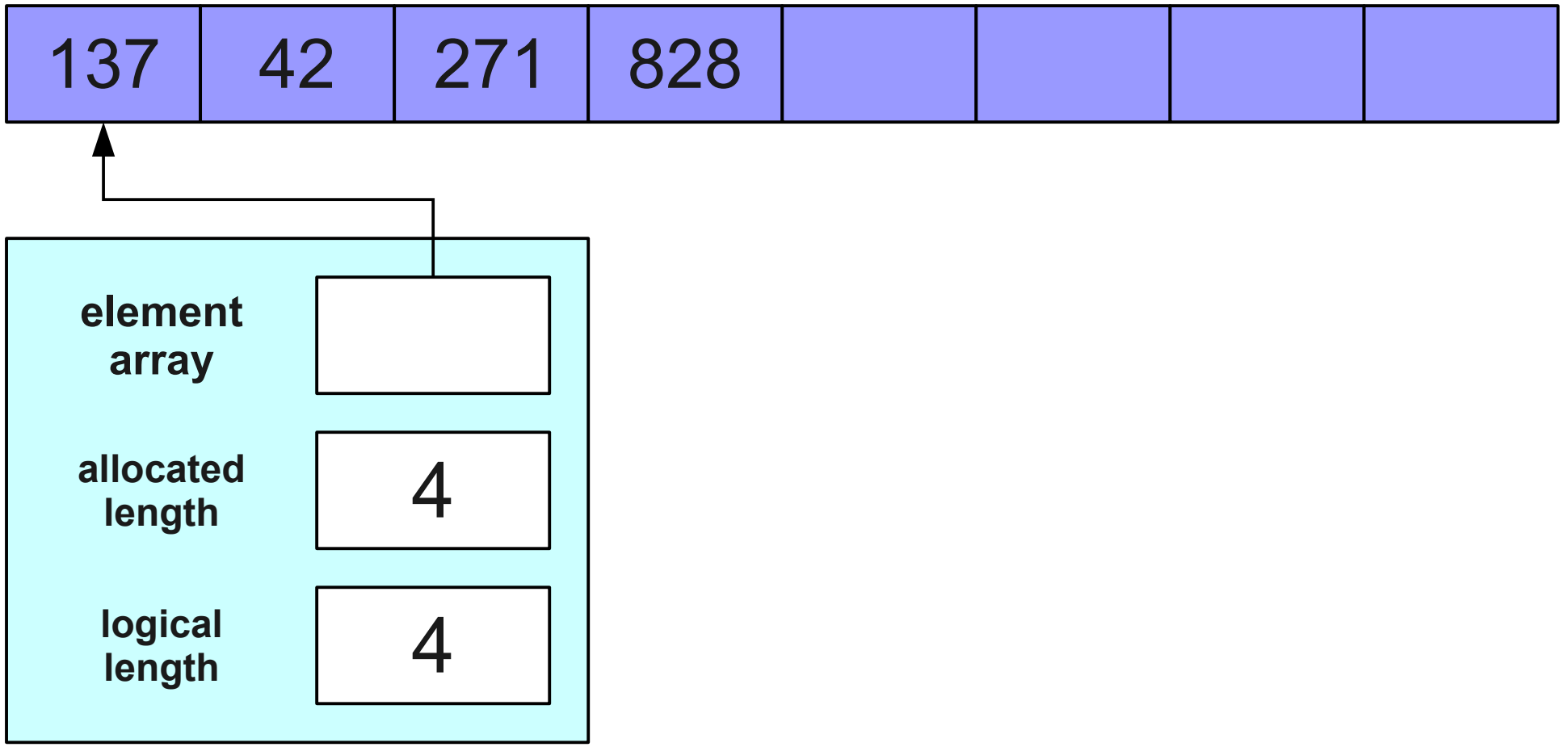

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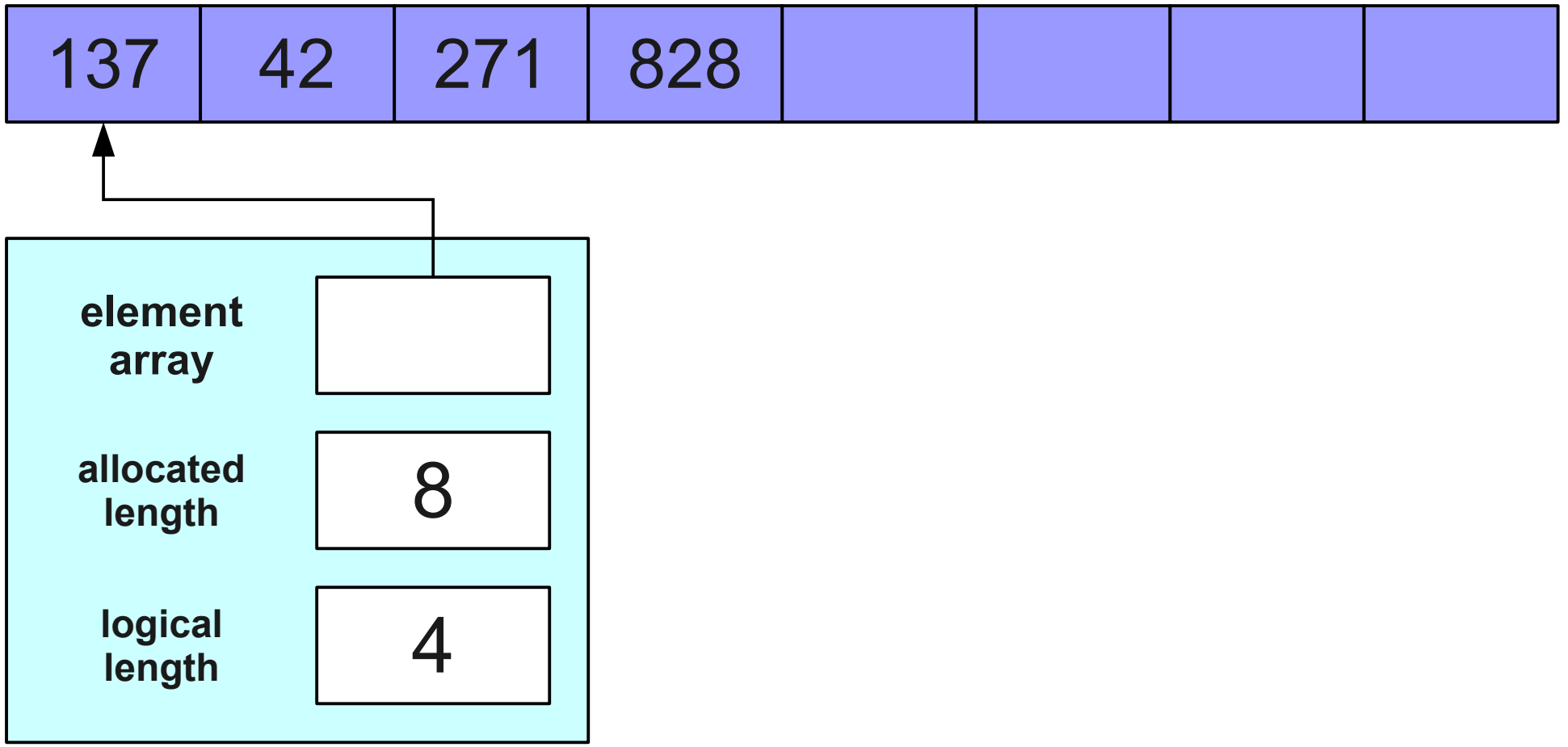

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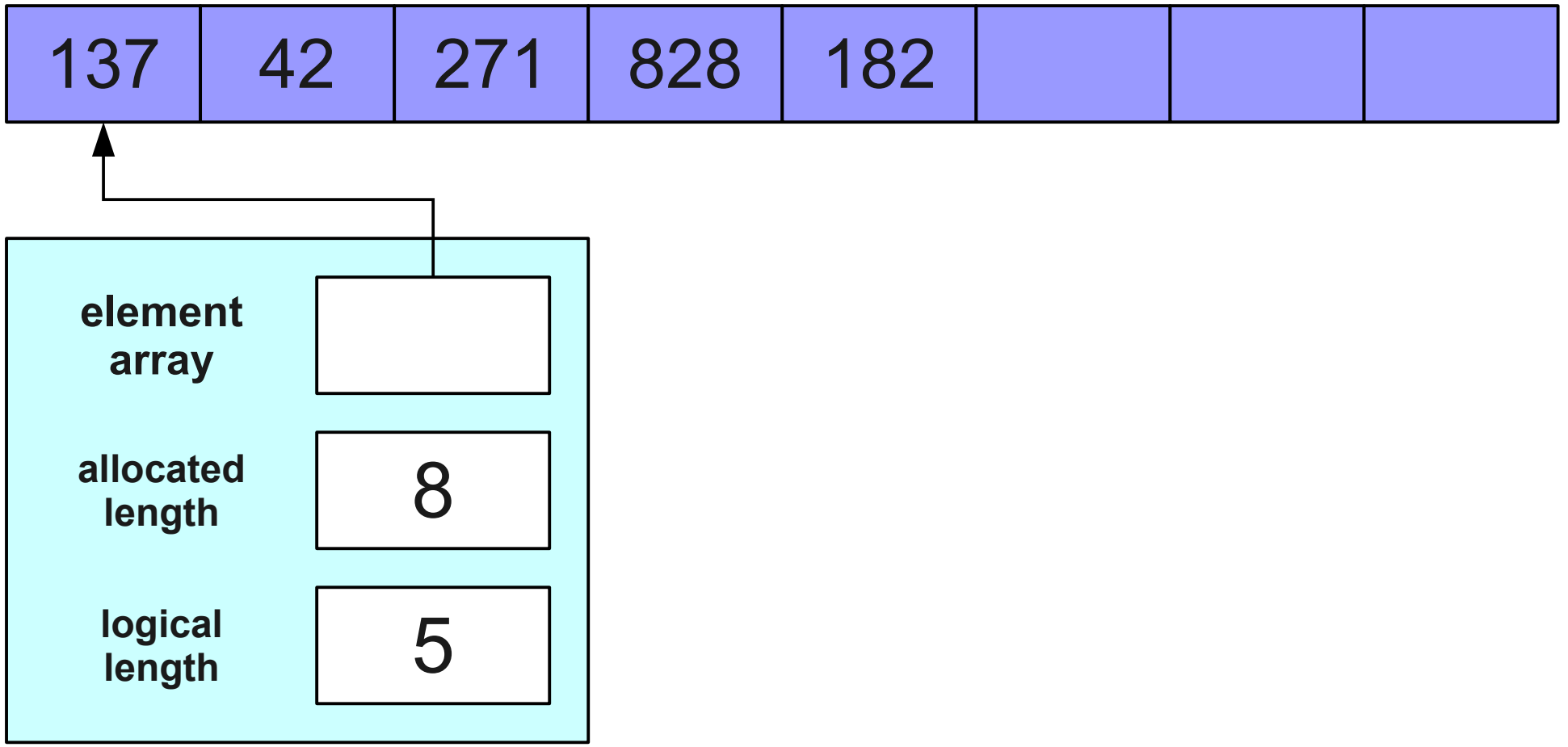

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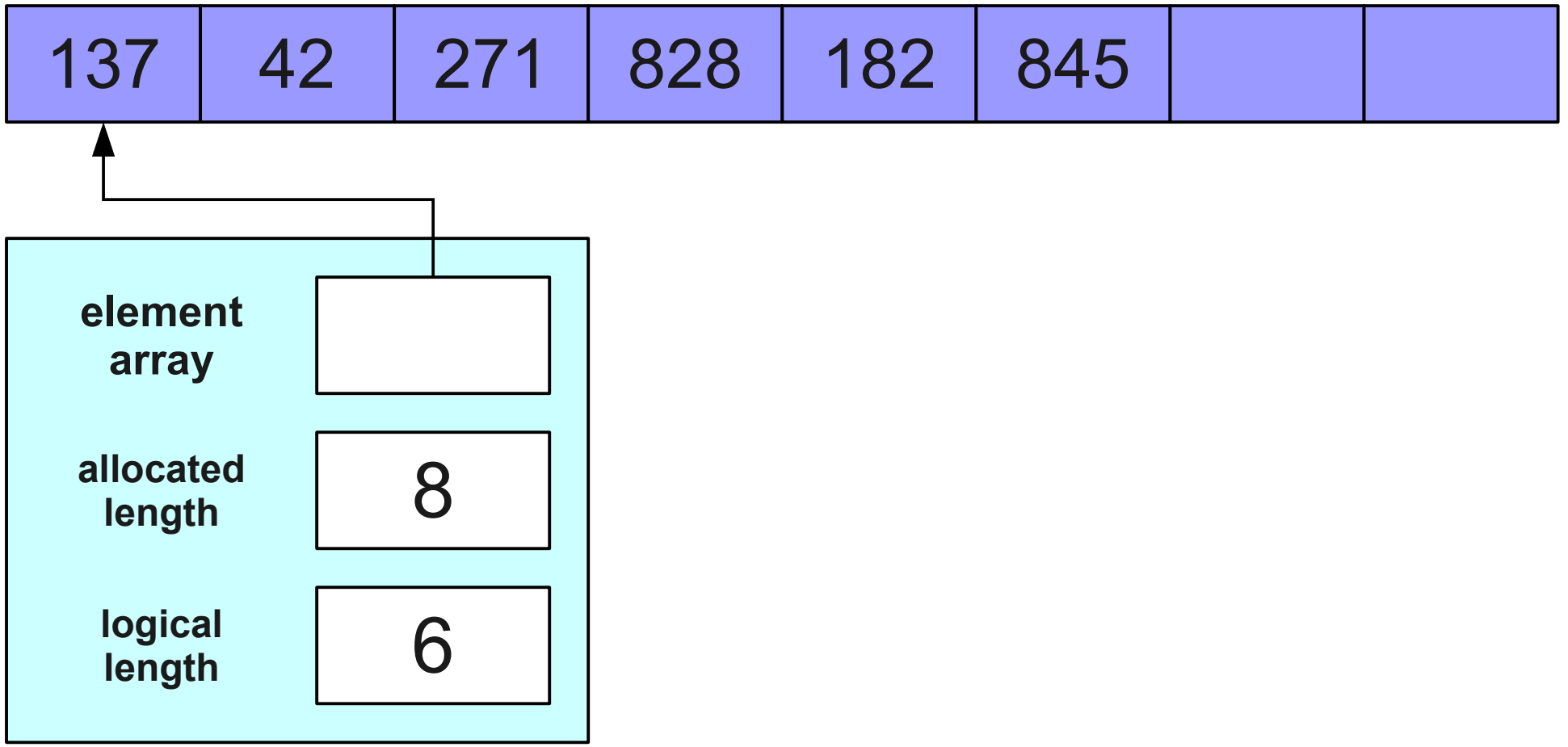

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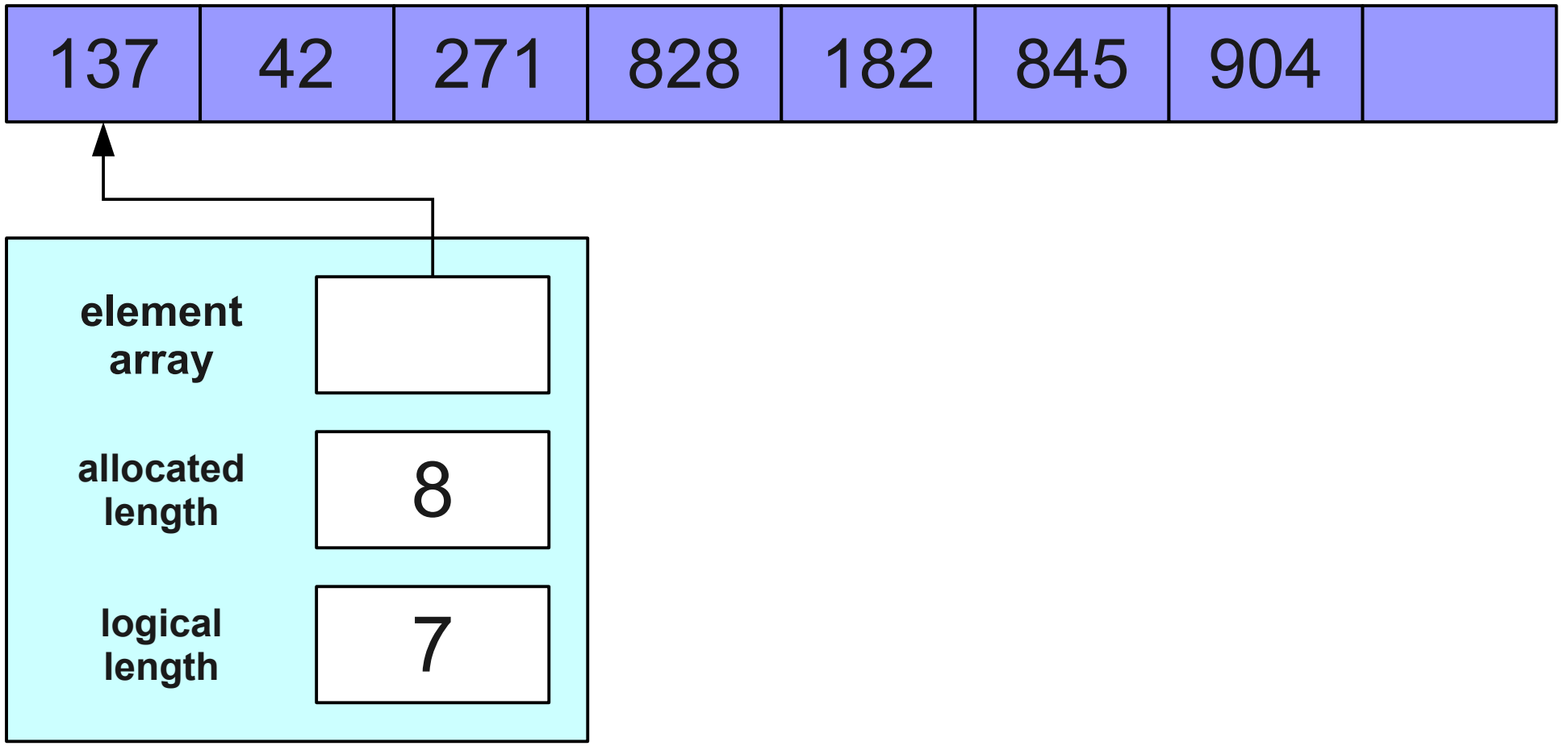

