Binary Search Trees

Friday Four Square!
4:15PM, Outside Gates

Implementing **`Set`**

- On Monday and Wednesday, we saw how to implement the **`Map`** and **`Lexicon`**, respectively.
- Let's now turn our attention to the **`Set`**.
- Major operations:
  - Insert
  - Remove
  - Contains

An Inefficient Implementation

- We could implement the `Set` as a list of all the values it contains.
- To add an element:
  - Check if the element already exists.
  - If not, append it.
- To remove an element:
  - Find and remove it from the list.
- To see if an element exists:
  - Search the list for the element.

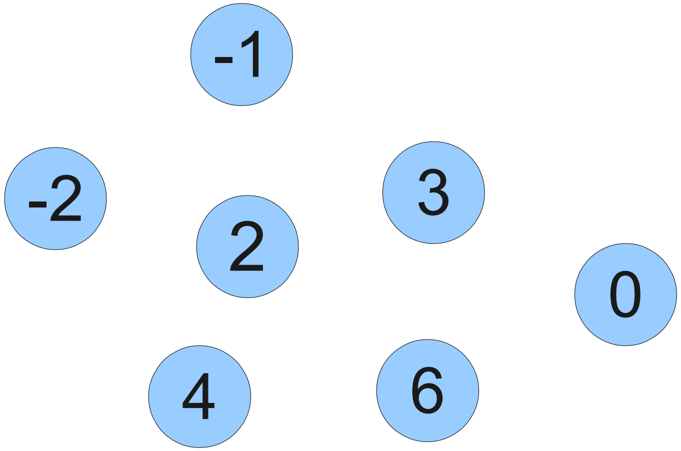
Using Hashing

- If we have a hash function for the elements being stored, we can implement a **`Set`** using a hash table.
- What is the expected time to insert a value?
- Answer: **O(1)**.
- What is the expected time to remove a value?
- Answer: **O(1)**.
- What is the expected time to check if a value exists?
- Answer: **O(1)**.
- However, writing a good hash function for a set of elements can be very tricky!

Using Tries

- If our keys are strings, we can store the set using a trie.
- Looking up or inserting a string with L letters takes time $O(L)$.
- Doesn't work for arbitrary values.