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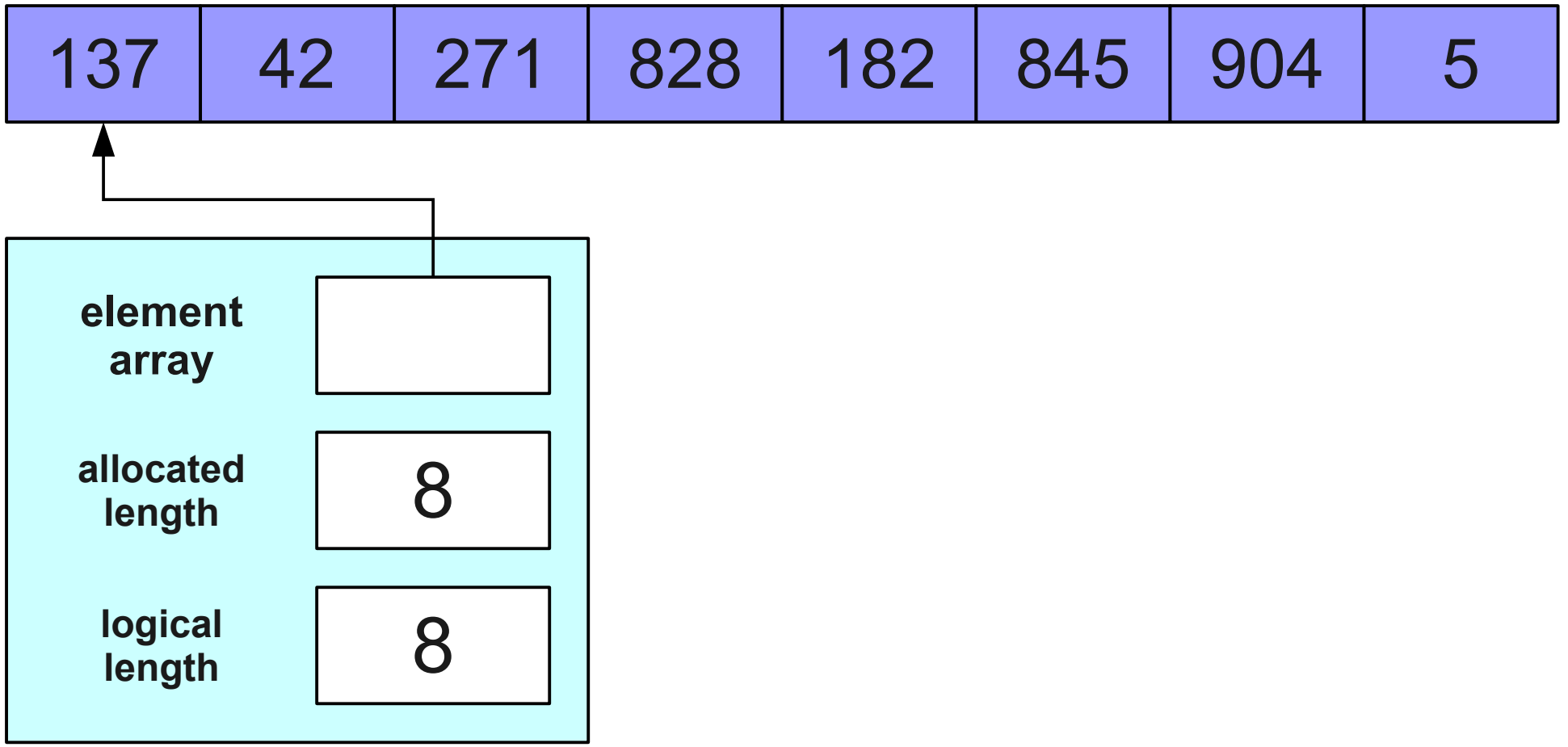
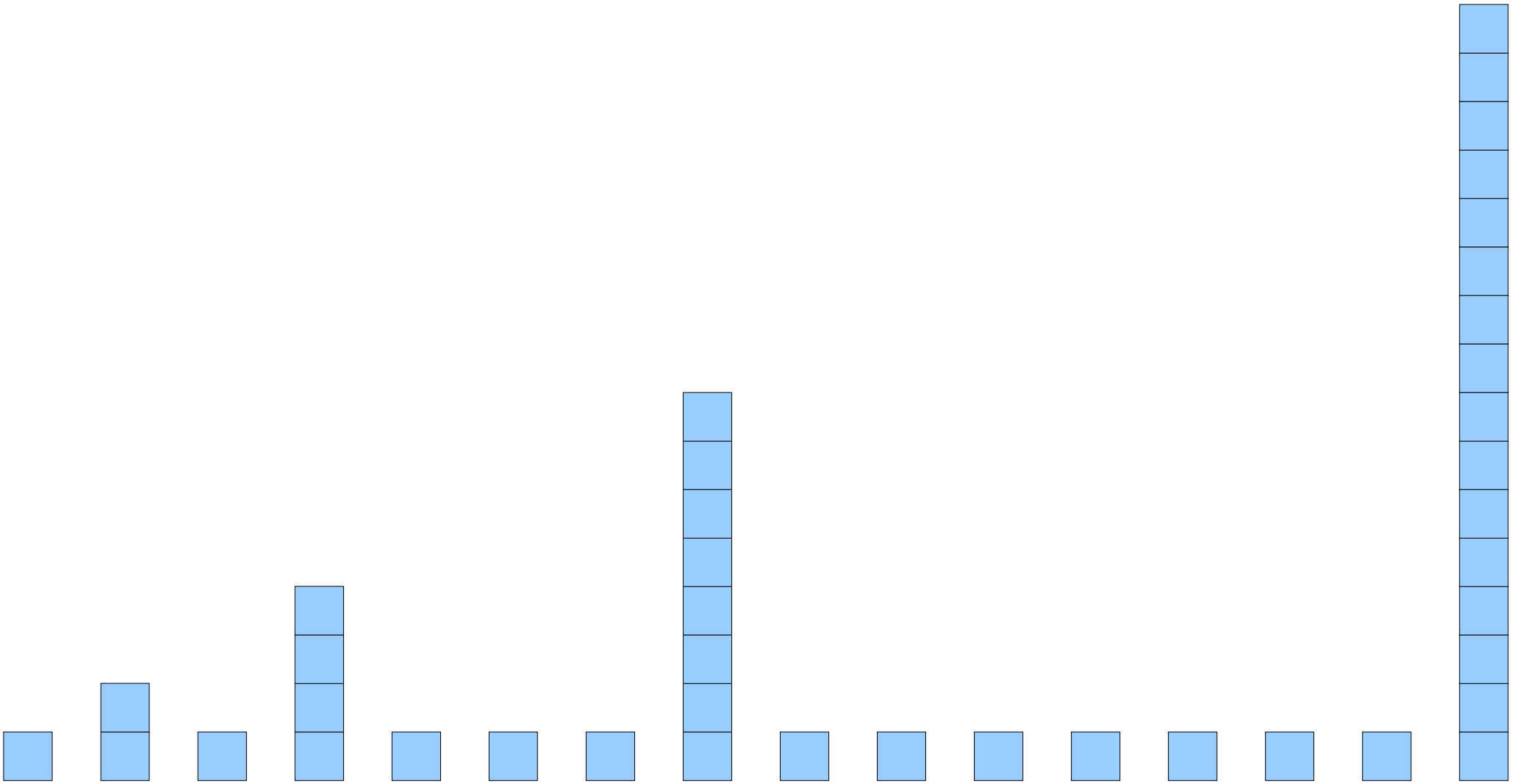


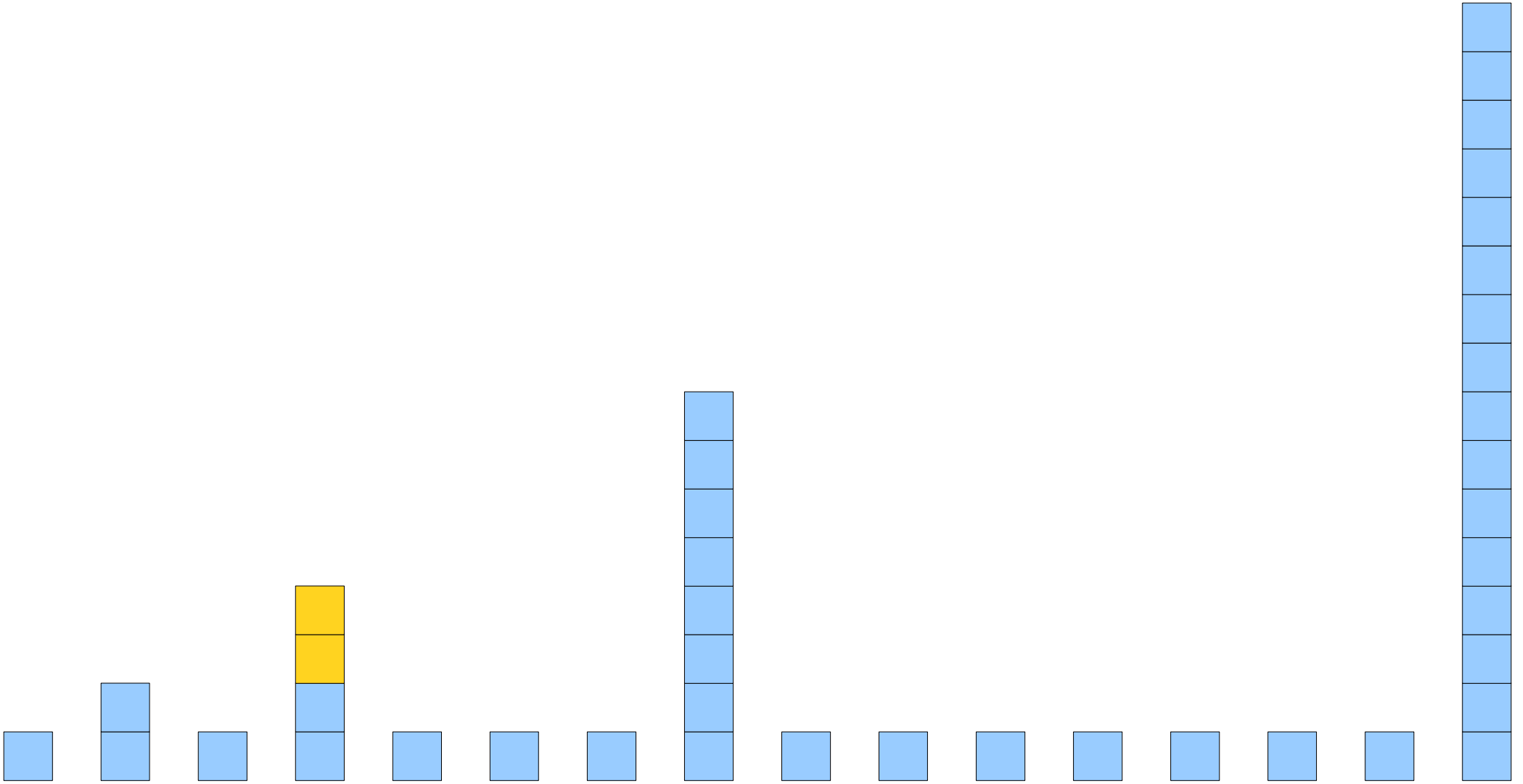
Let's Give it a Try!

How do we analyze this?

Spreading the Work



Spreading the Work



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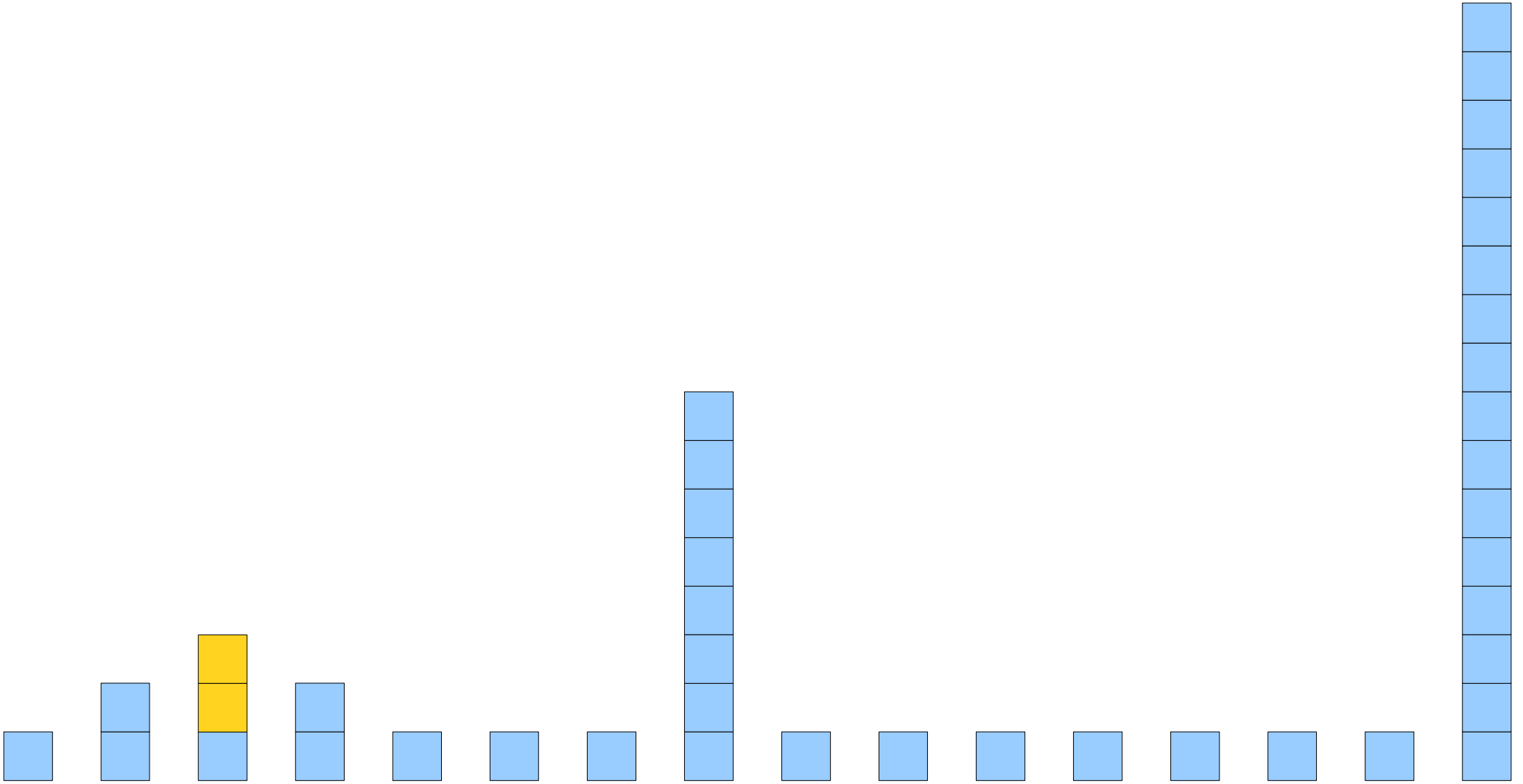

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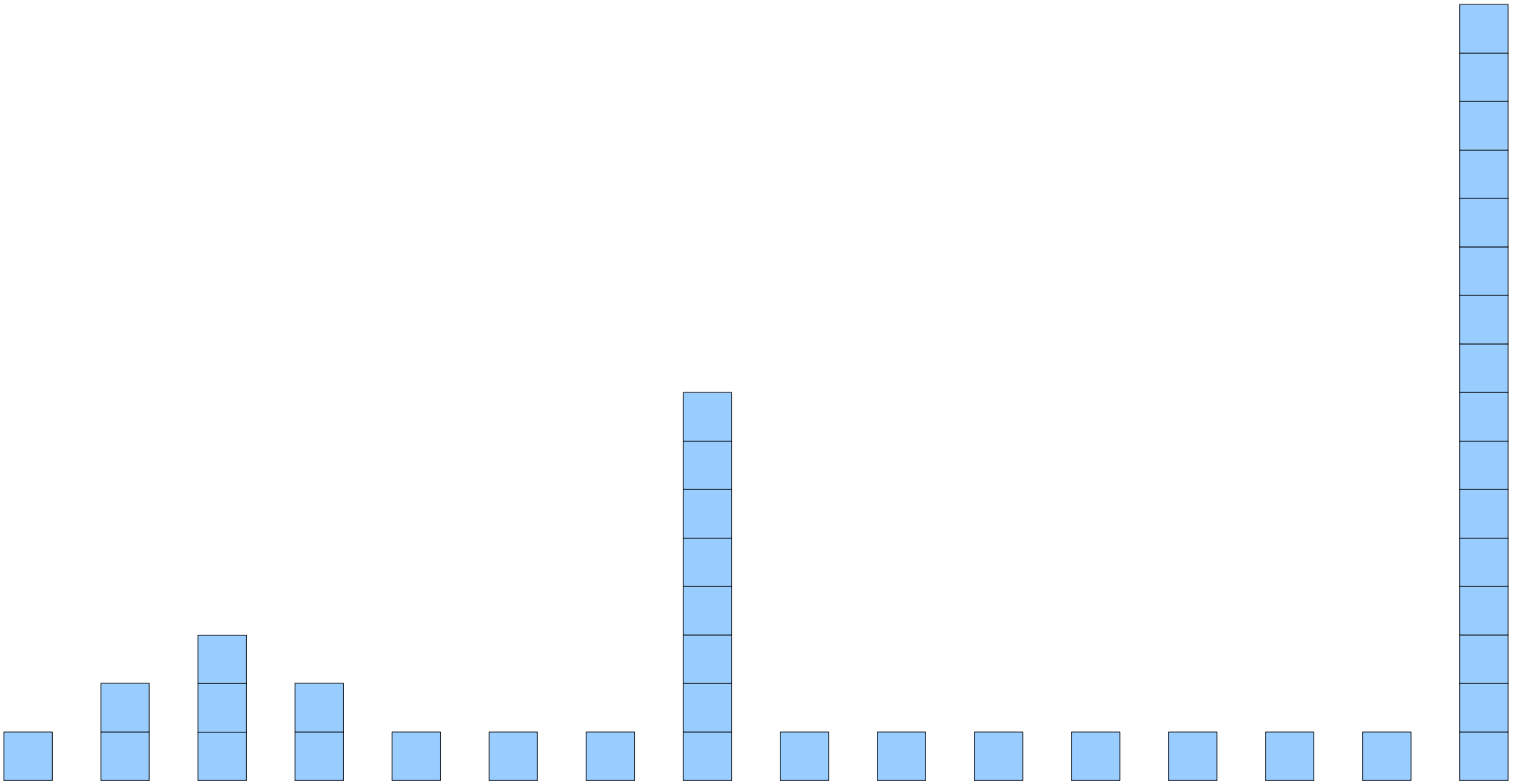

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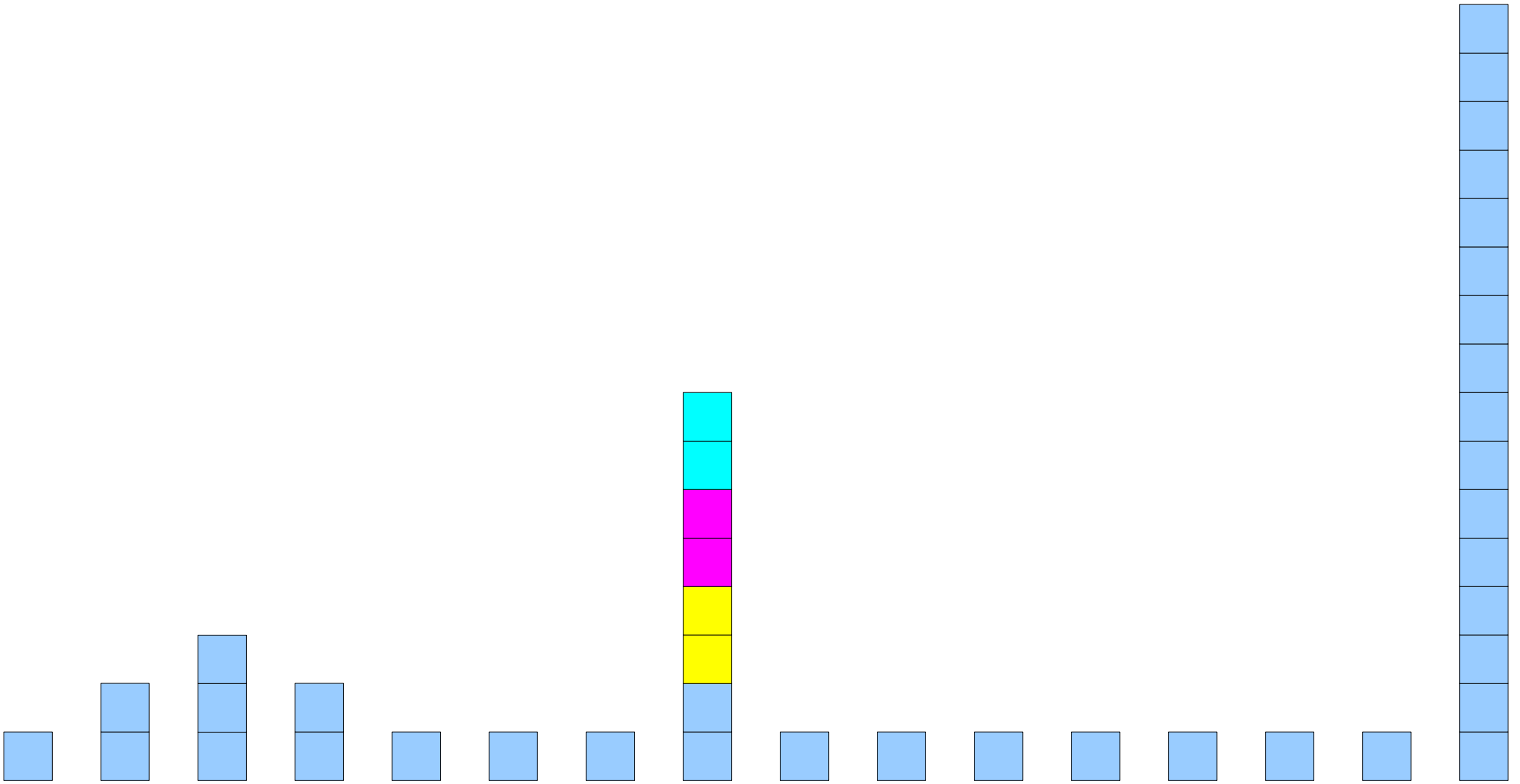

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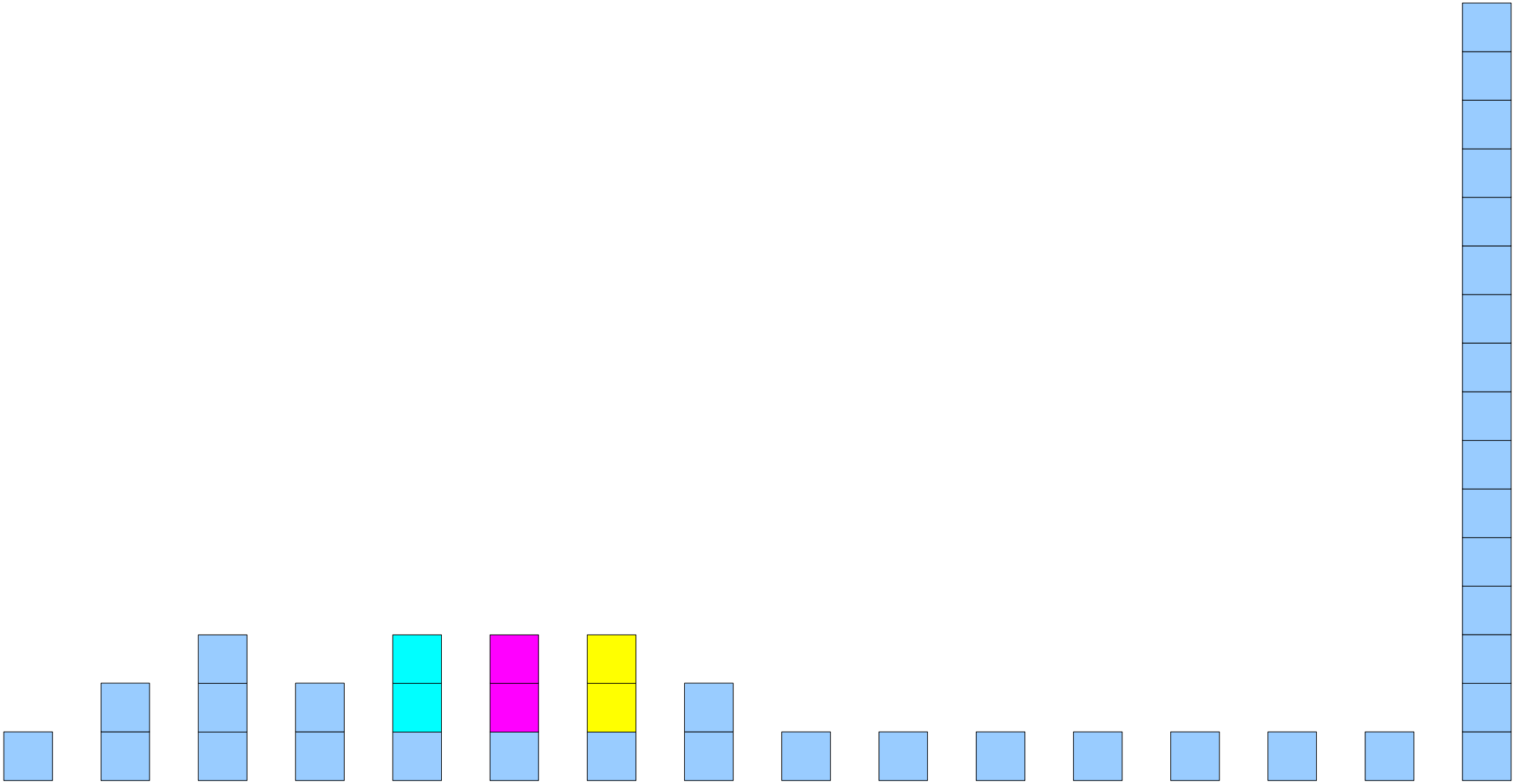

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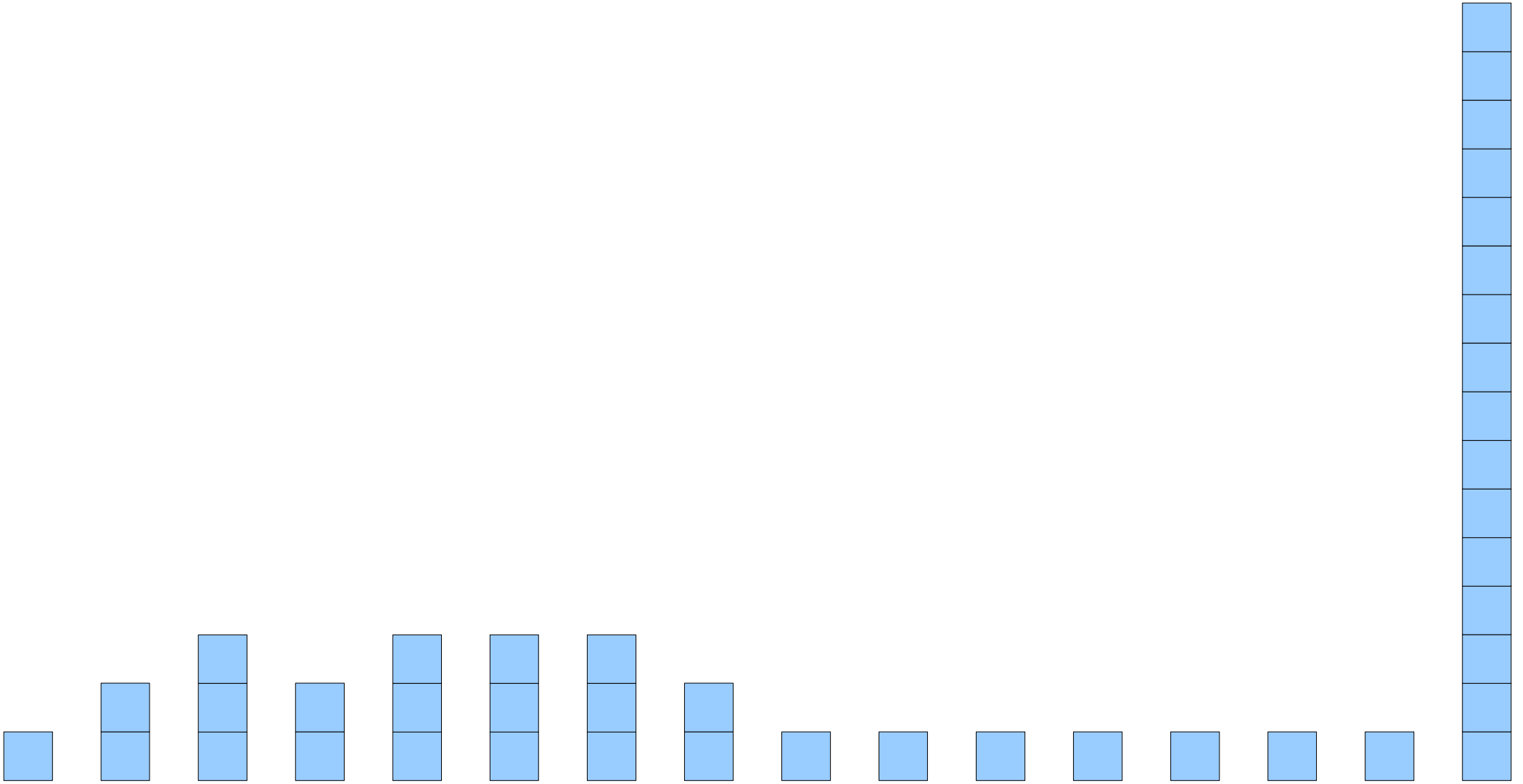

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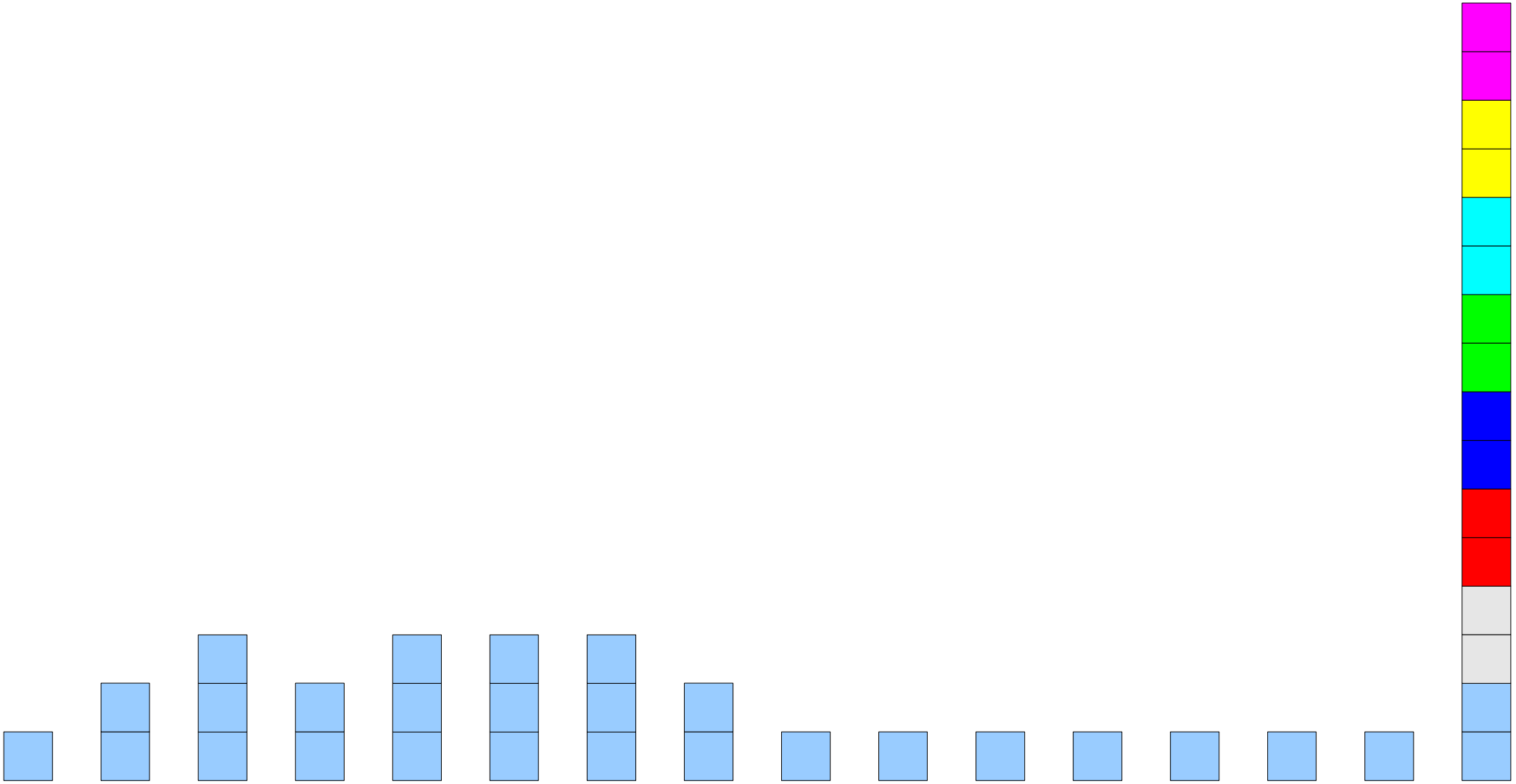

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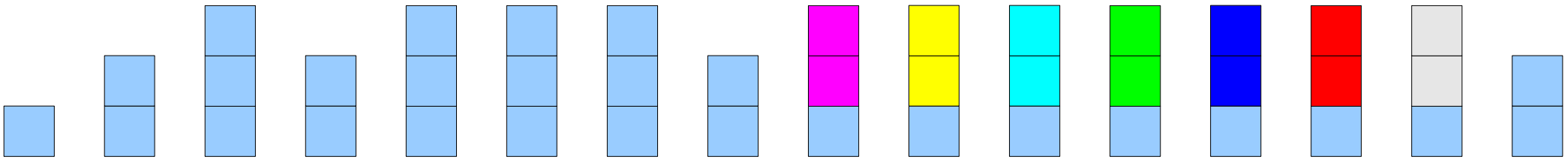
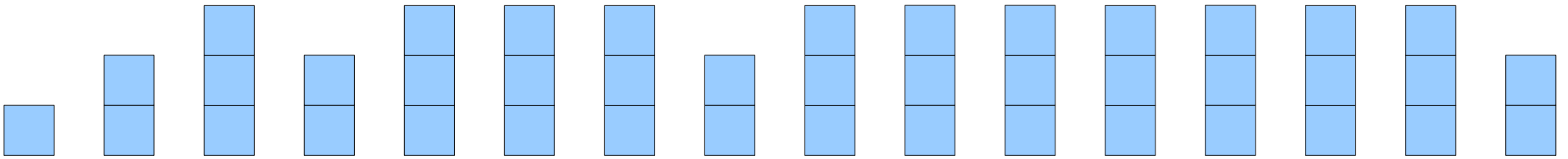


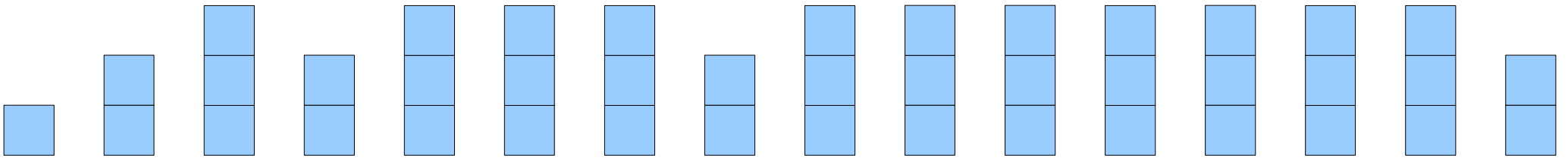
Spreading the Work



Spreading the Work

On average, we do just 3 units of work!

This is O(1) work on average!



Sharing the Burden

- We still have “heavy” pushes taking time O(n) and “light” pushes taking time O(1).
- Worst-case time for a push is O(n).
- Heavy pushes become so rare that the **average** time for a push is O(1).
- Can we confirm this?

Amortized Analysis

- The analysis we have just done is called an **amortized analysis**.
- Reason about the total amount of work done, not the word done per operation.
- In an amortized sense, our implementation of the stack is extremely fast!
- This is one of the most common approaches to implementing **`Stack`**.