An Entirely Different Approach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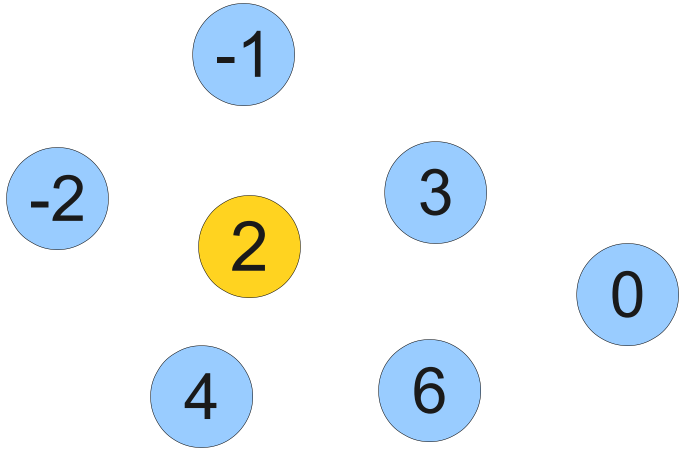
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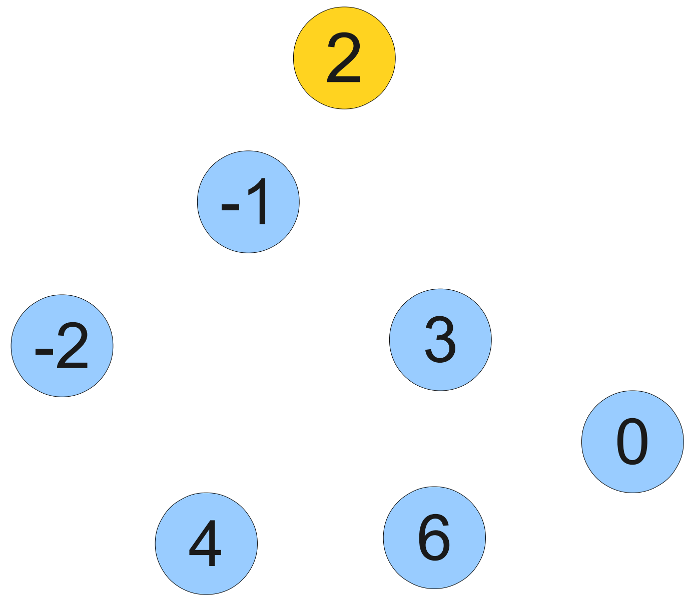
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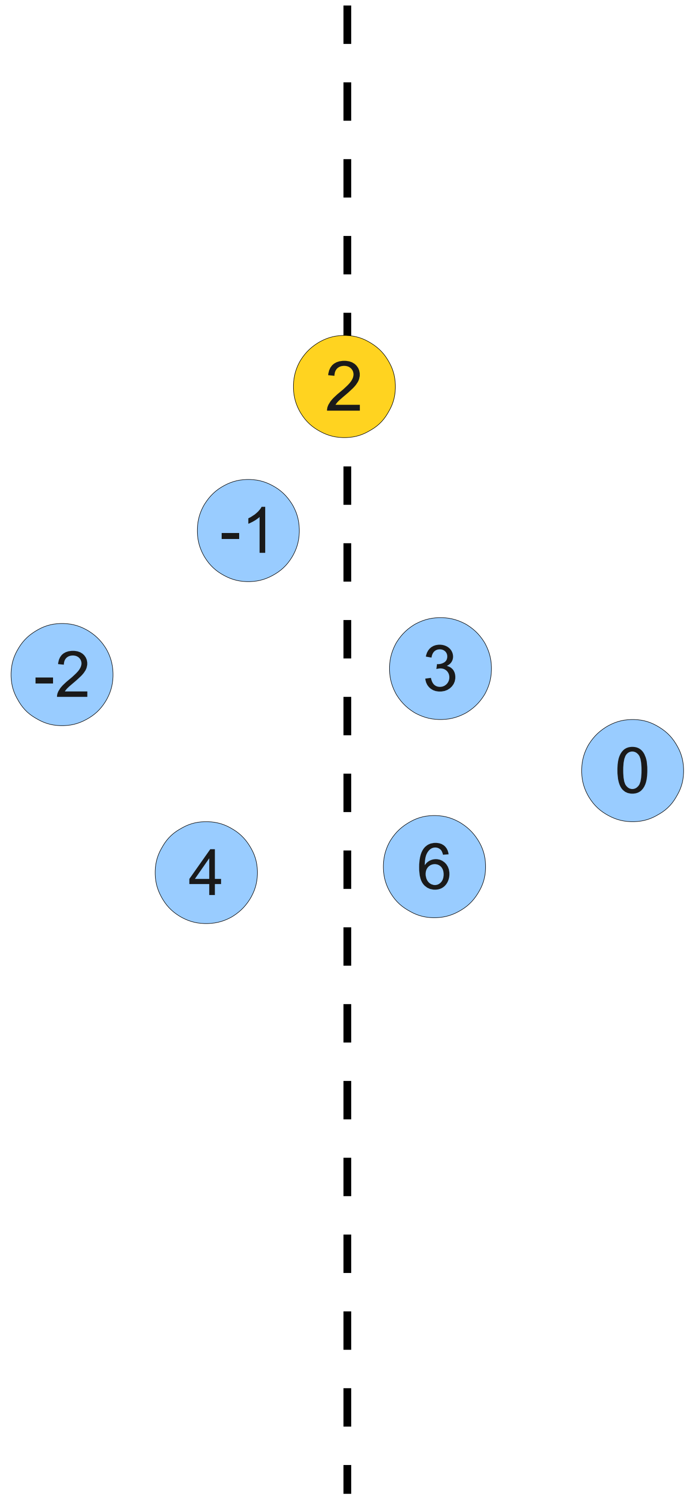
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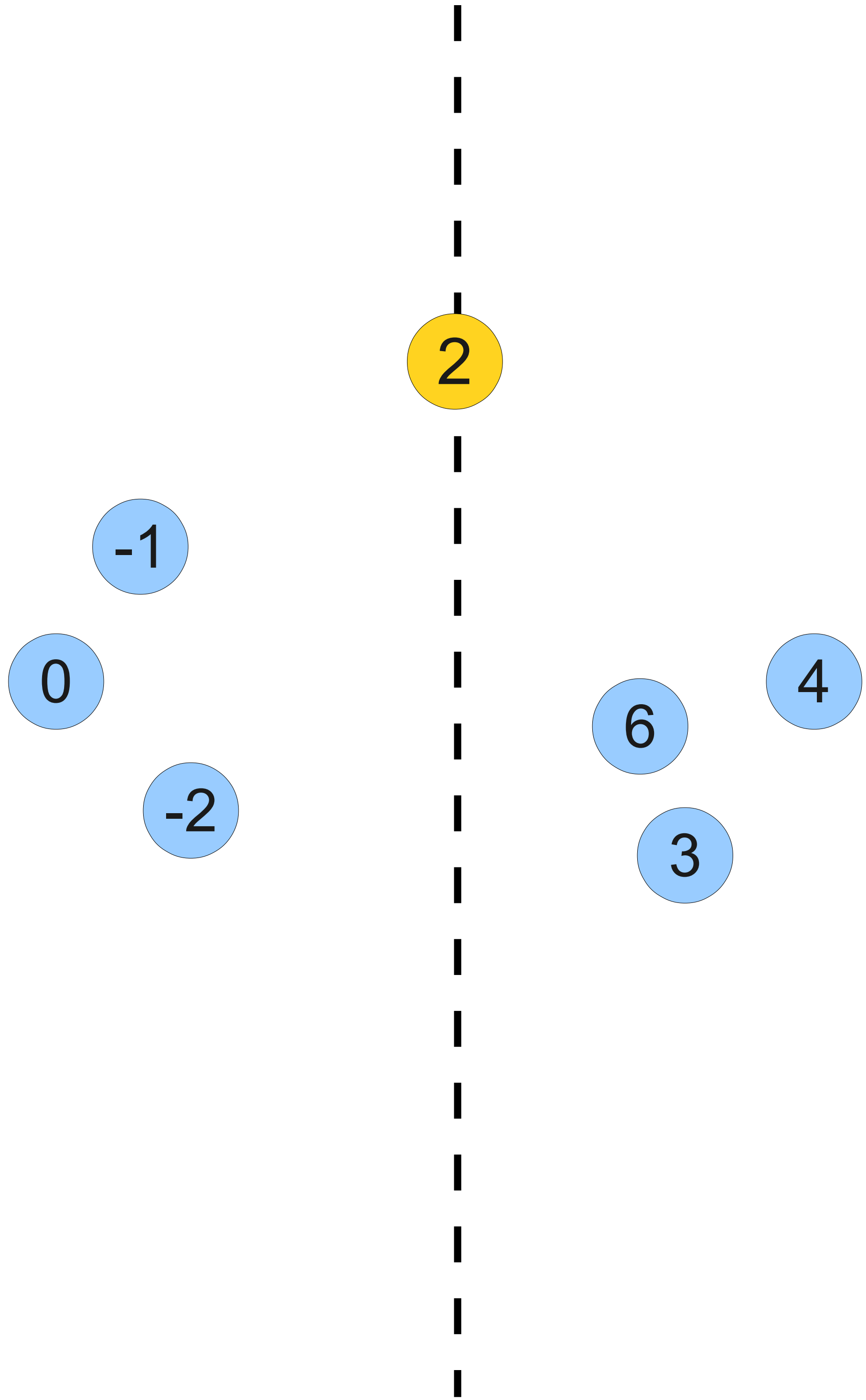
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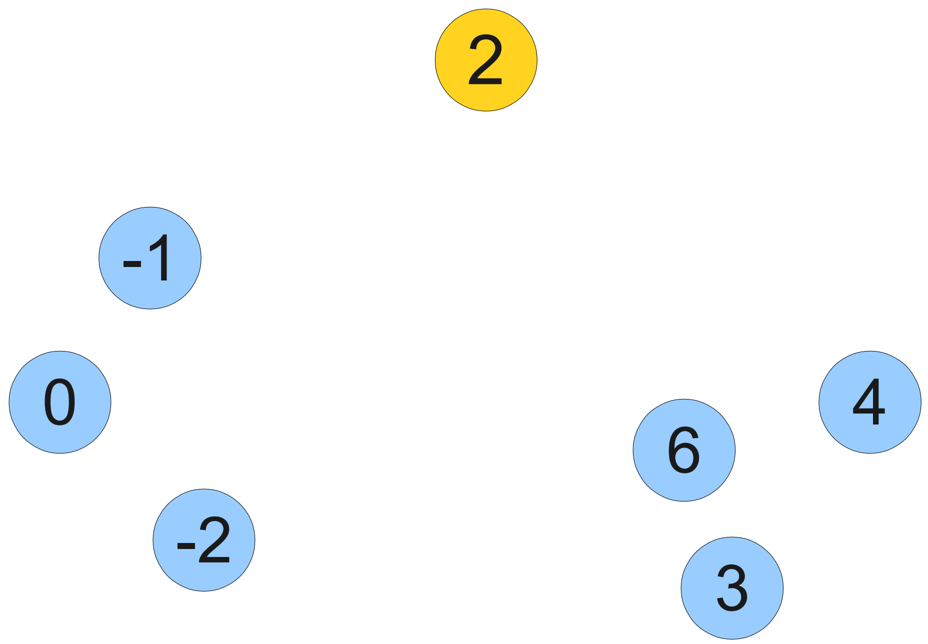
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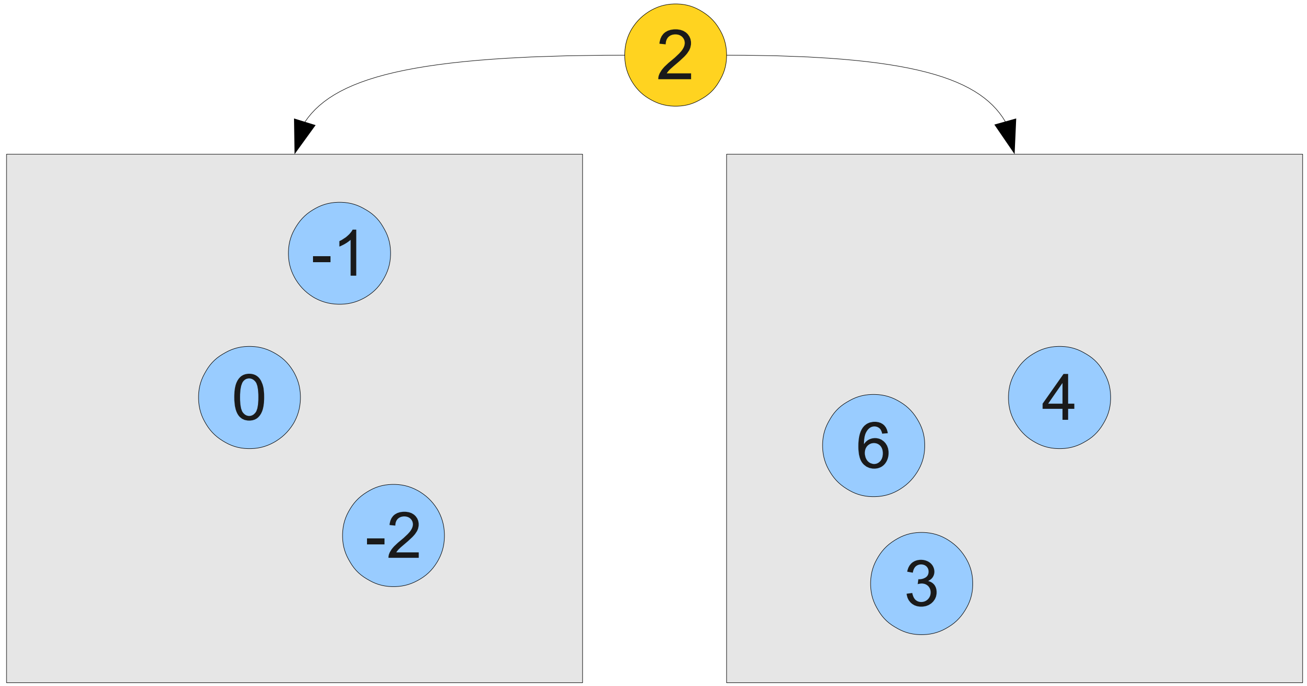