Implementing **`Queue`**

Implementing `Queue`

- We've just used dynamic arrays to implement a stack. Could we use them to implement a queue?
- Yes, but here's a better idea: **could we use our stack to implement a queue?**

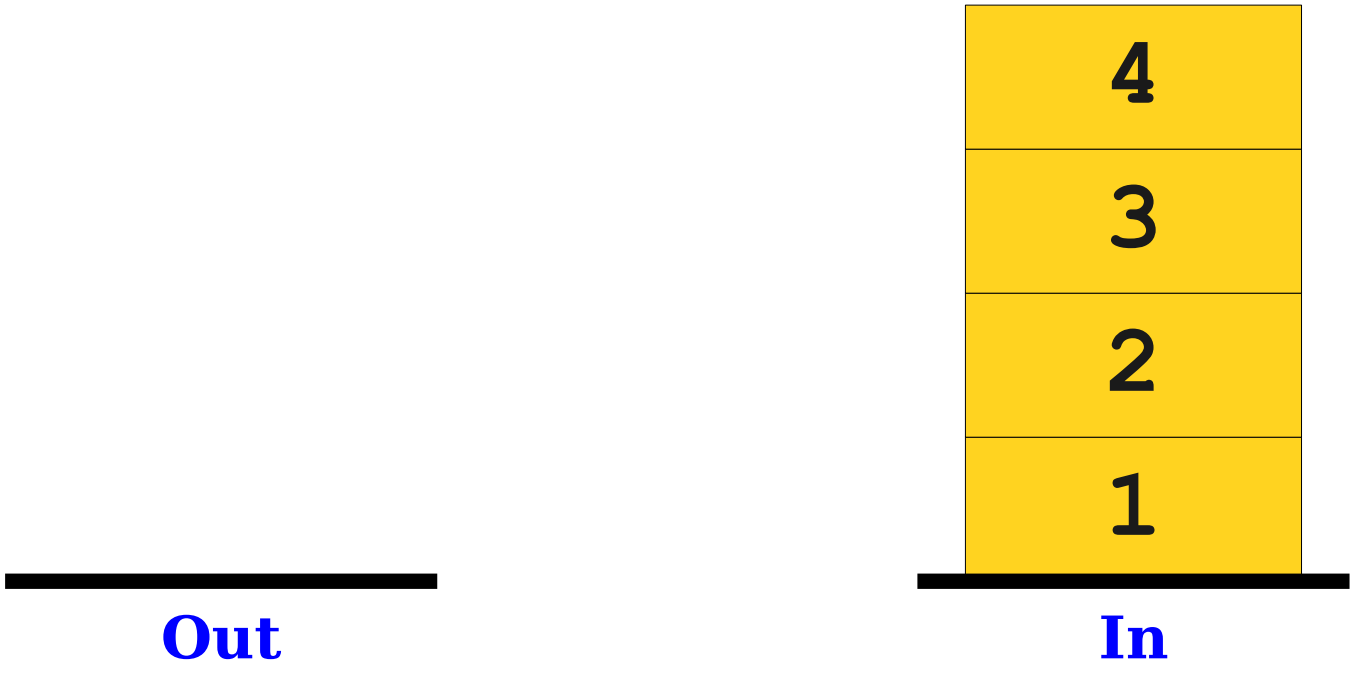
The Two-Stack Queue

The Two-Stack Queue

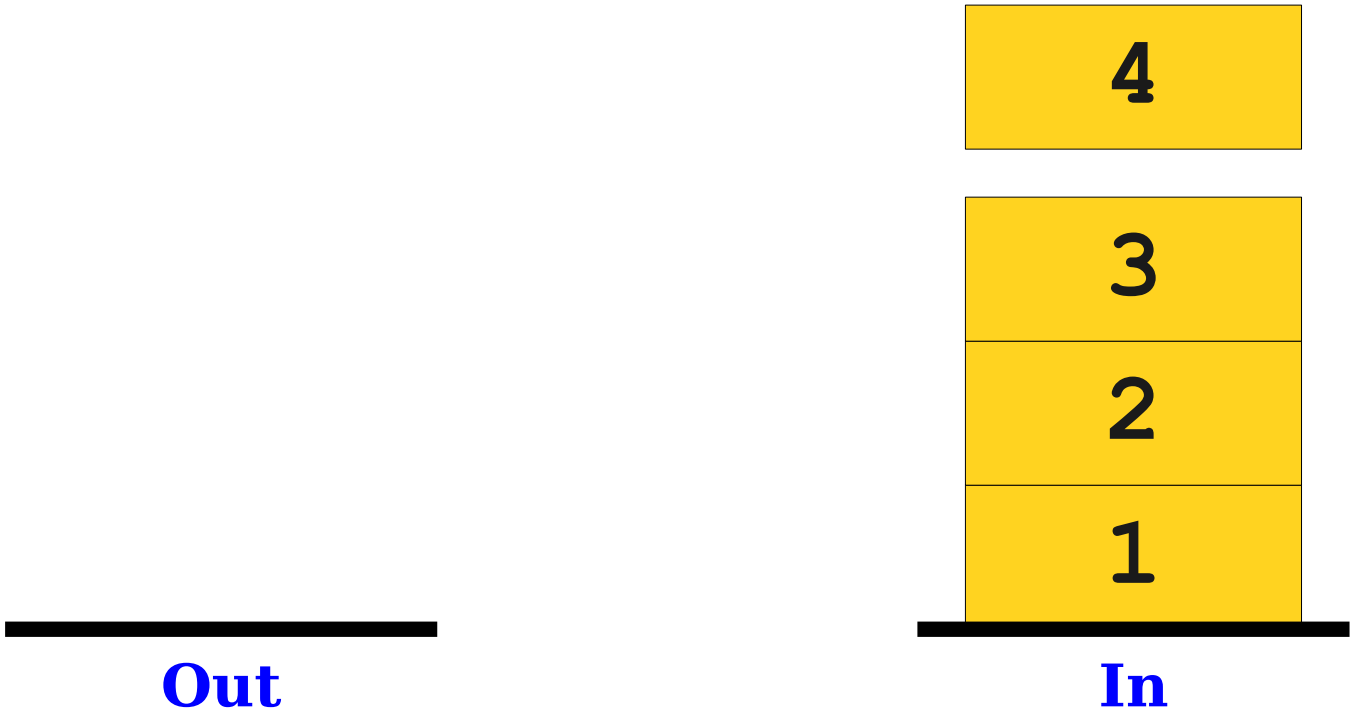
The Two-Stack Queue

The Two-Stack Queue

The Two-Stack Queue



The Two-Stack Queue



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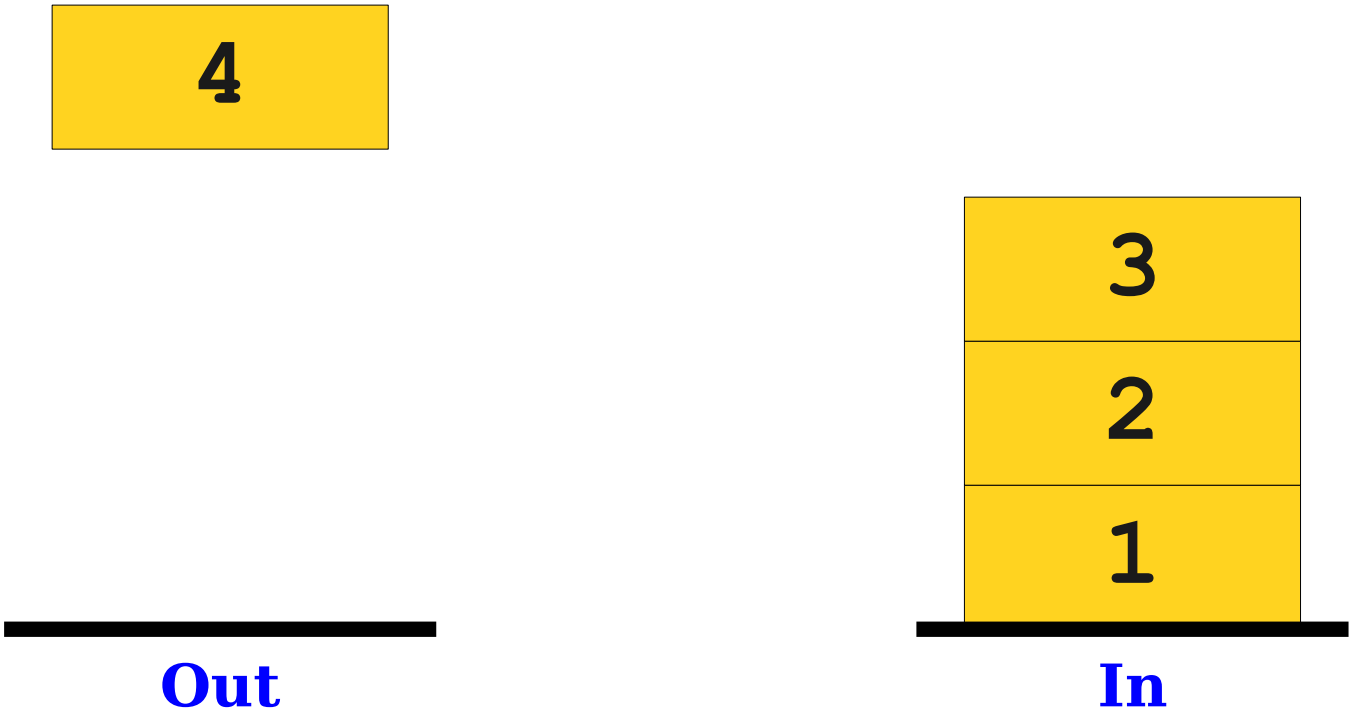

The Two-Stack Queue

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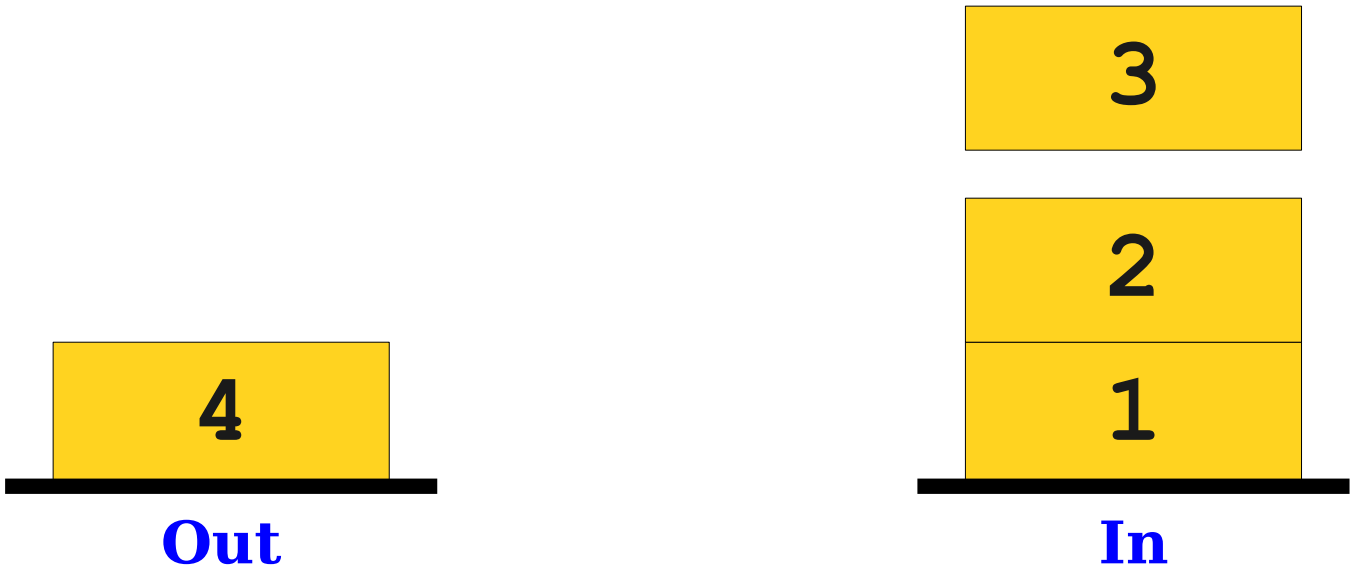

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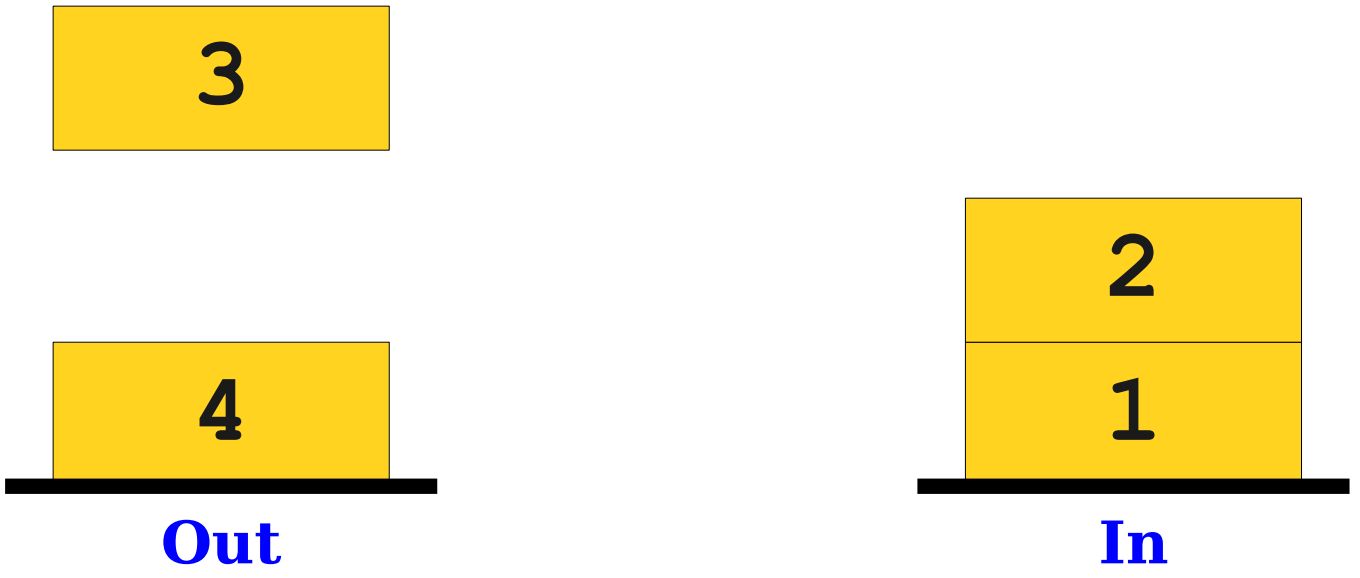

The Two-Stack Queue

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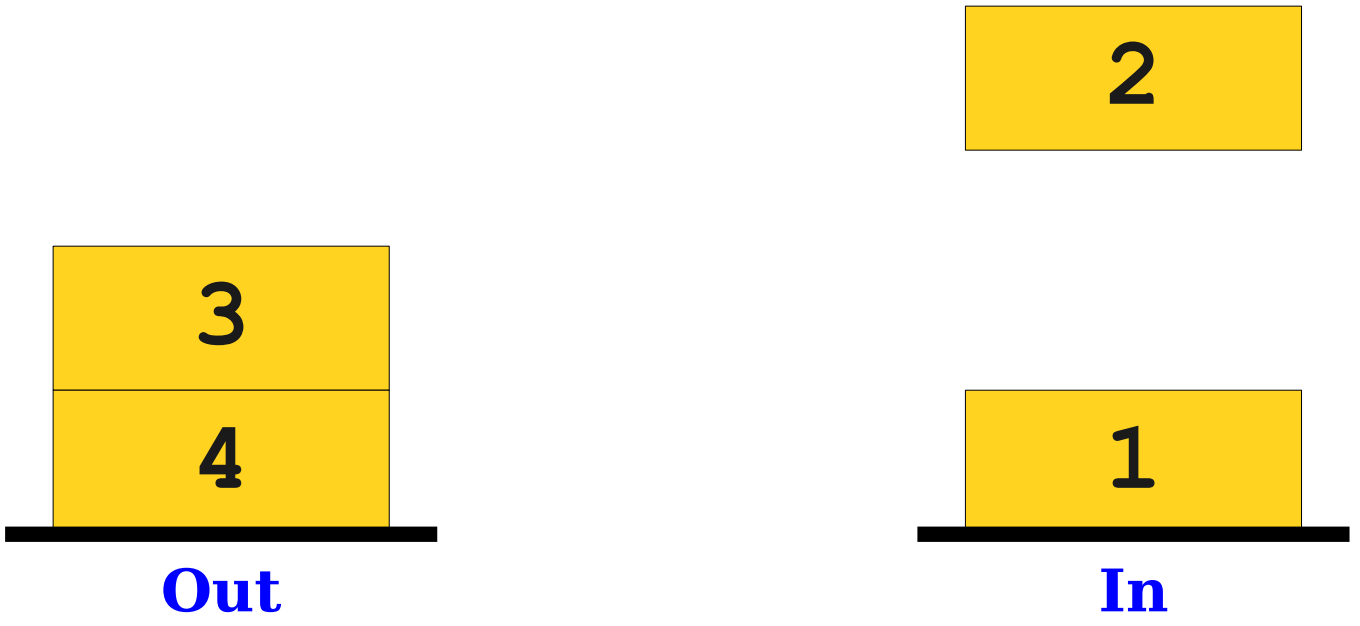

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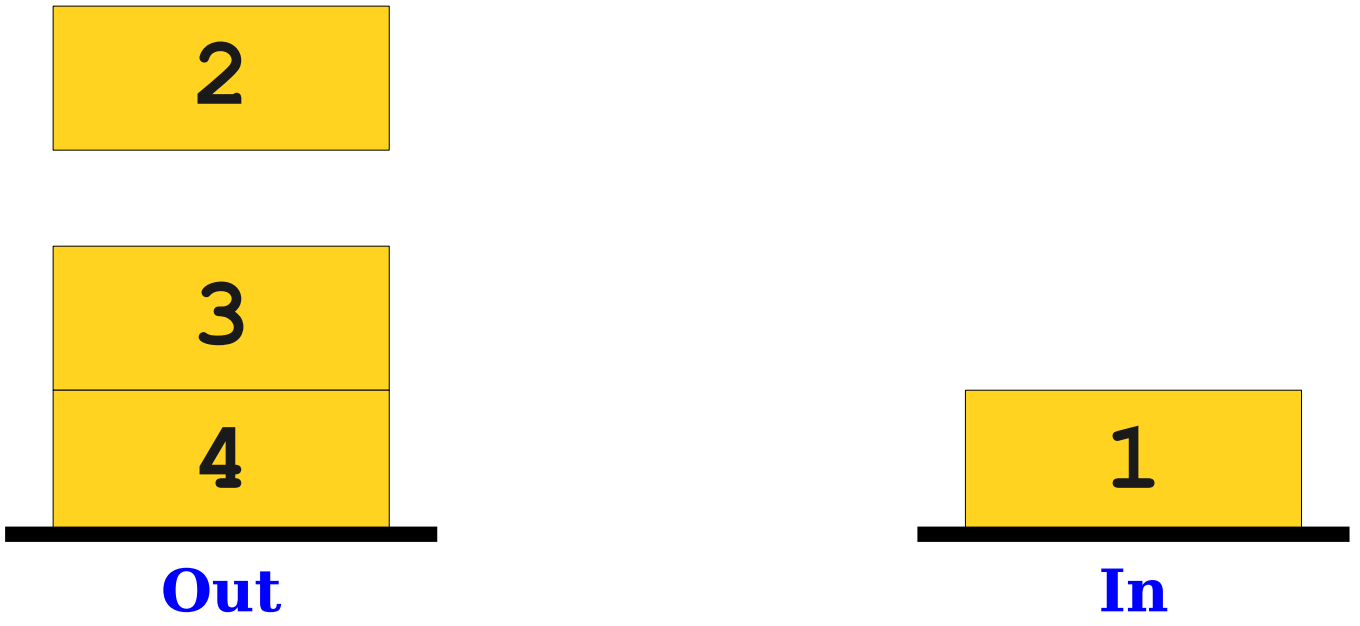