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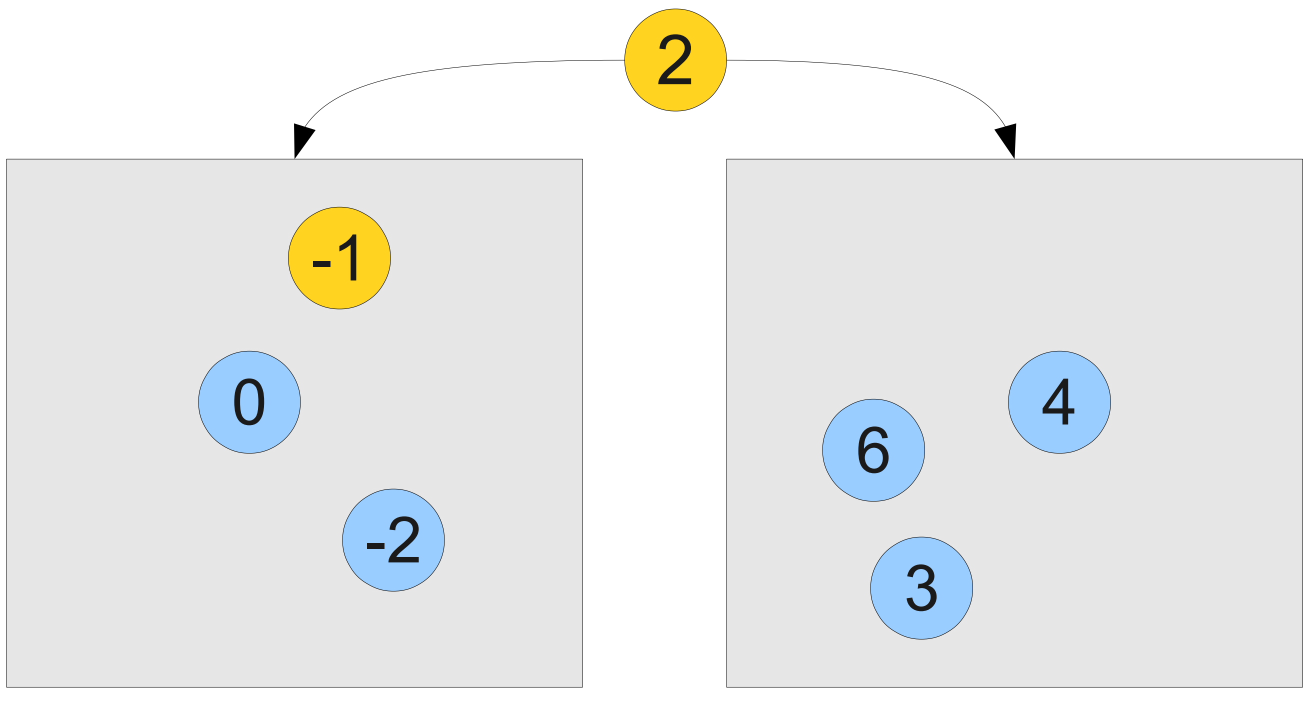
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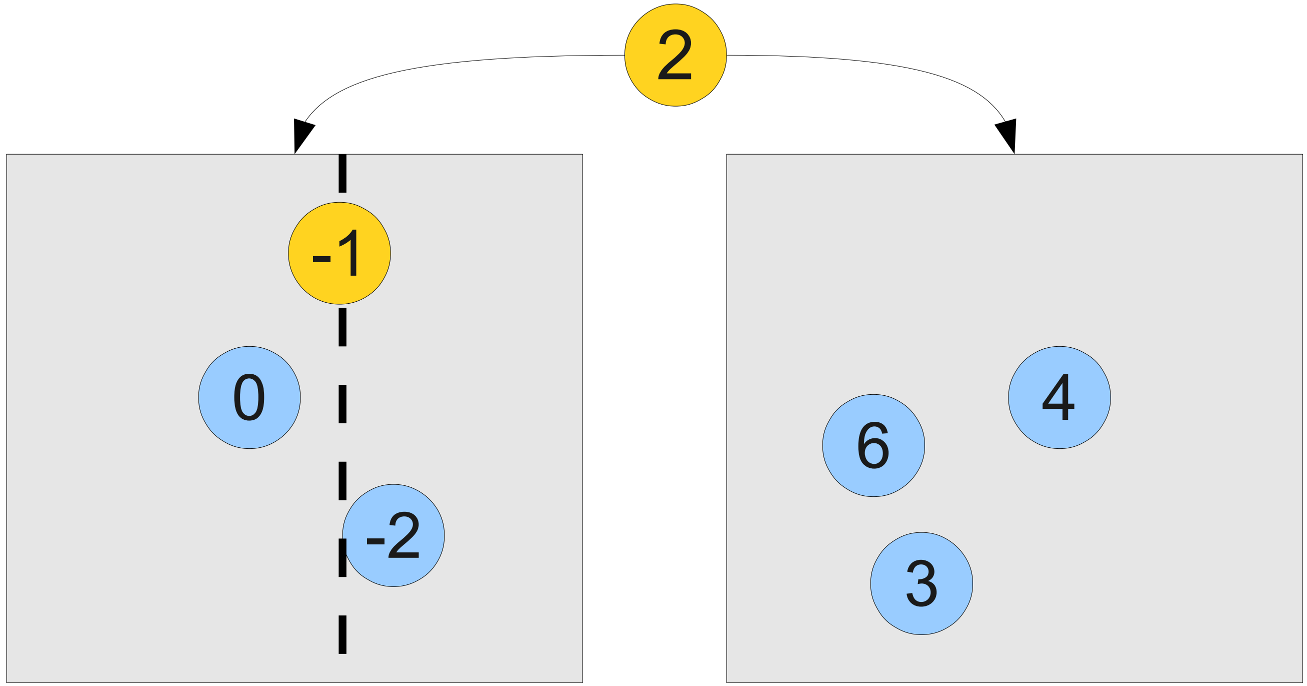
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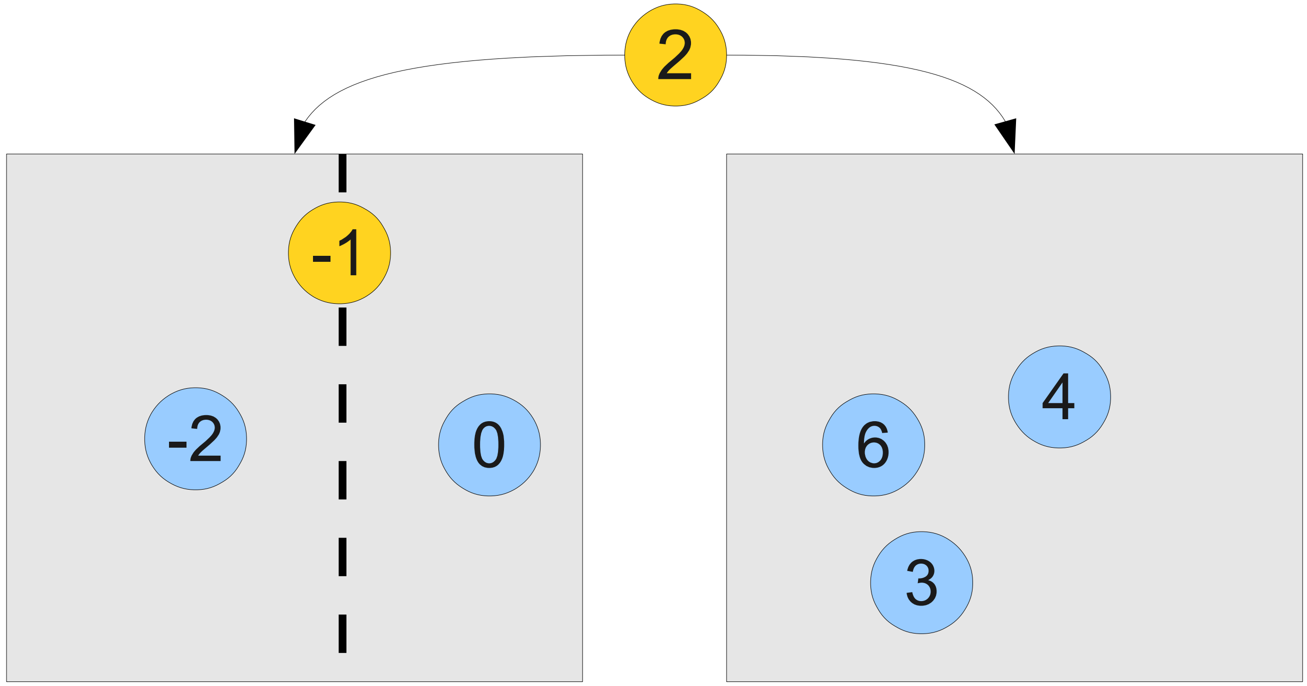
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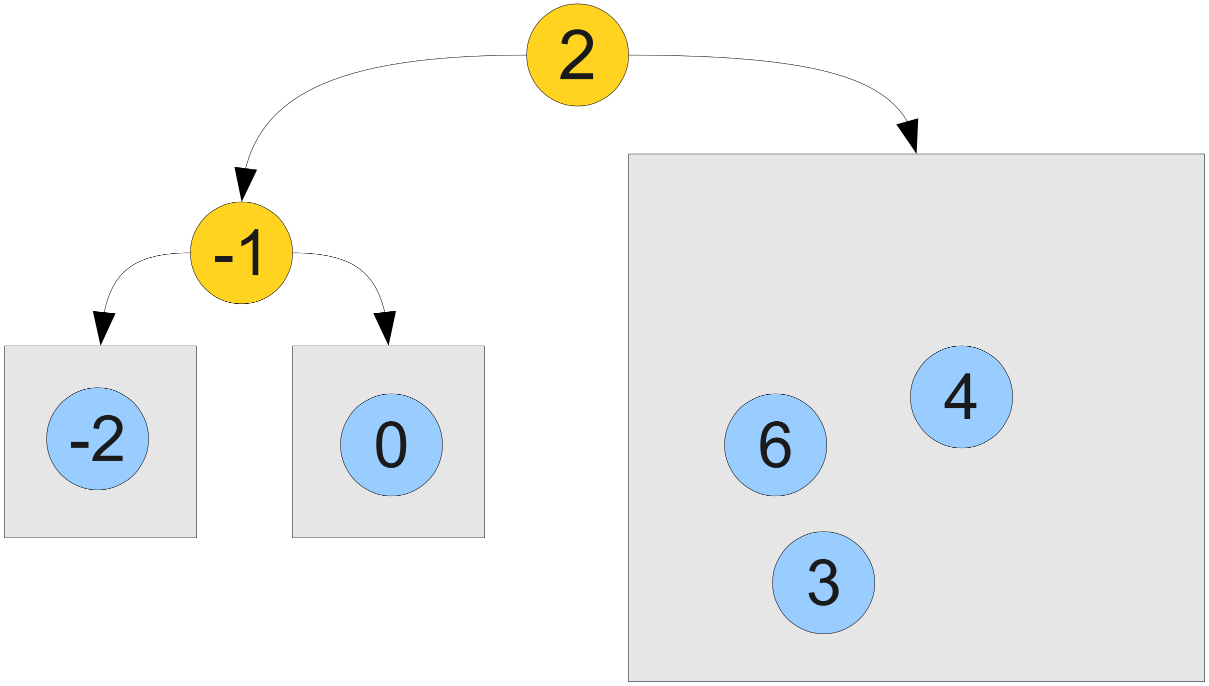
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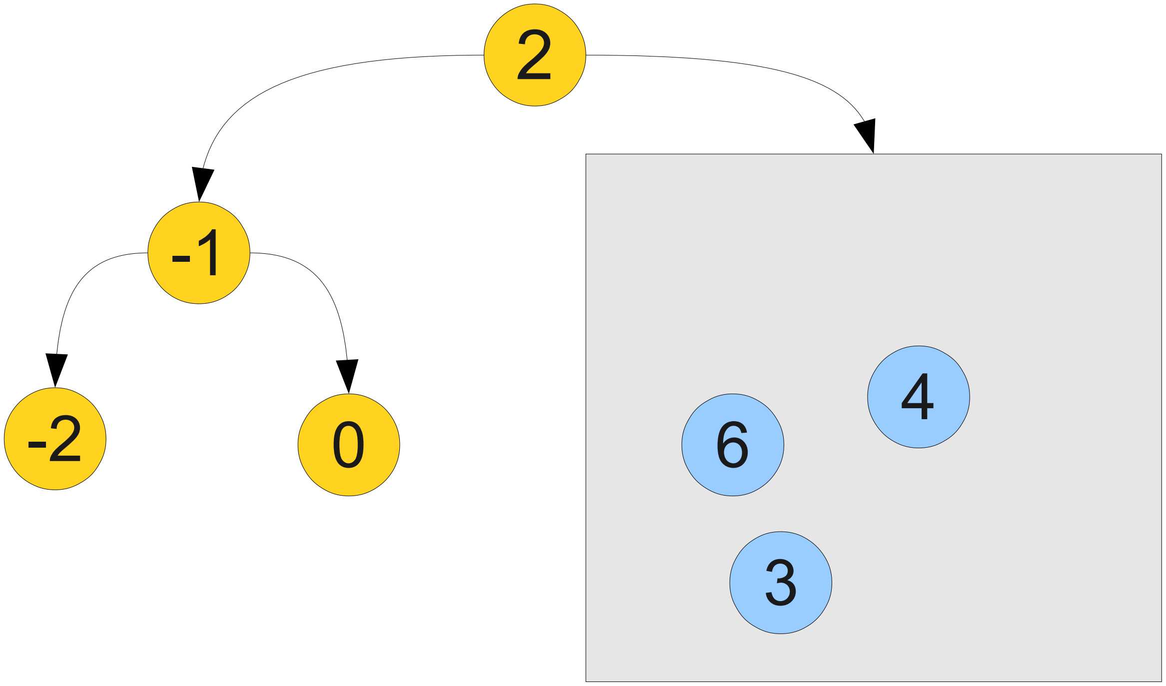
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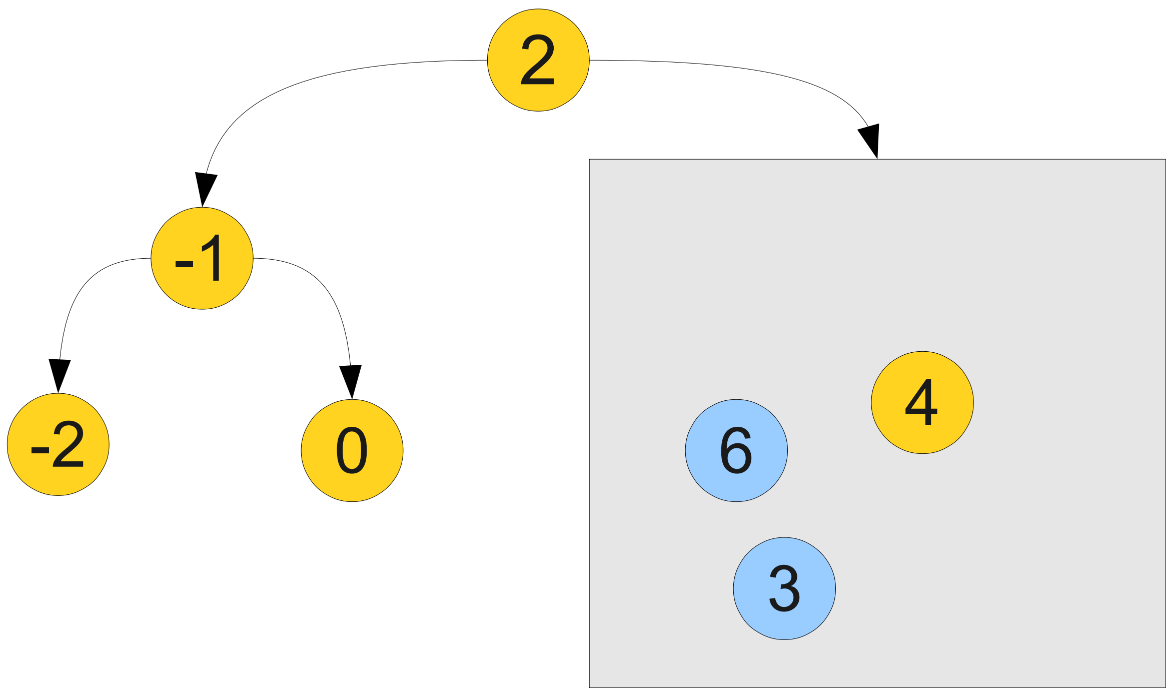
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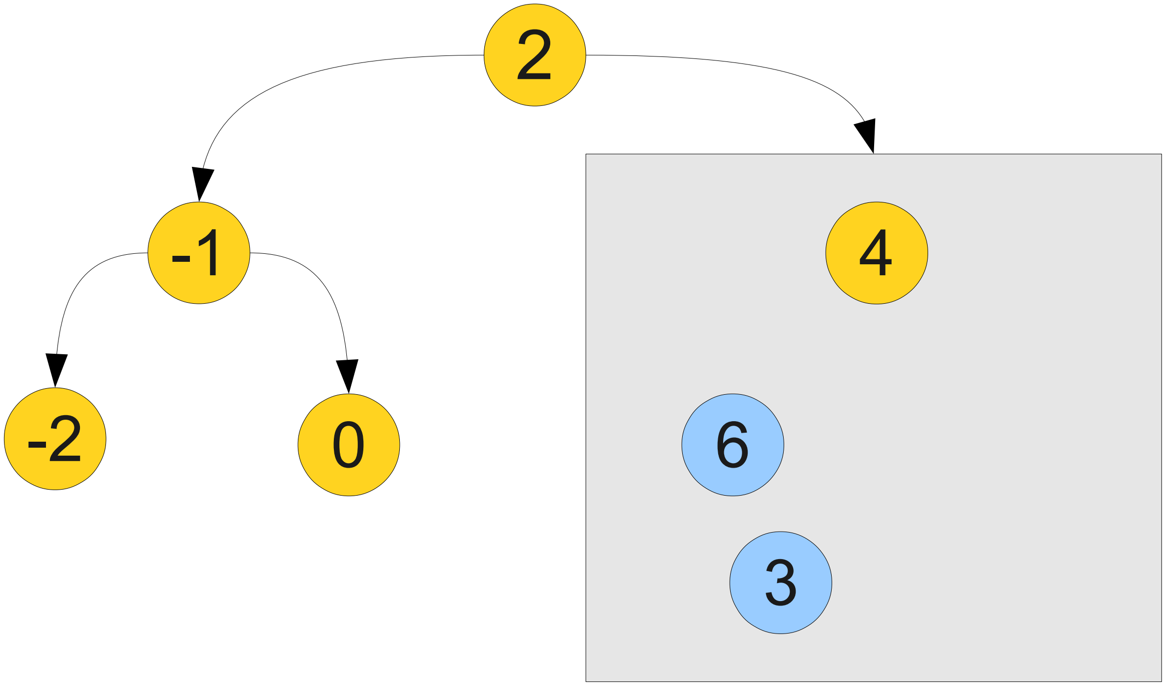
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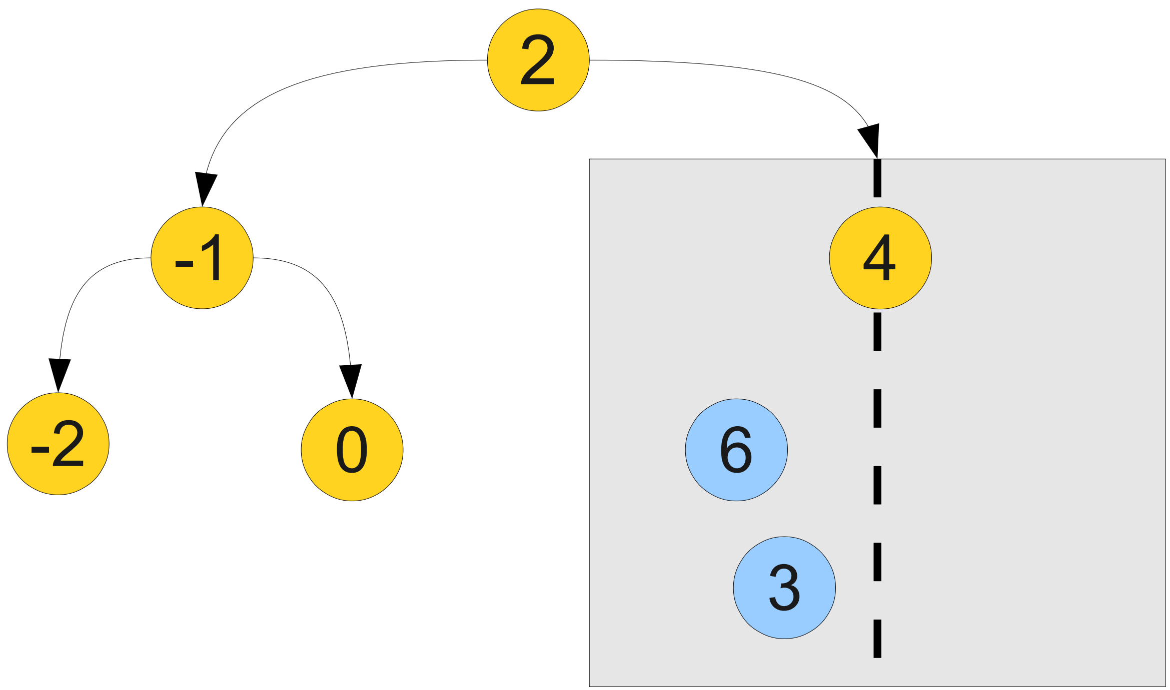
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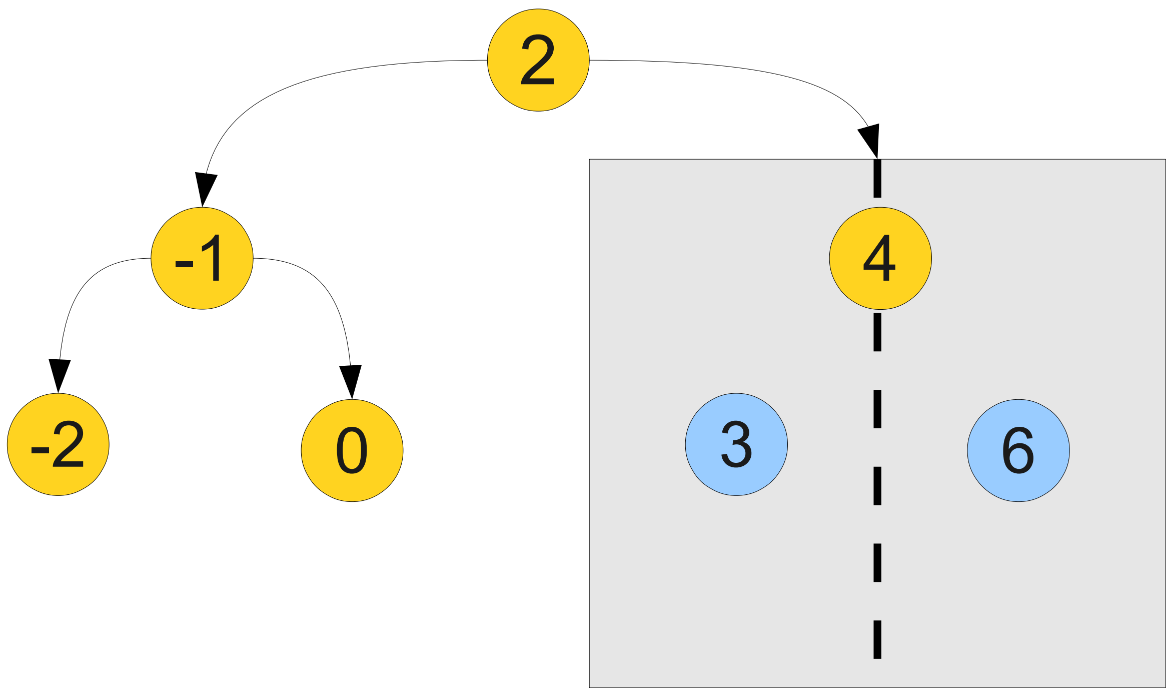
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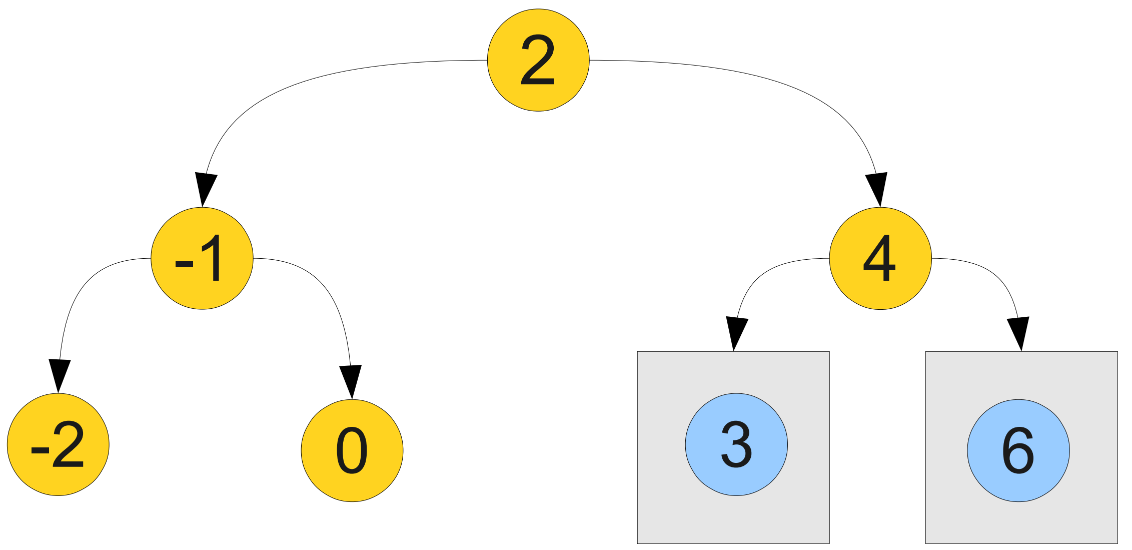
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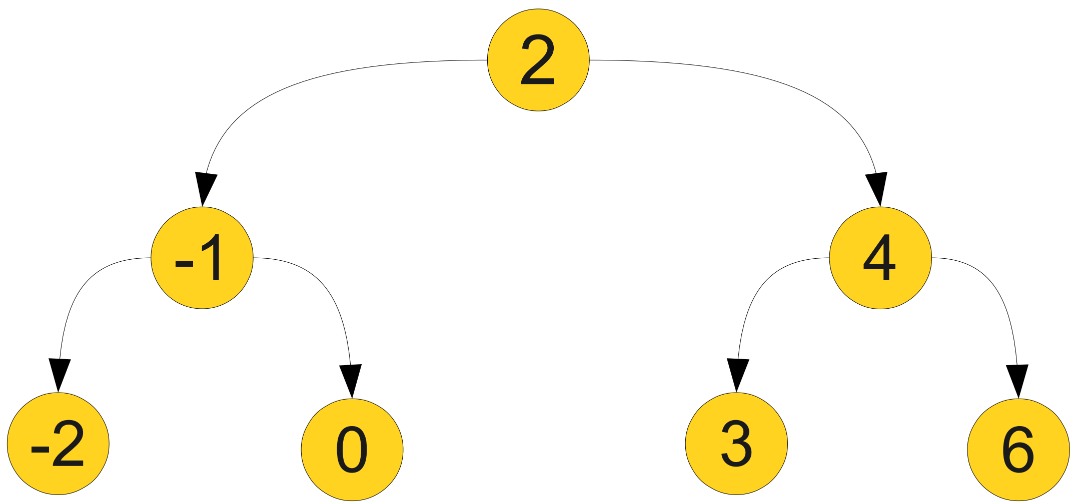
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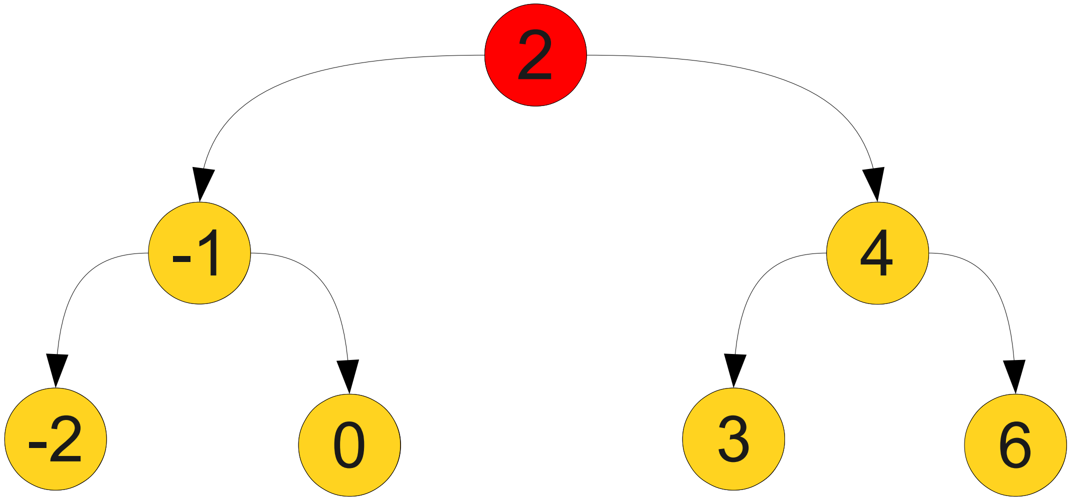
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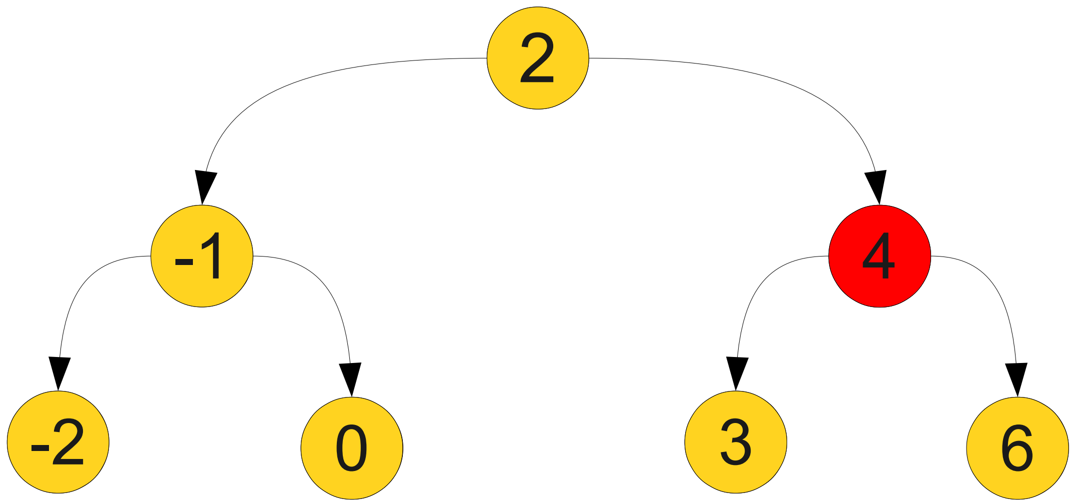