The Two-Stack Queue

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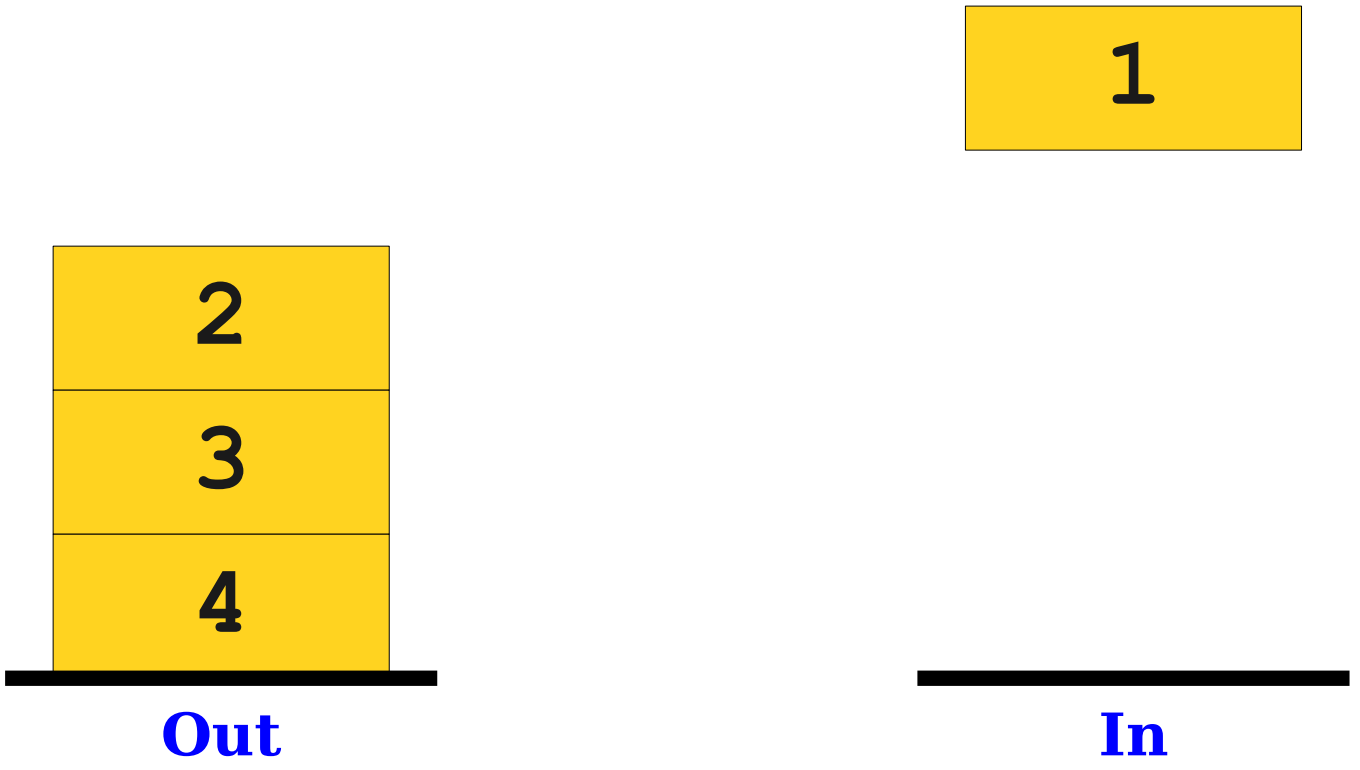

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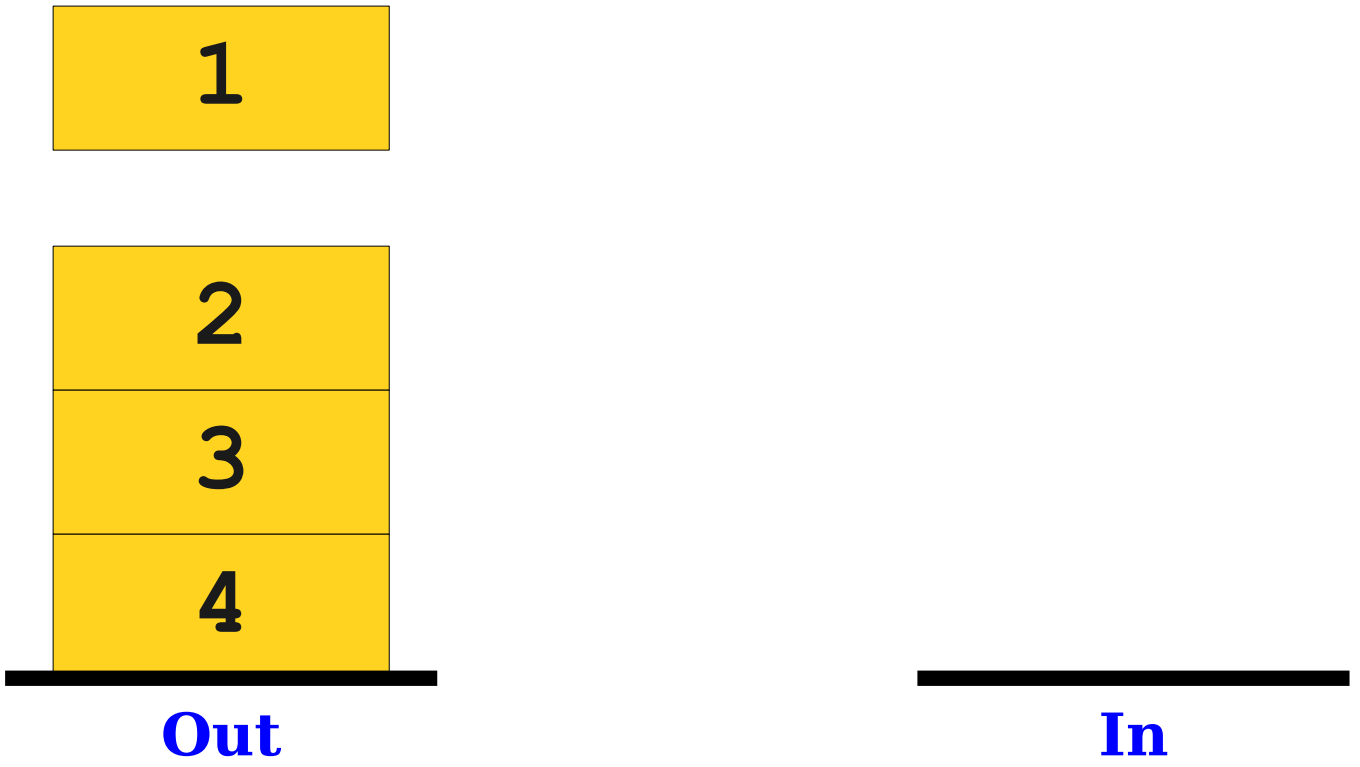

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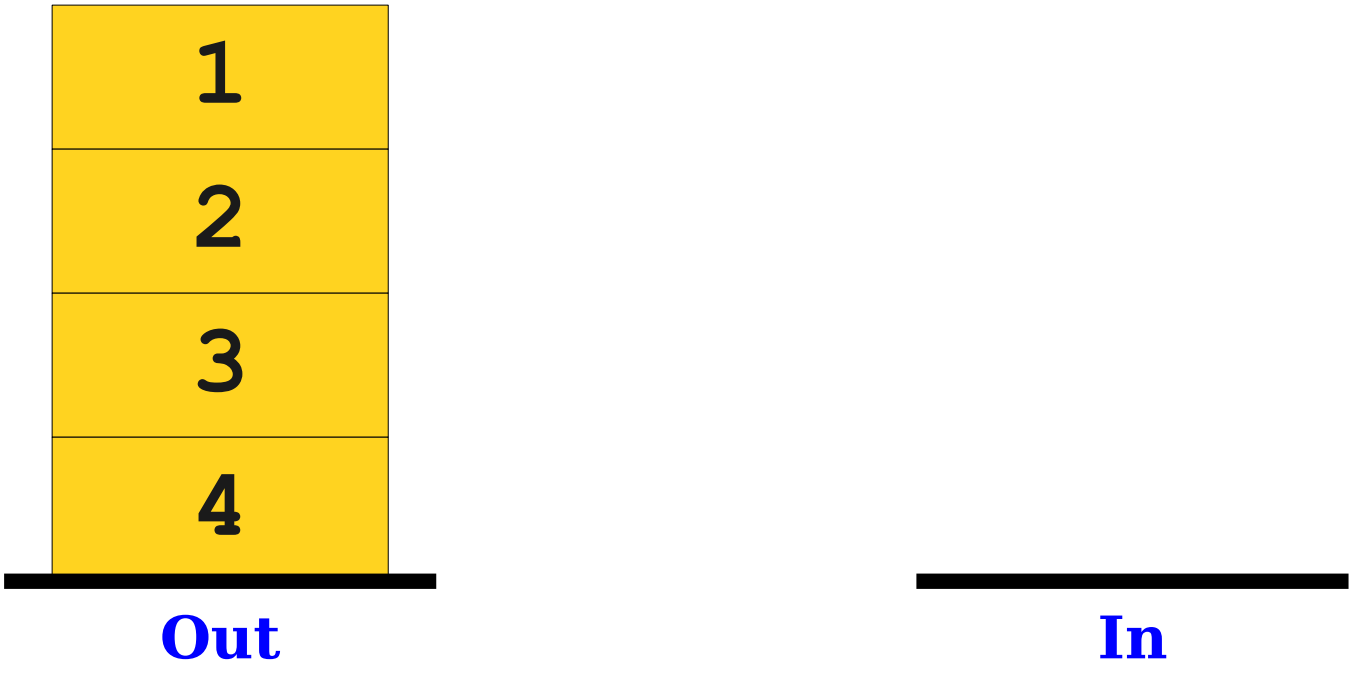

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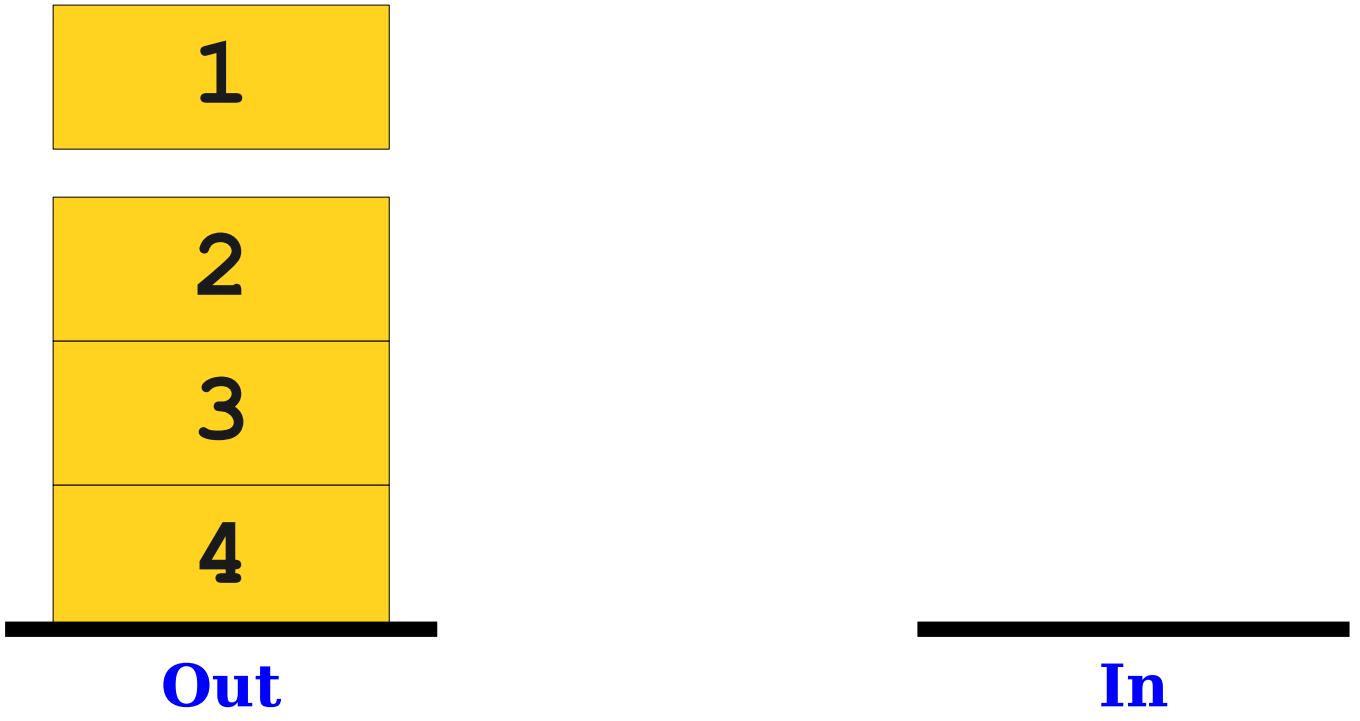
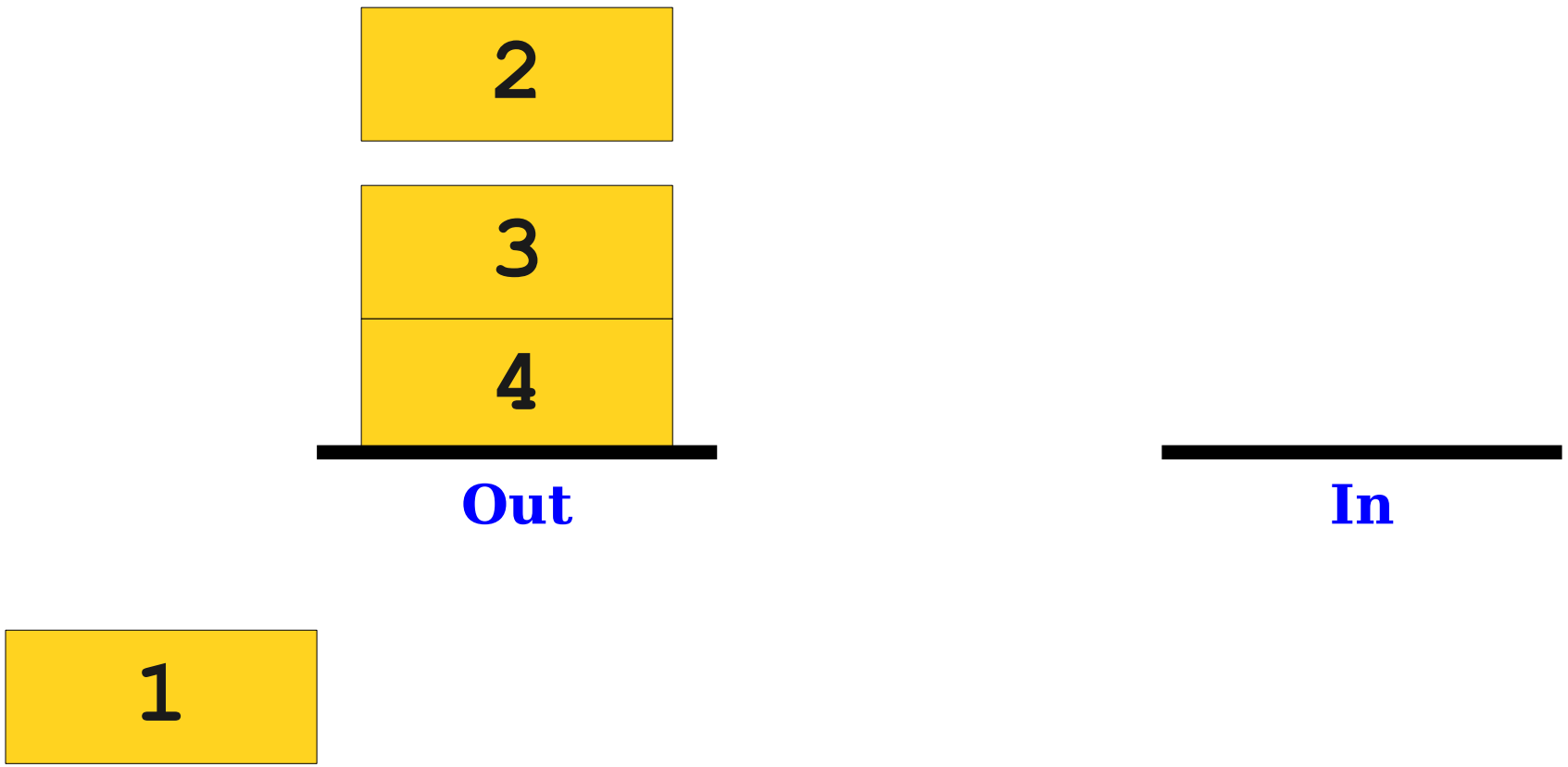