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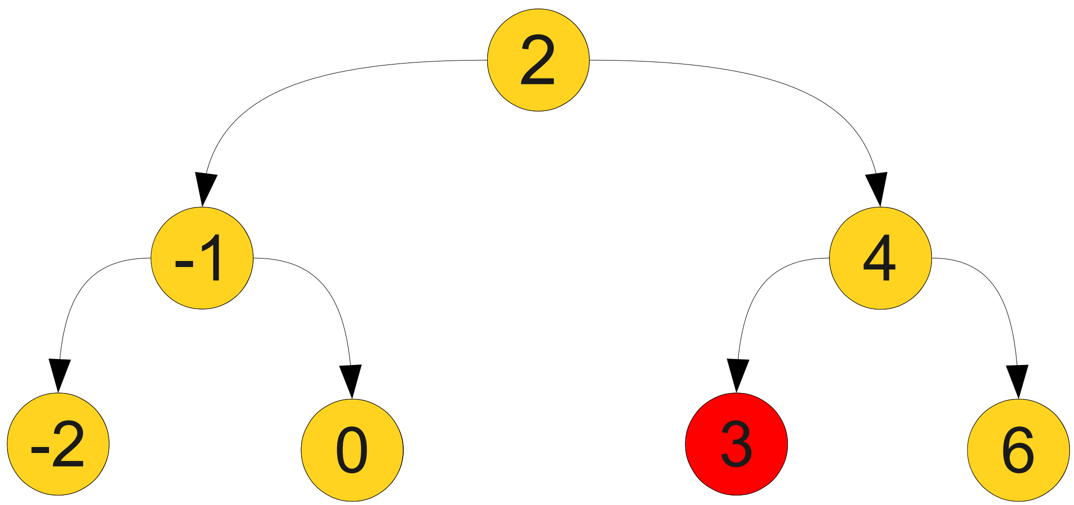
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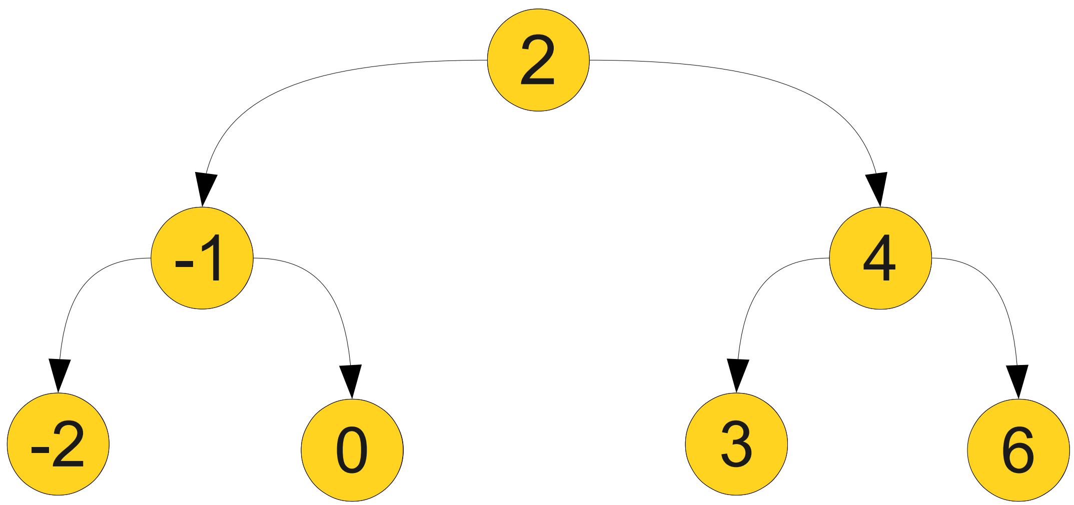
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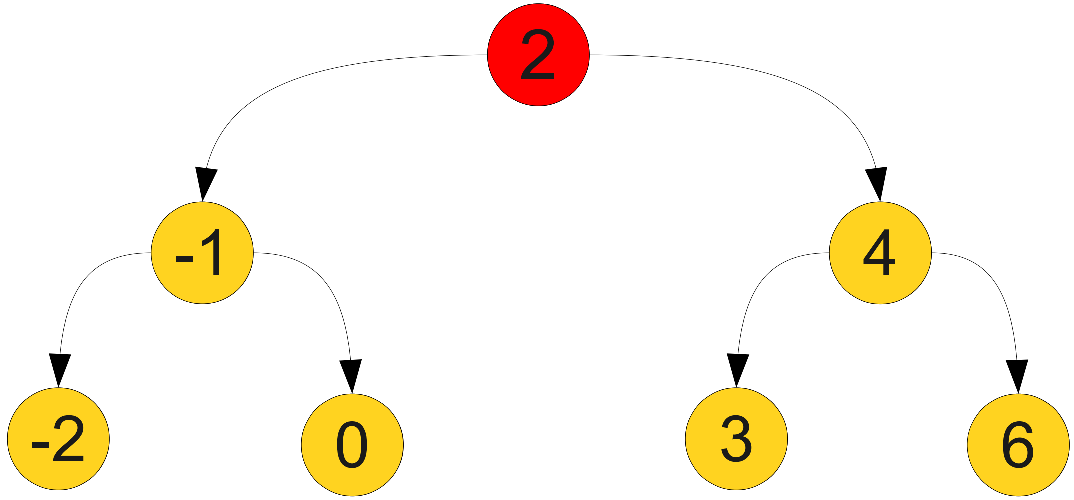
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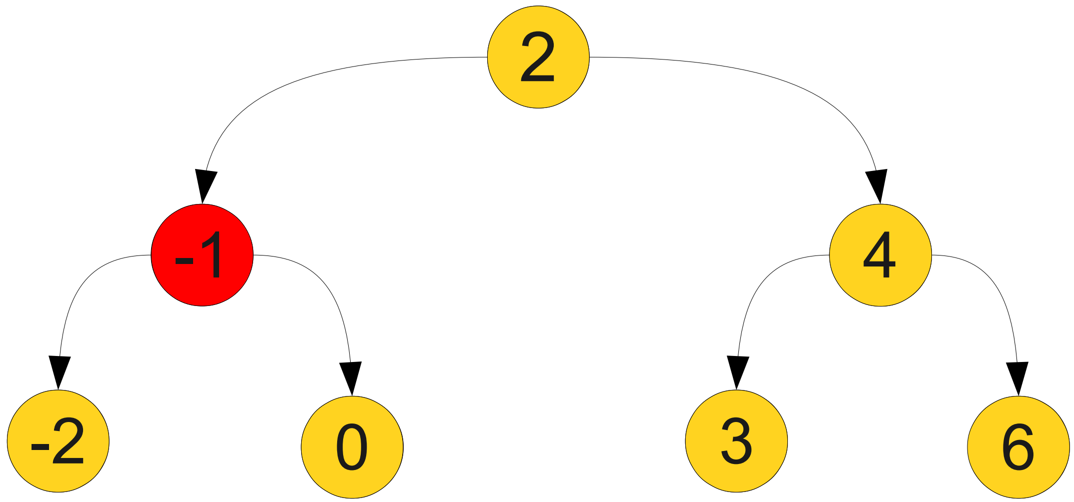
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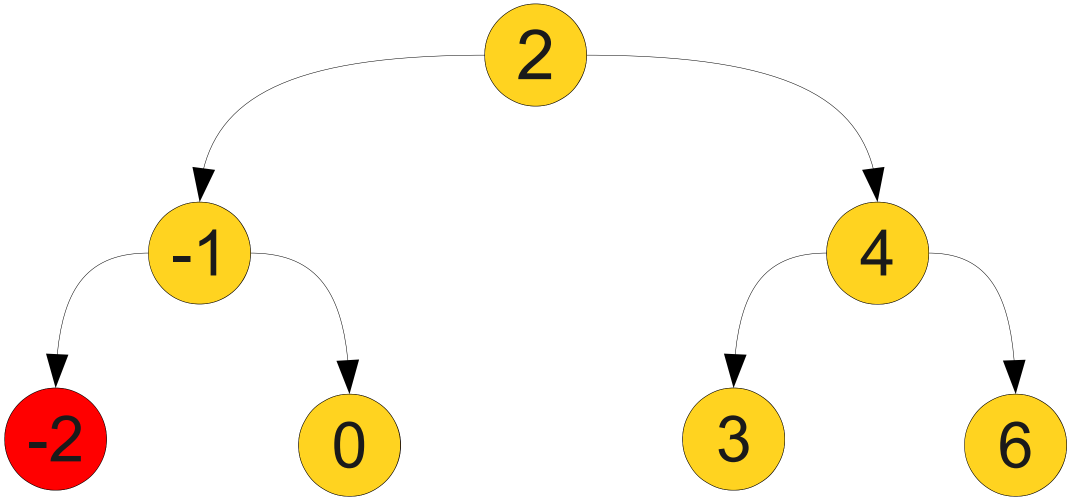
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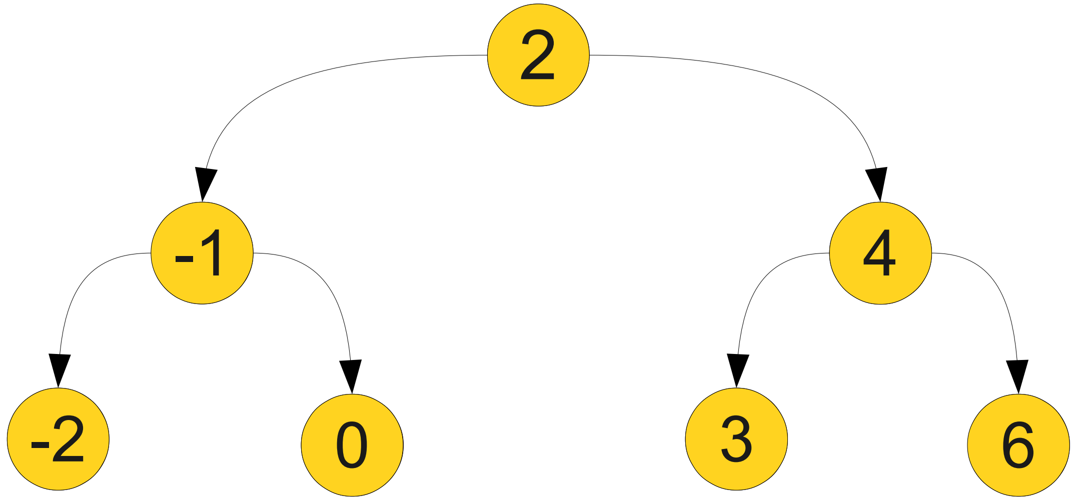
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Binary Search Trees