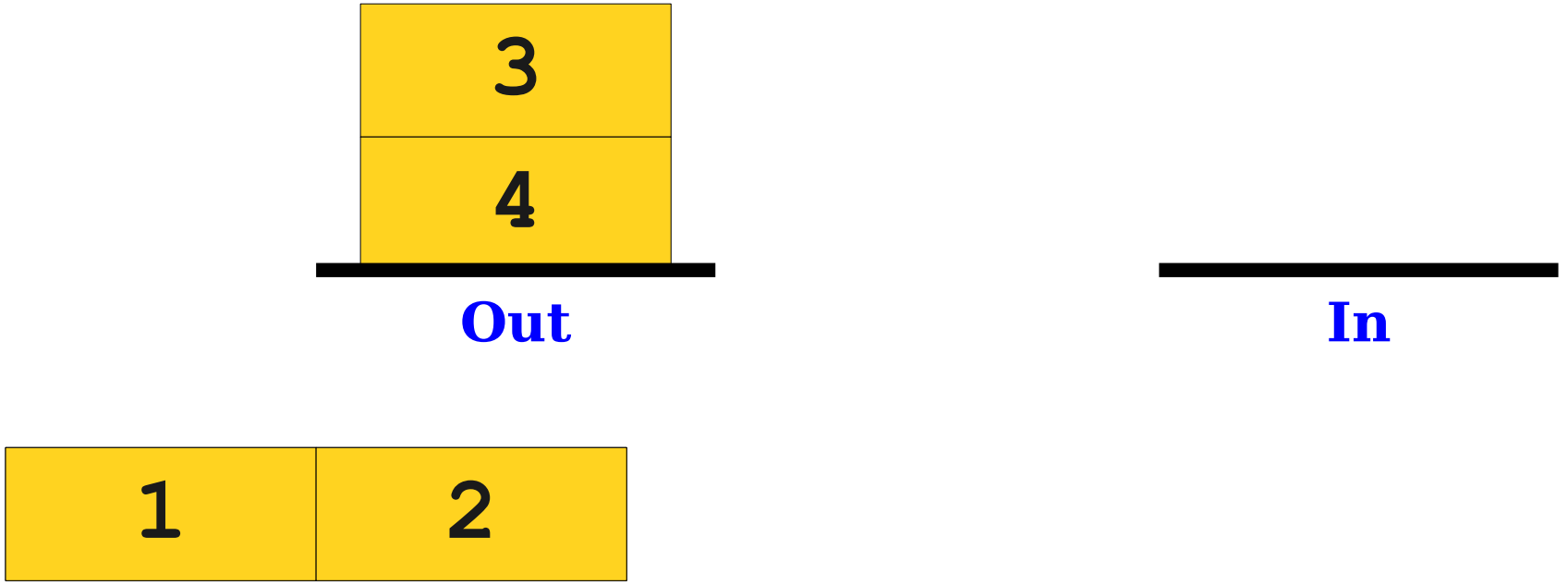
The Two-Stack Queue

1

The Two-Stack Queue



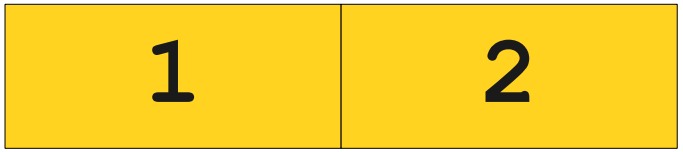
The Two-Stack Queue



The Two-Stack Queue

Out

In



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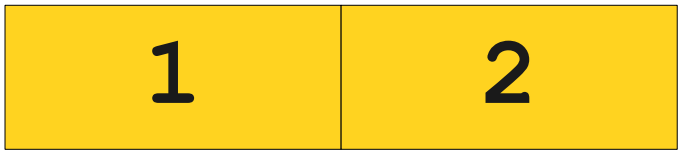

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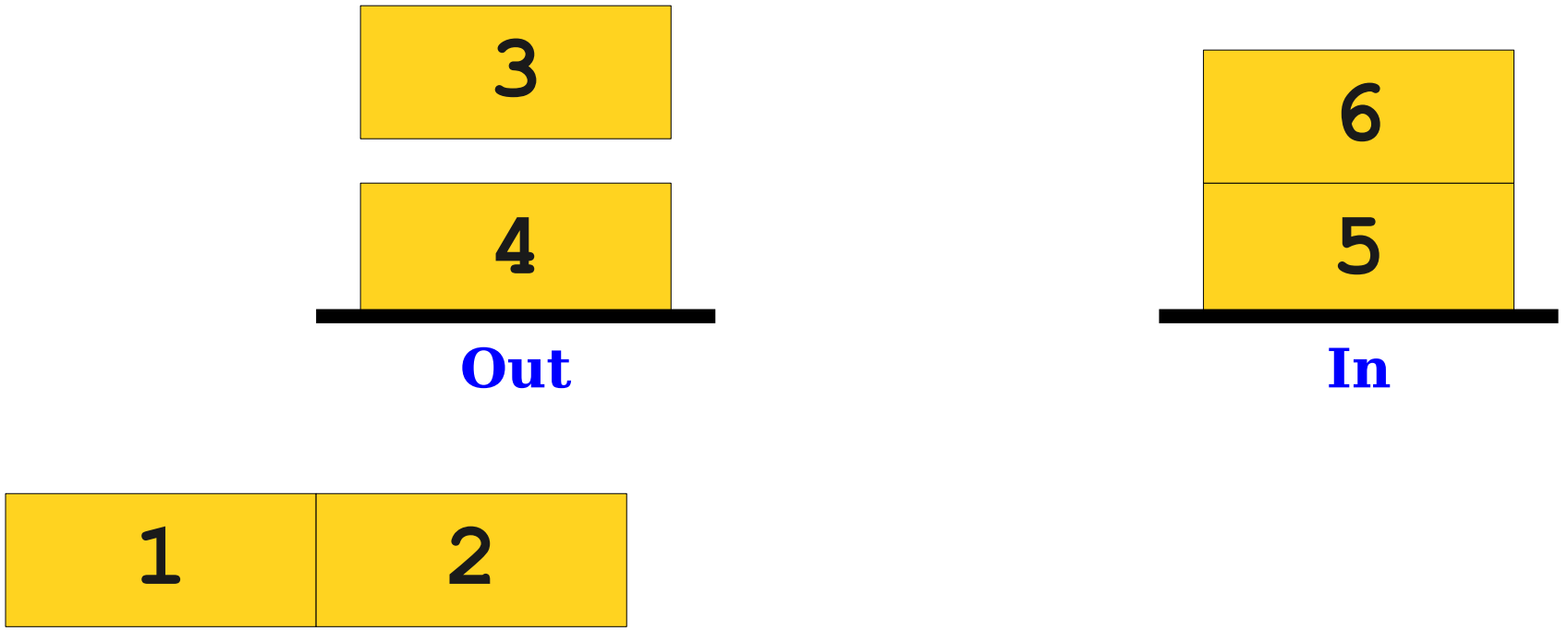

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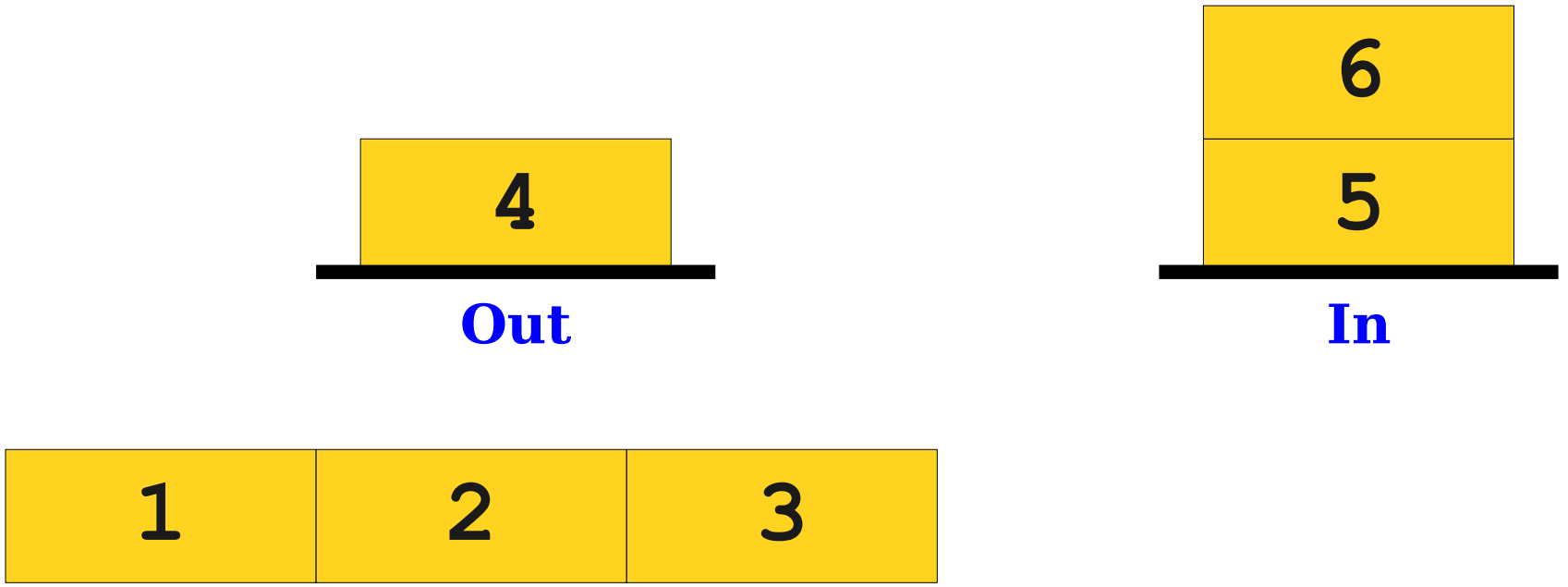

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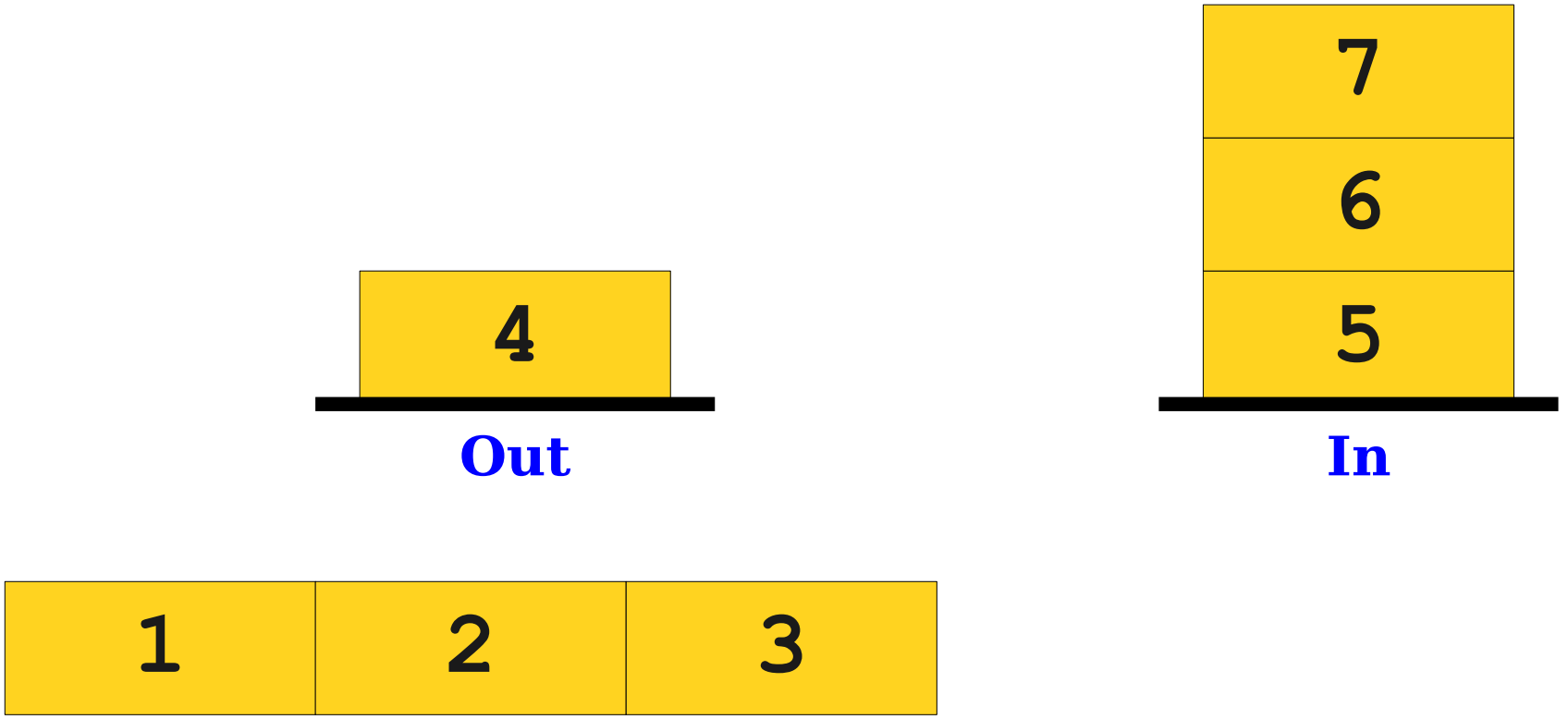

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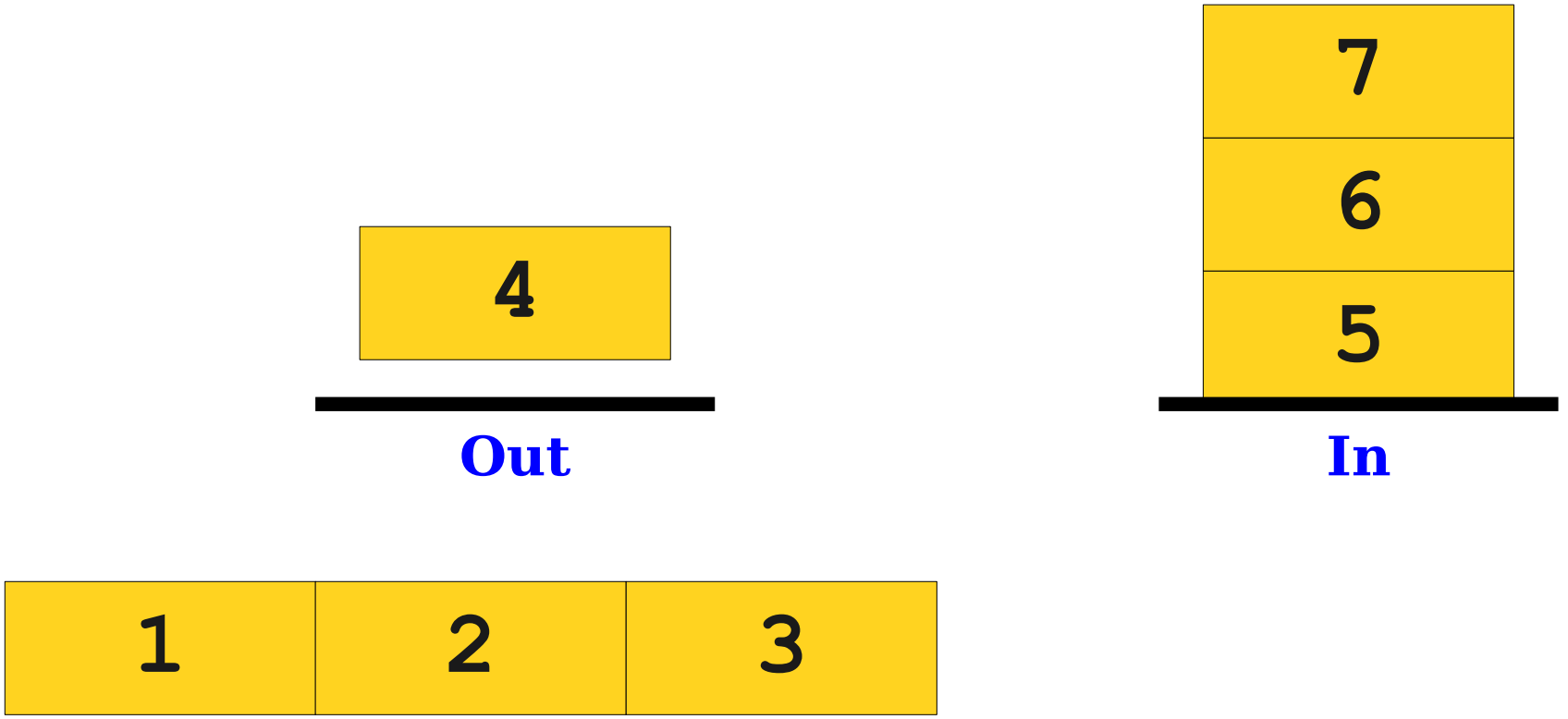

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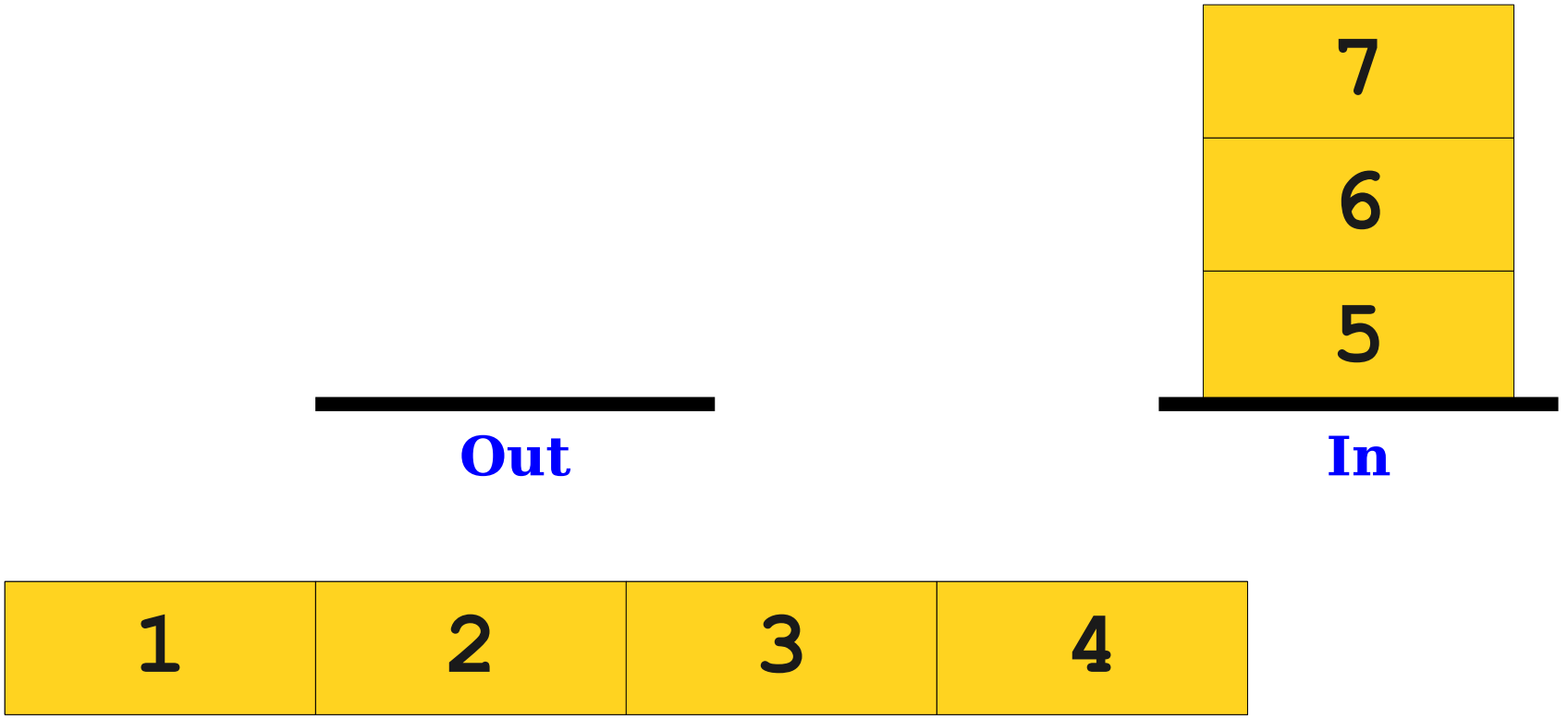