- The data structure we have just seen is called a **binary search tree** (or **BST**).
- Uses comparisons between elements to store elements efficiently.
- No need for a complex hash function, or the ability to work one symbol at a time.

The Intuition

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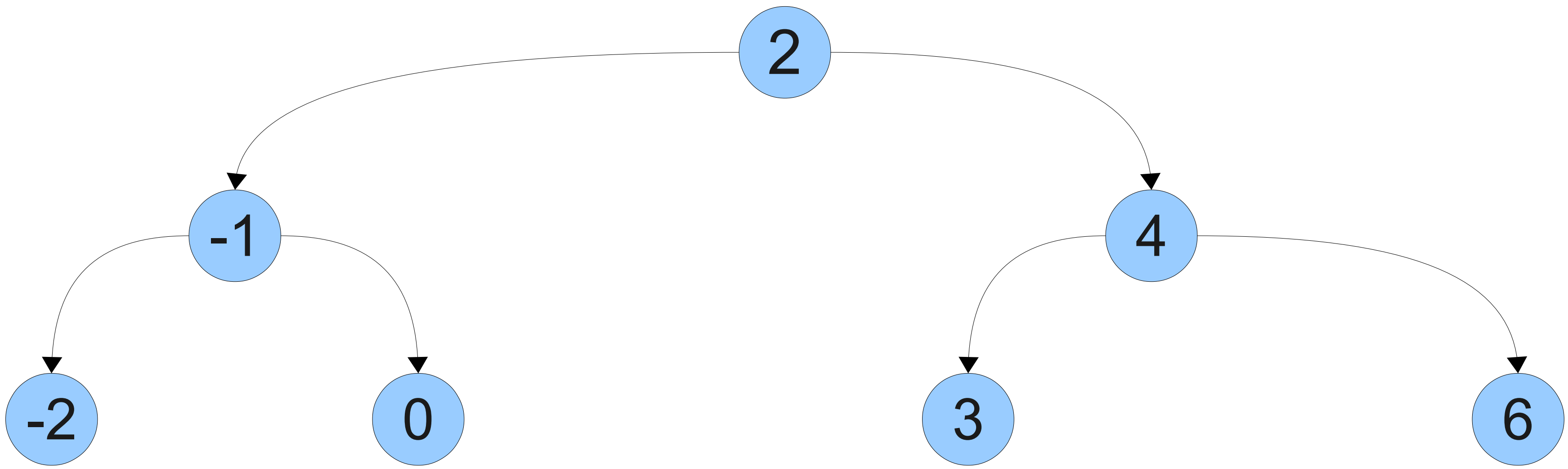

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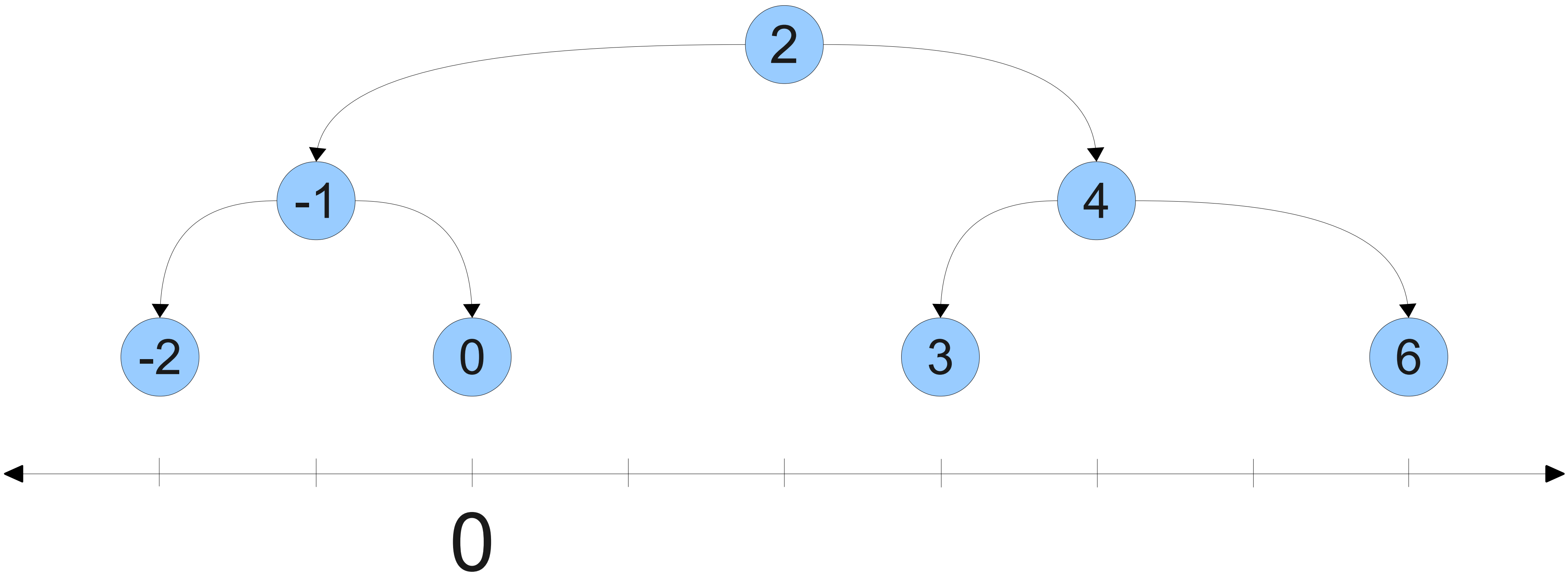

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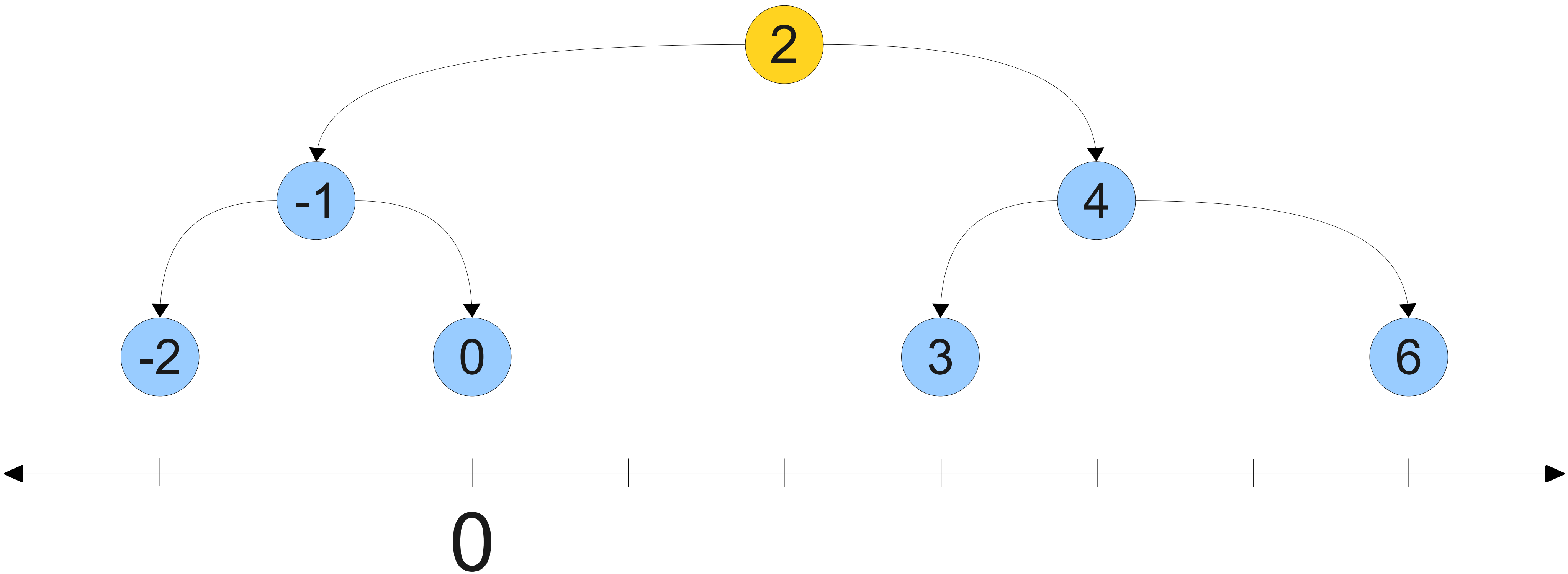