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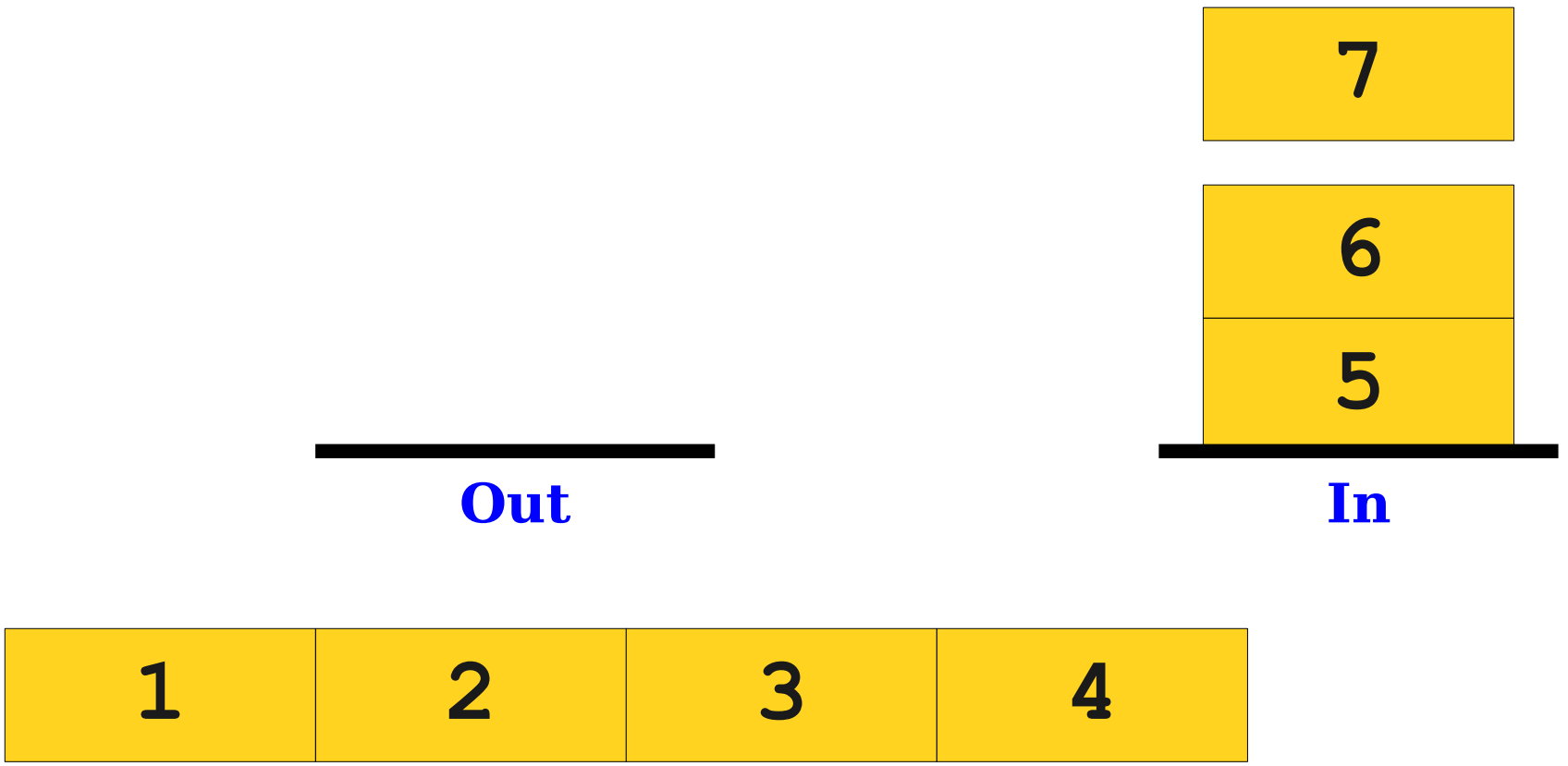

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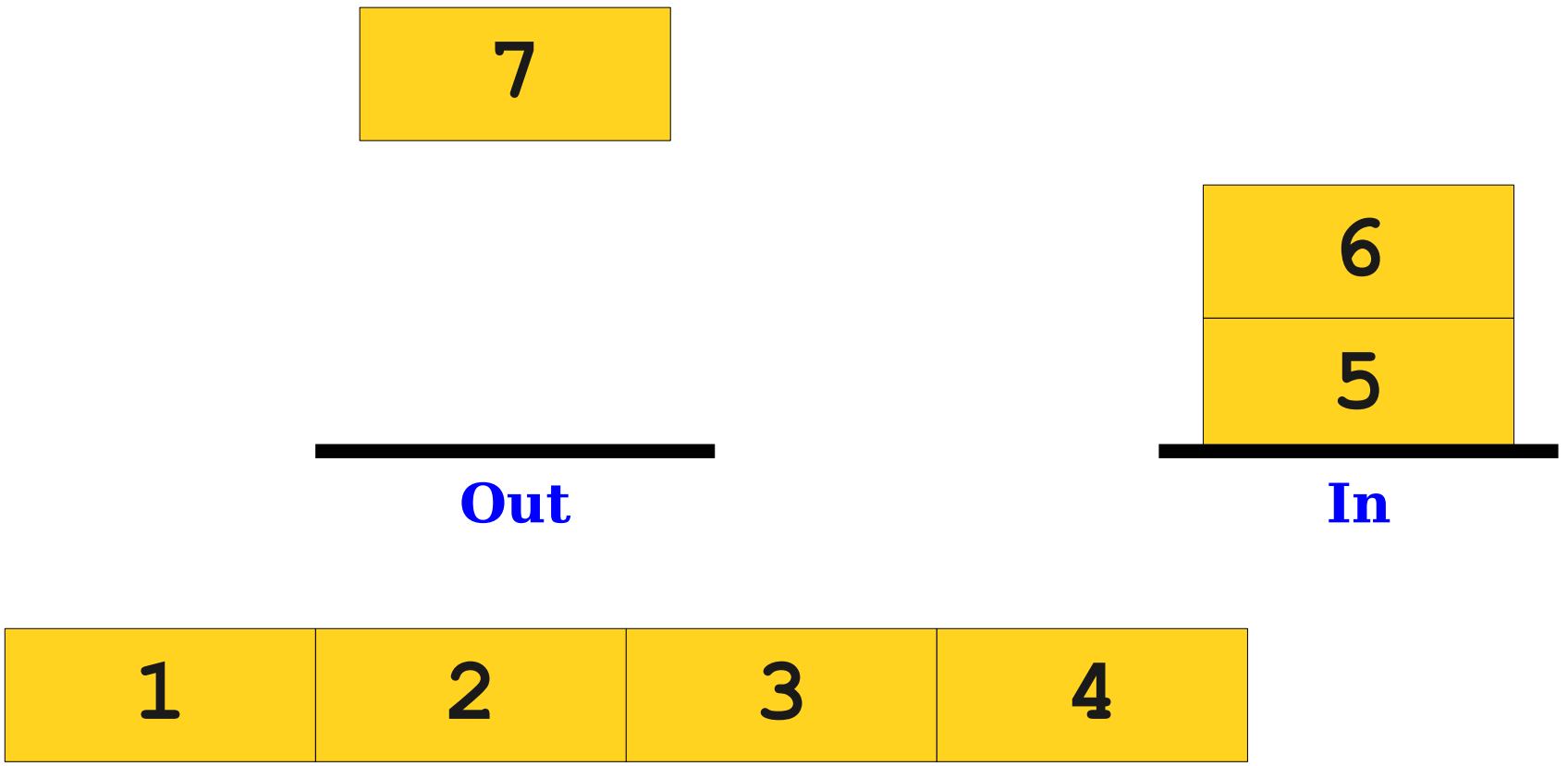

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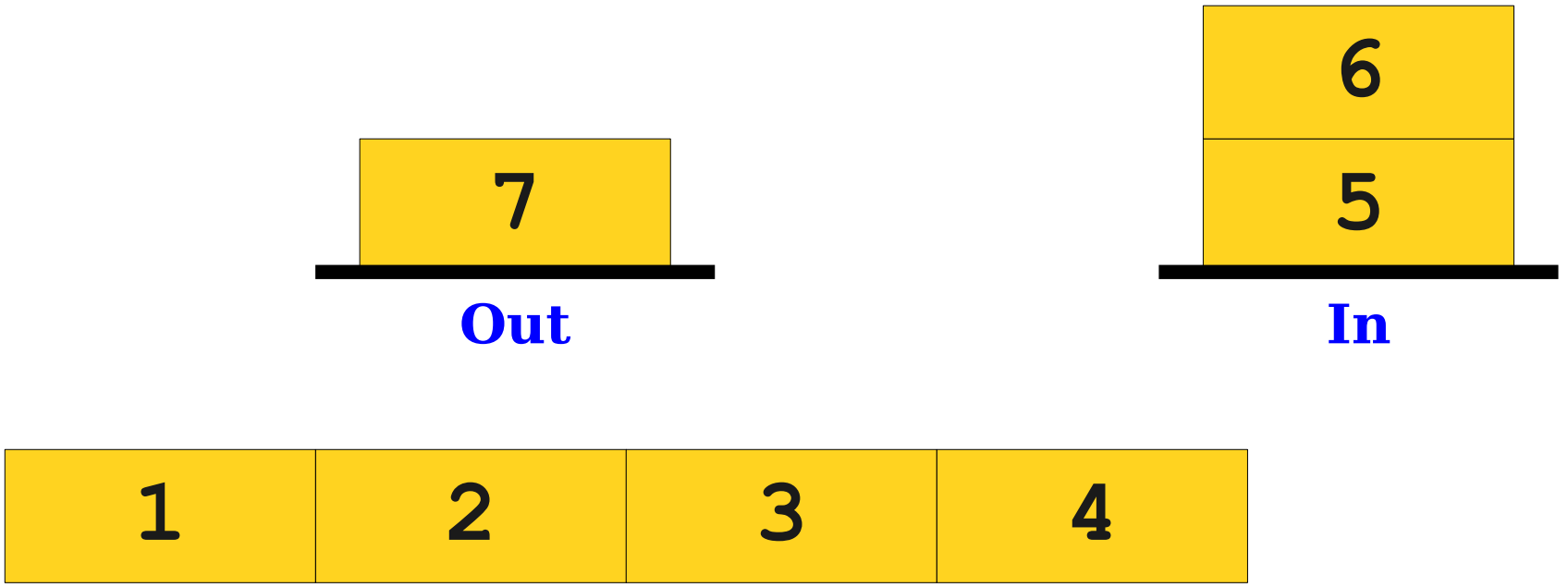

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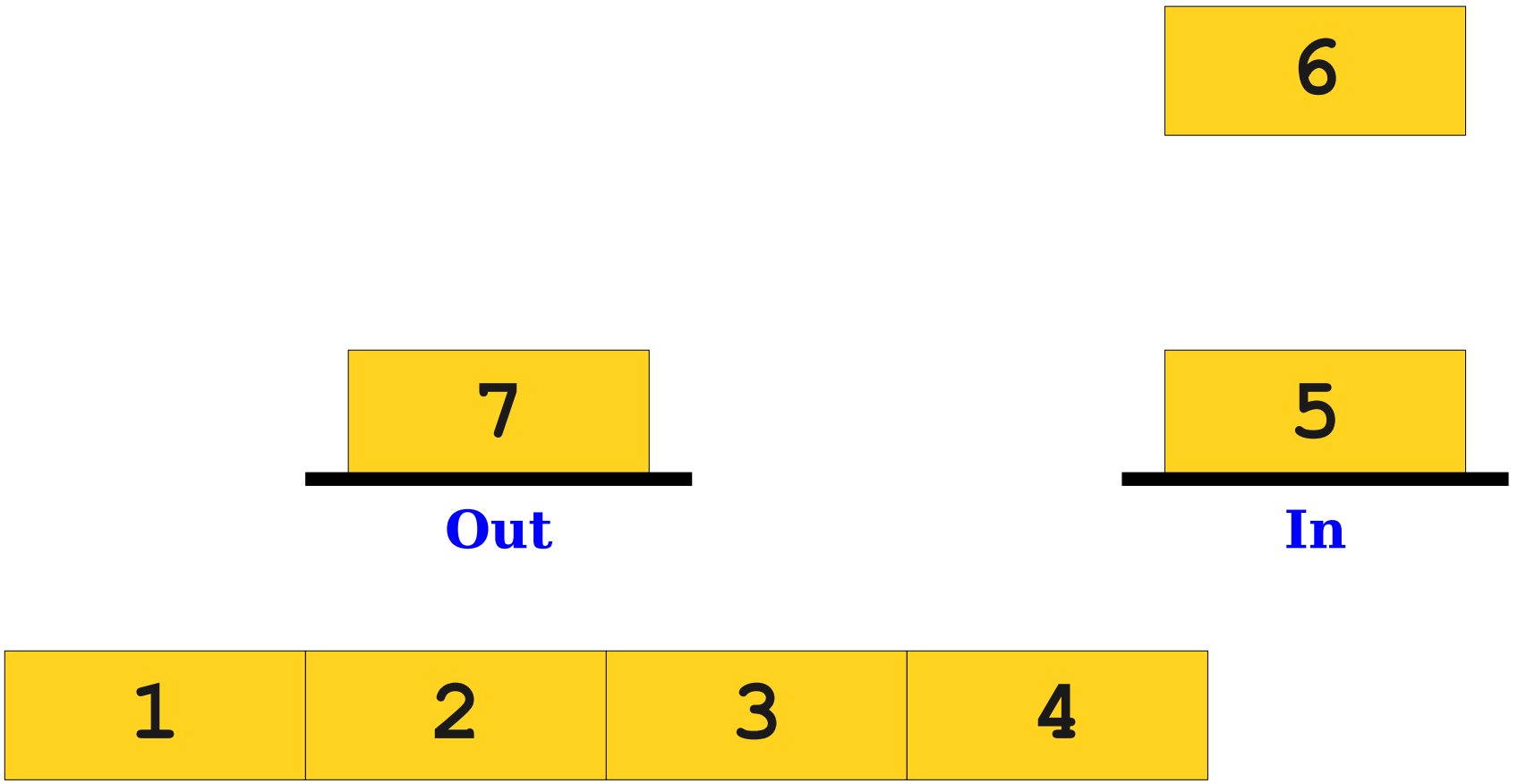

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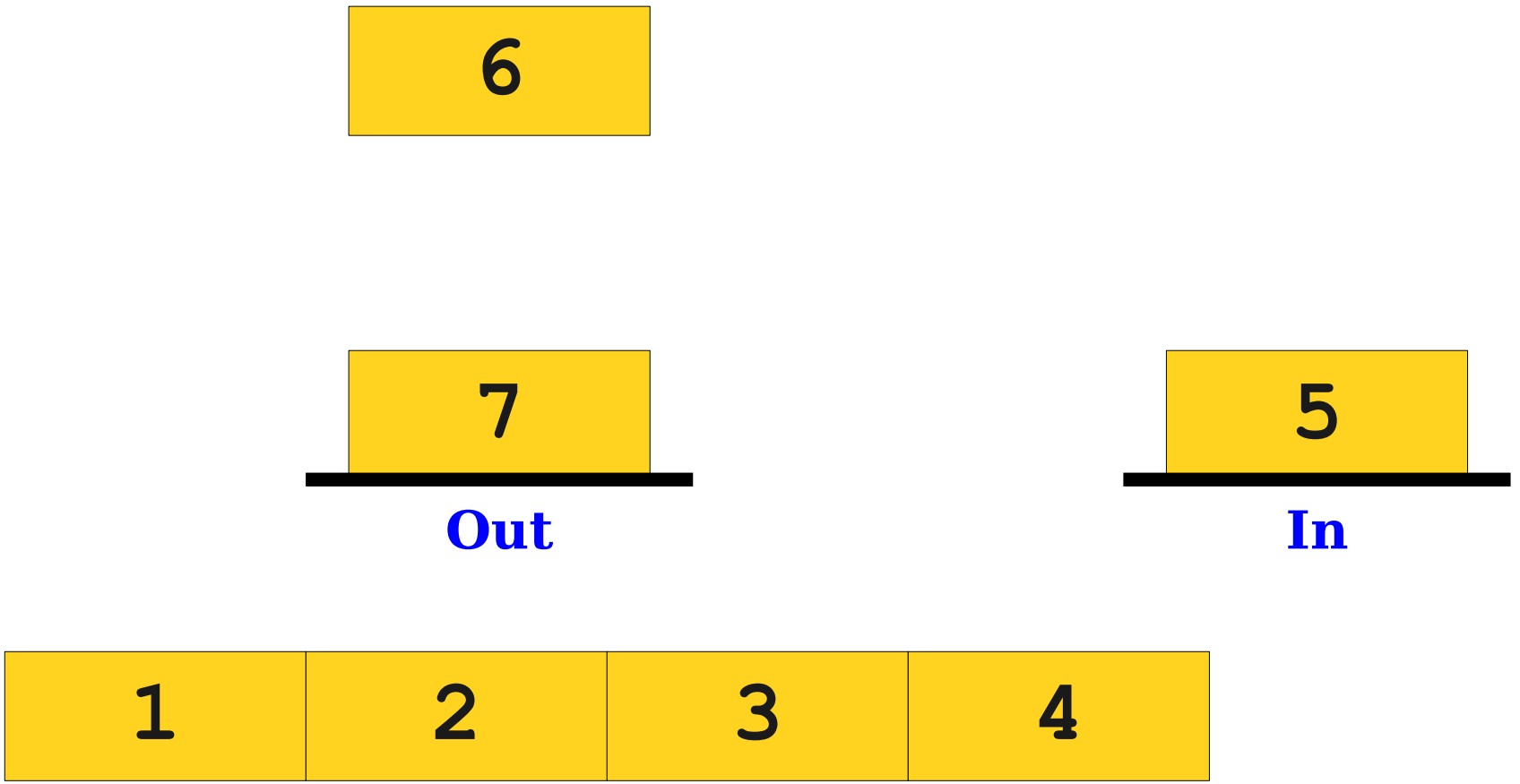