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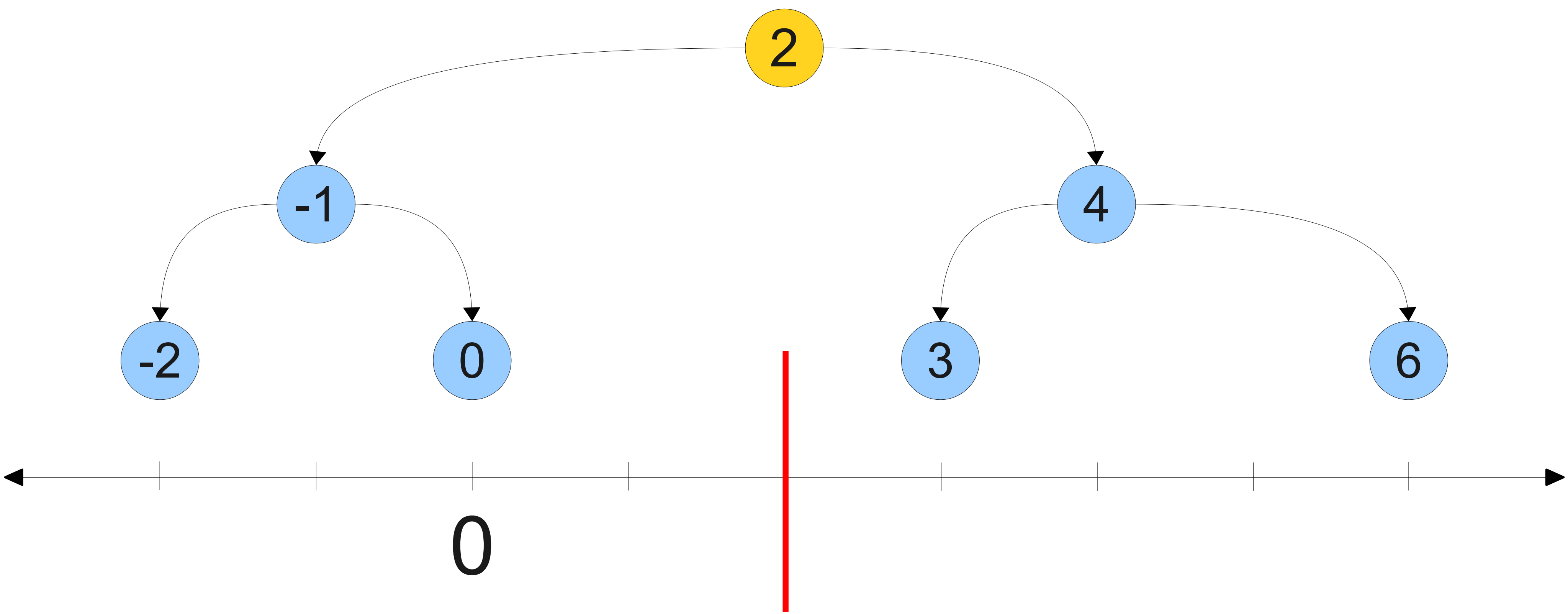

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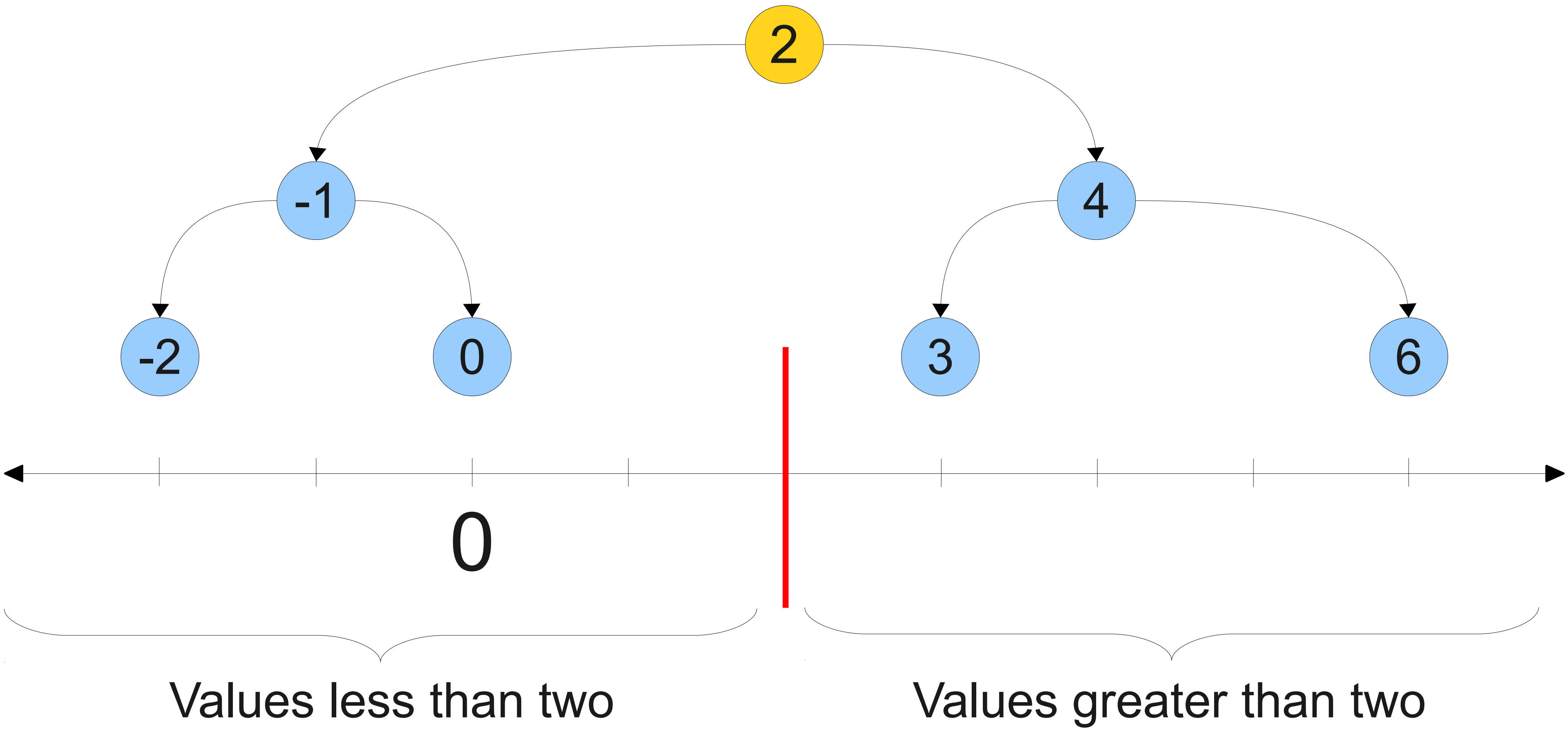

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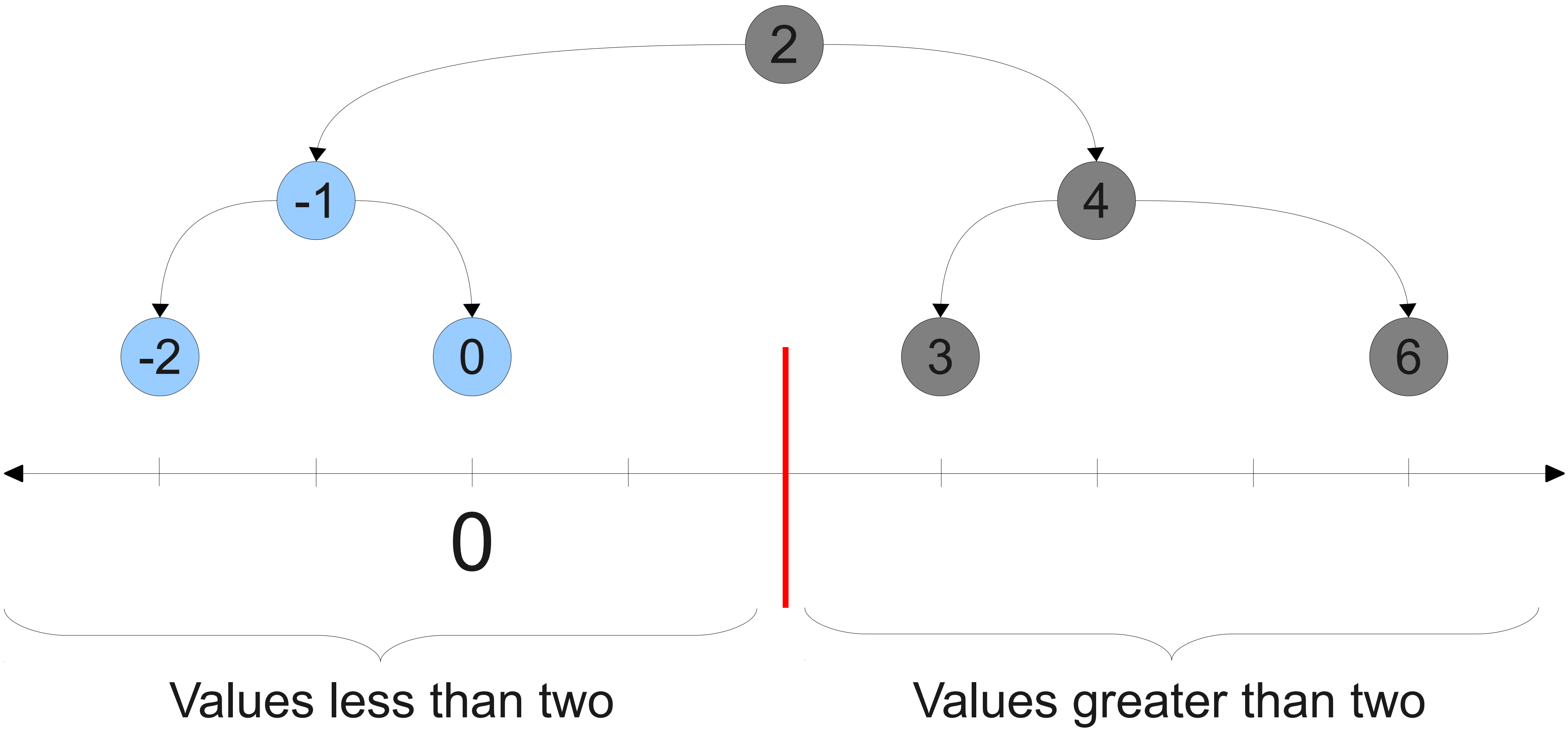