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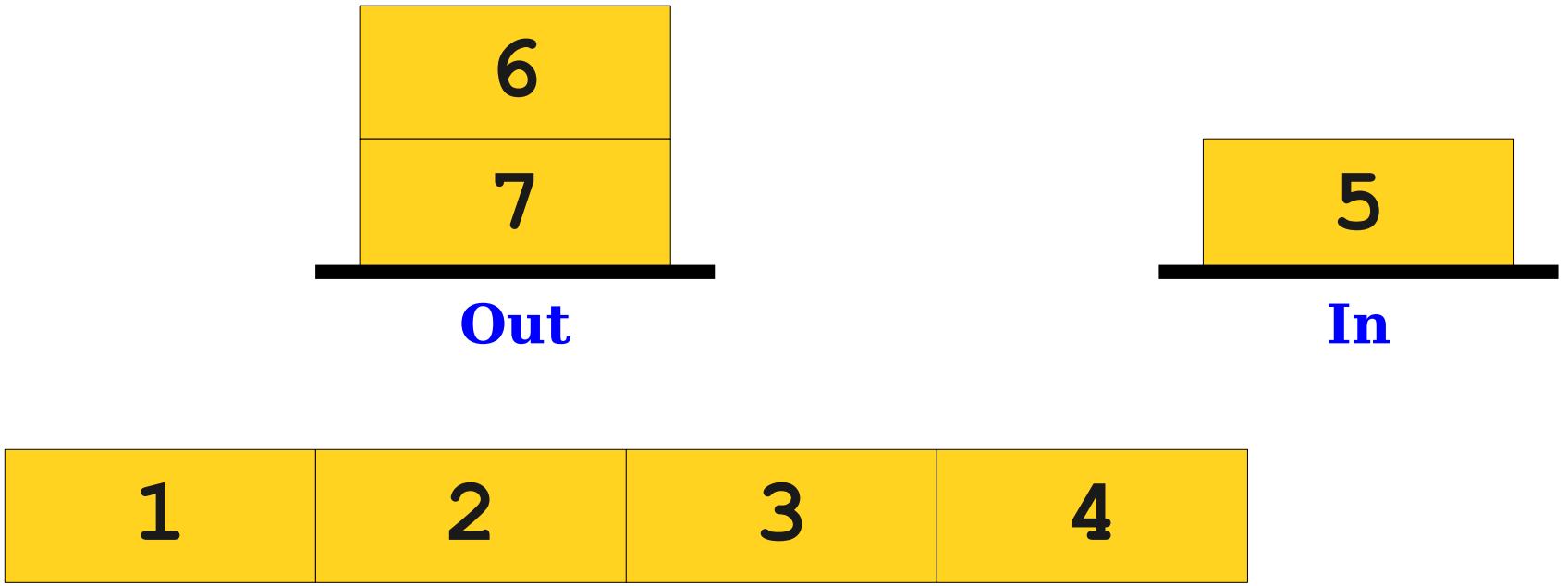

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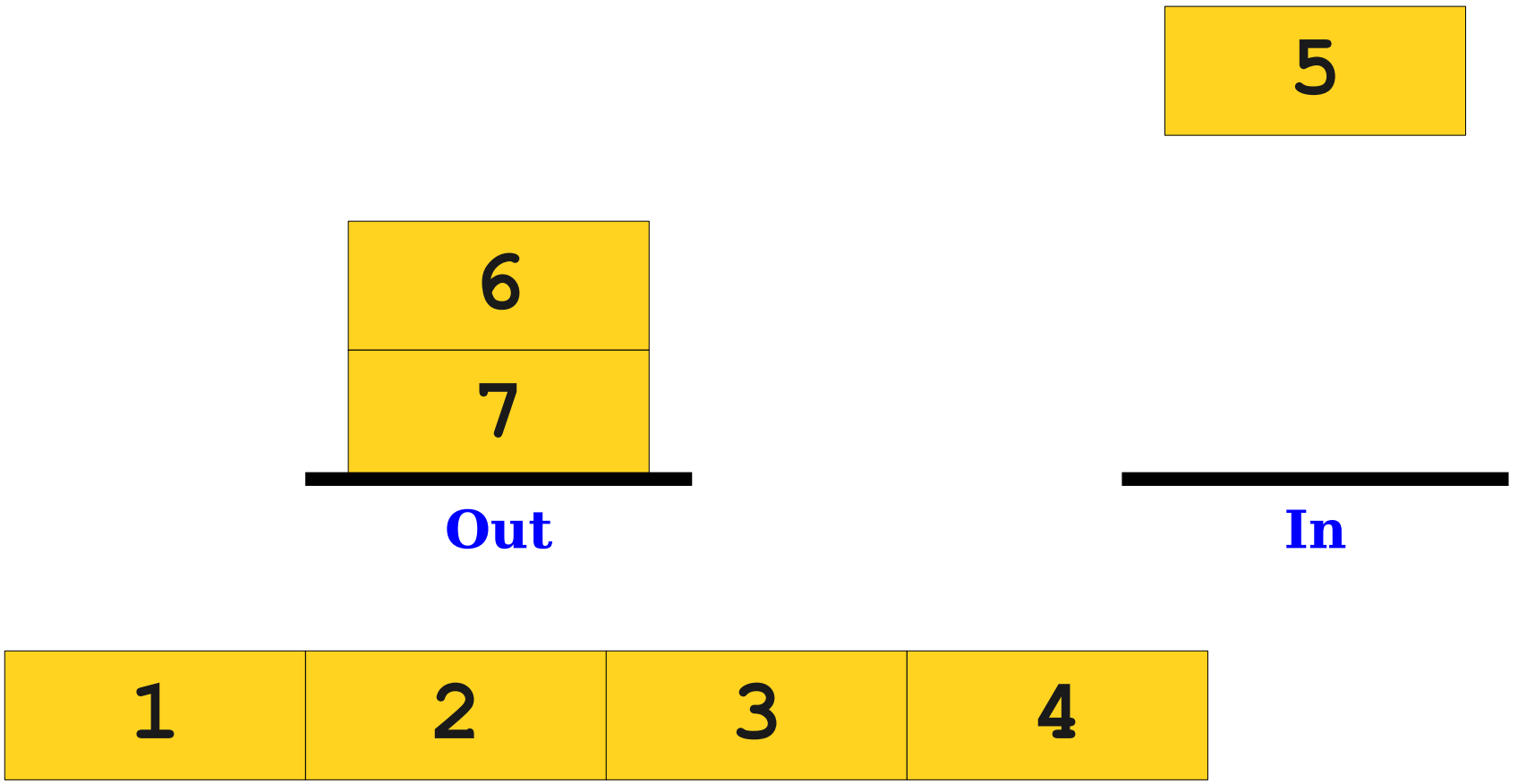

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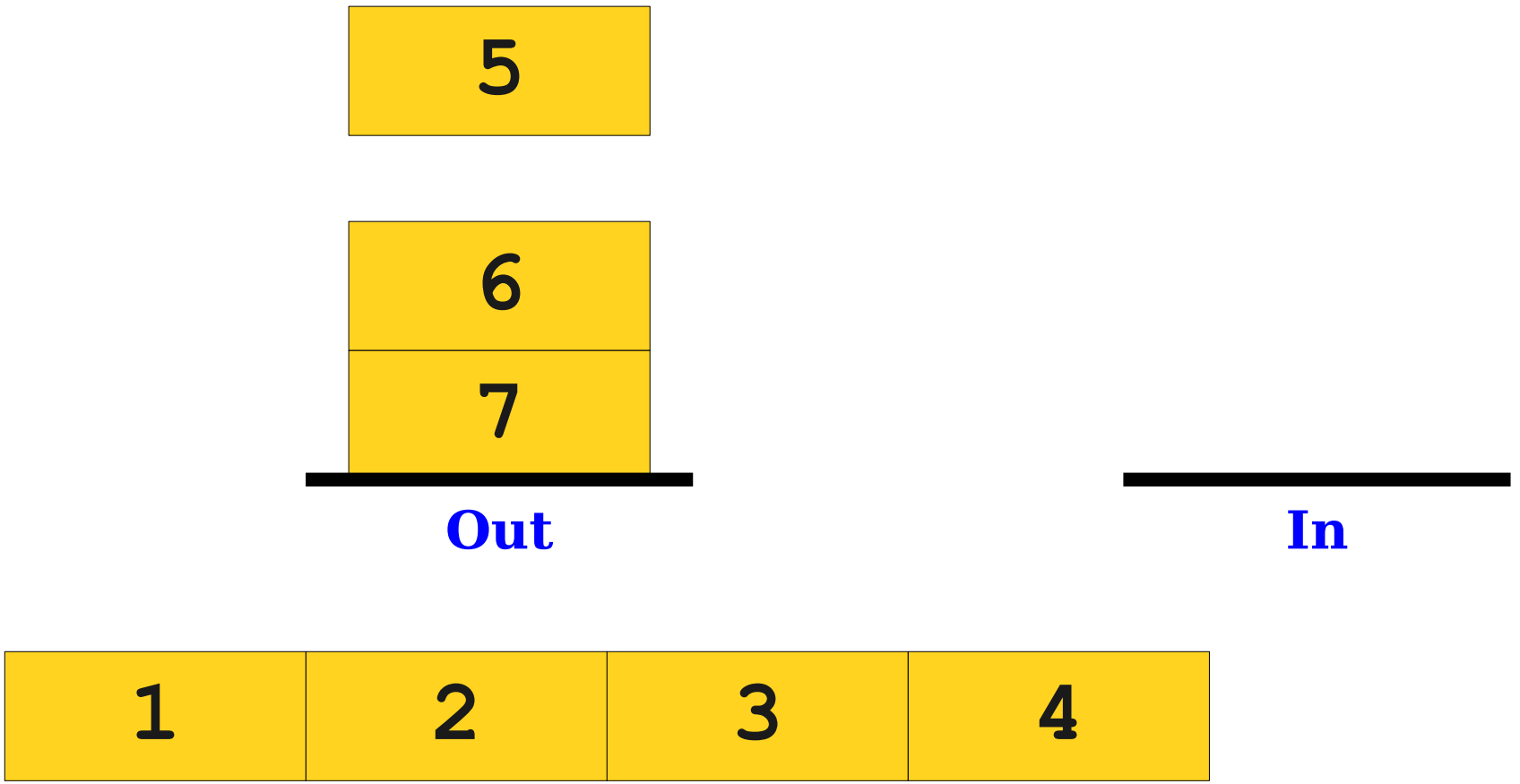

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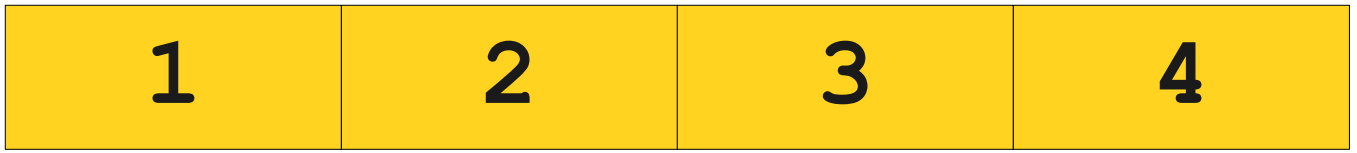

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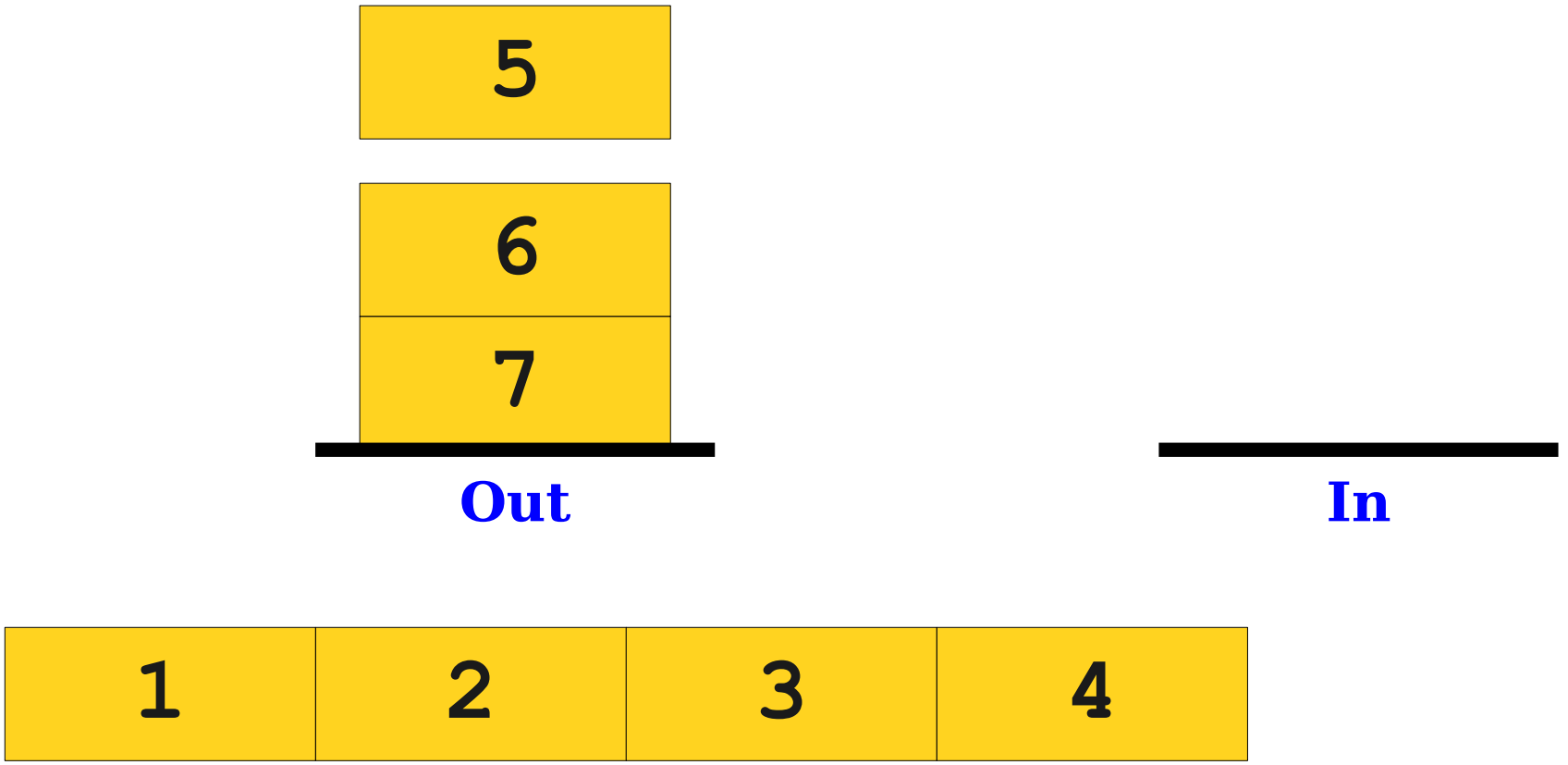

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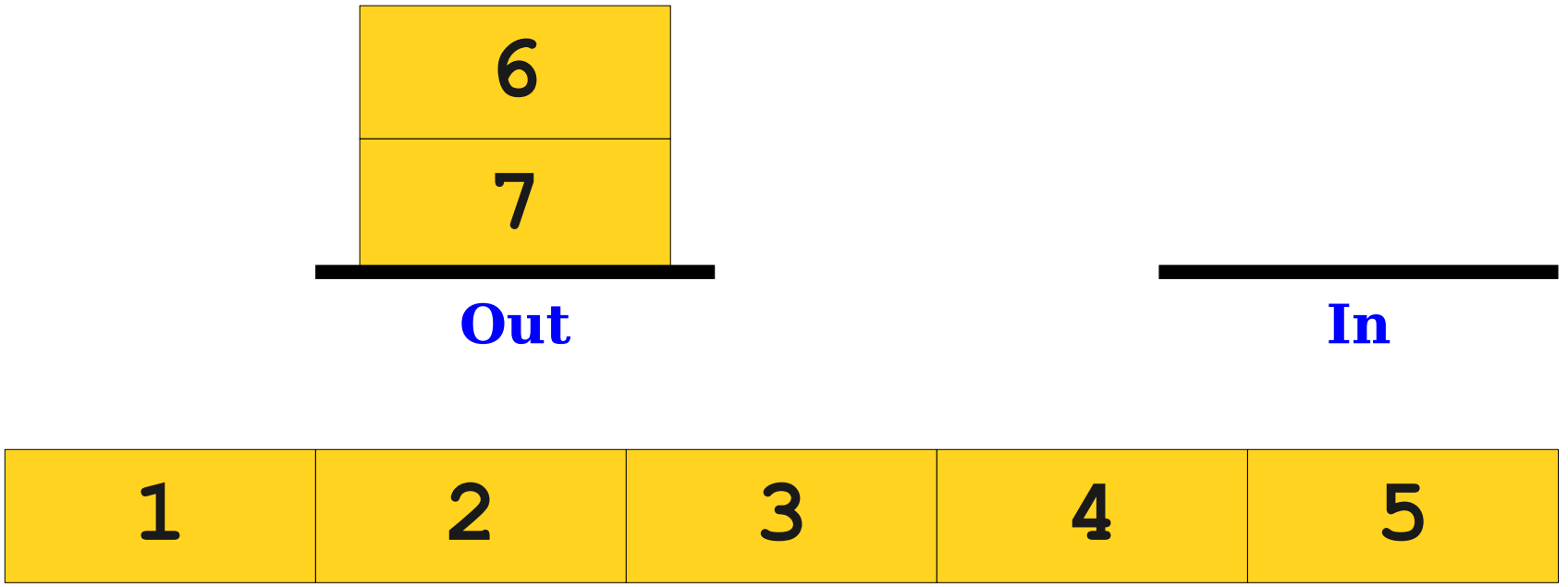

The Two-Stack Queue

- Maintain two stacks, an **In** stack and an **Out** stack.
- To enqueue an element, push it onto the **In** stack.
- To dequeue an element:
  - If the **Out** stack is empty, pop everything off the **In** stack and push it onto the **Out** stack.
  - Pop the **Out** stack and return its value.

Analyzing Efficiency

- How efficient is our two-stack queue?
- All enqueues just do one push.
- A dequeue might do a lot of pushes *and* a lot of pops.
- However, let's do an amortized analysis:
  - Each element is pushed at most twice and popped at most twice.
  - n enqueues and n dequeues thus do at most $4n$ pushes and pops.
  - Any $4n$ pushes / pops takes $O(n)$ amortized time.
  - Amortized cost: **O(1)** per operation.

Next Time

- **Linked Lists**
  - A different way to represent sequences of elements.
- **Dynamic Allocation Revisited**
  - What else can we allocate?