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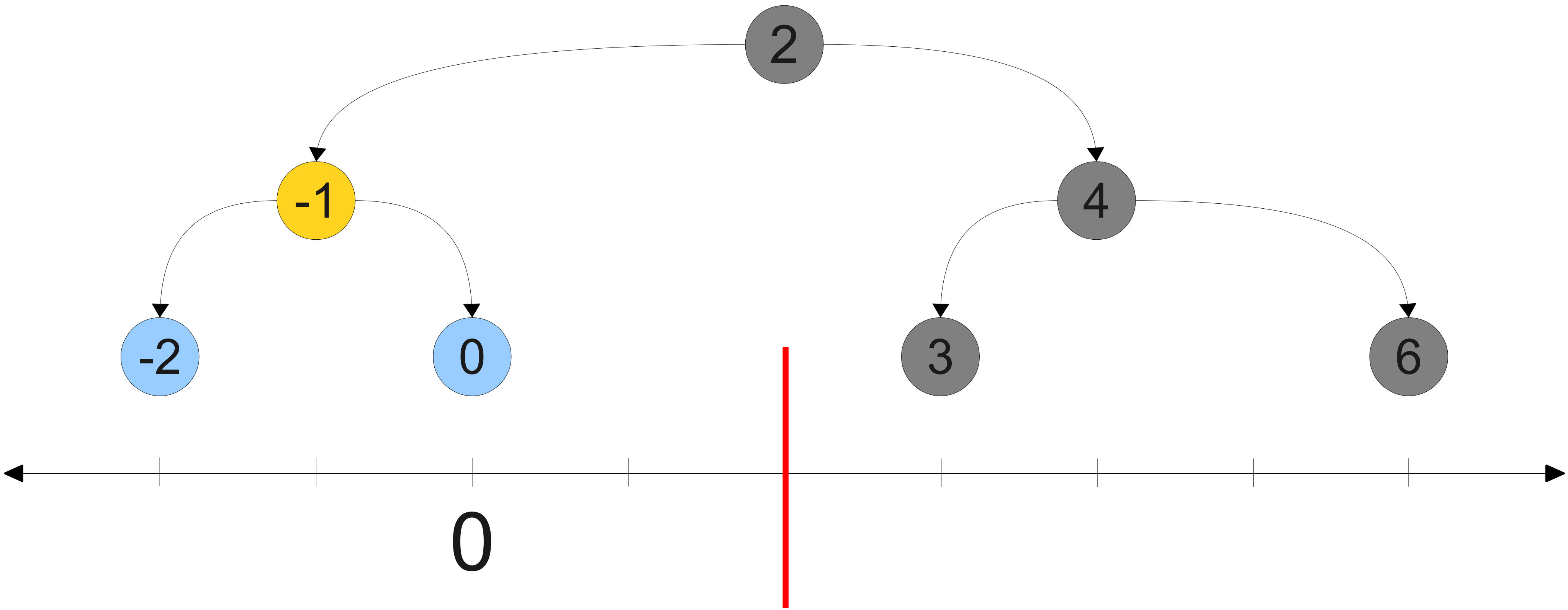

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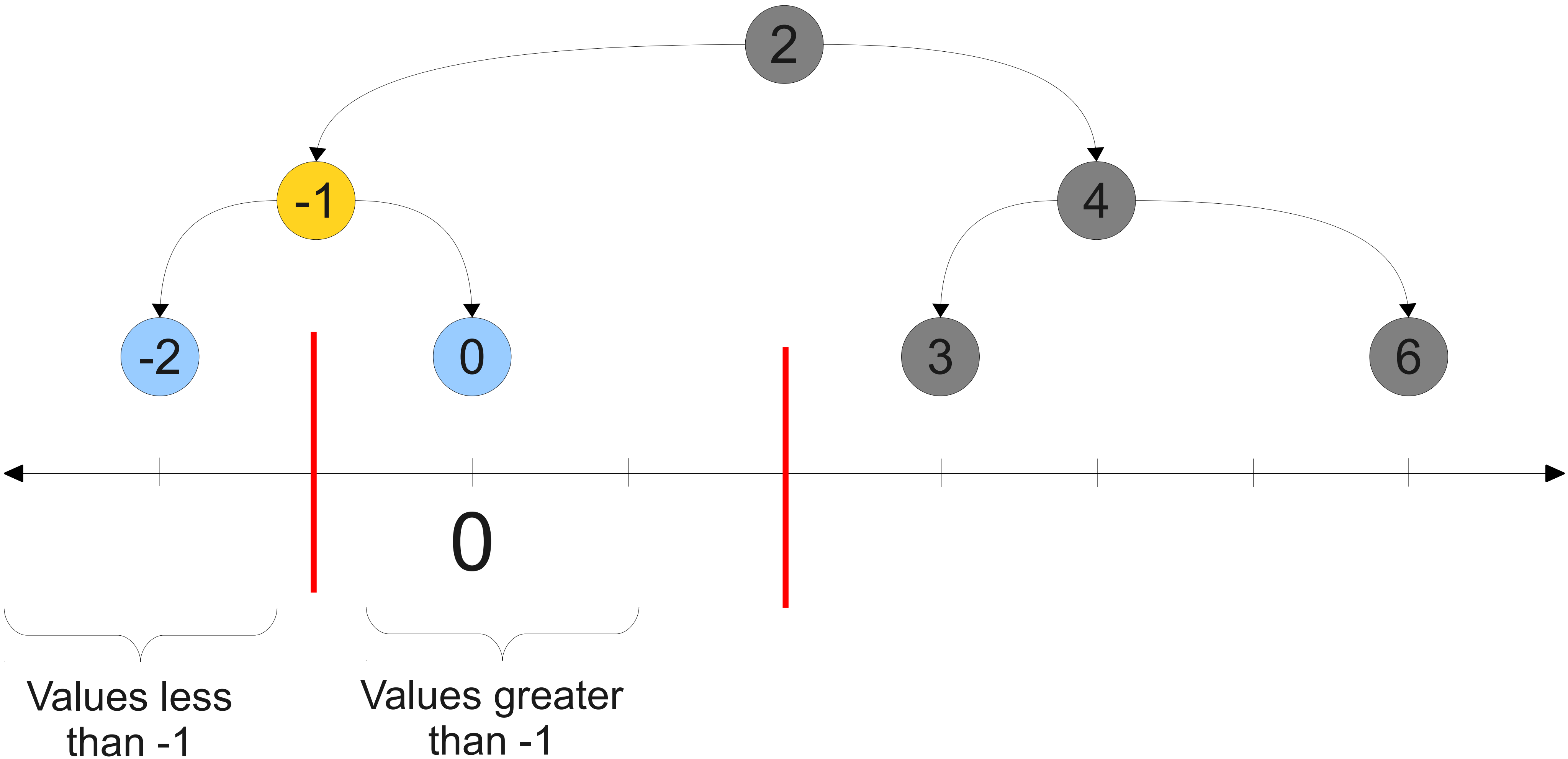

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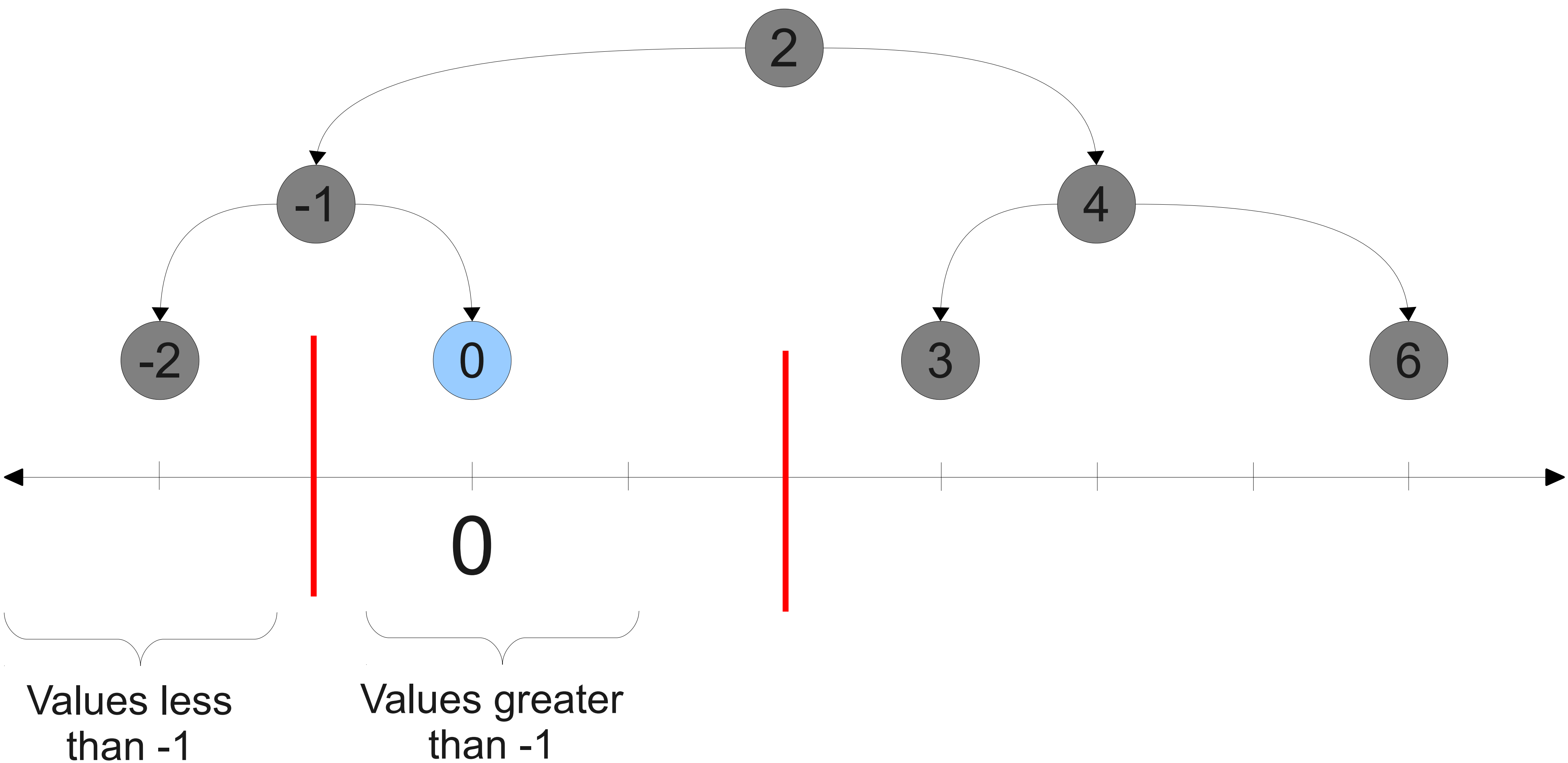

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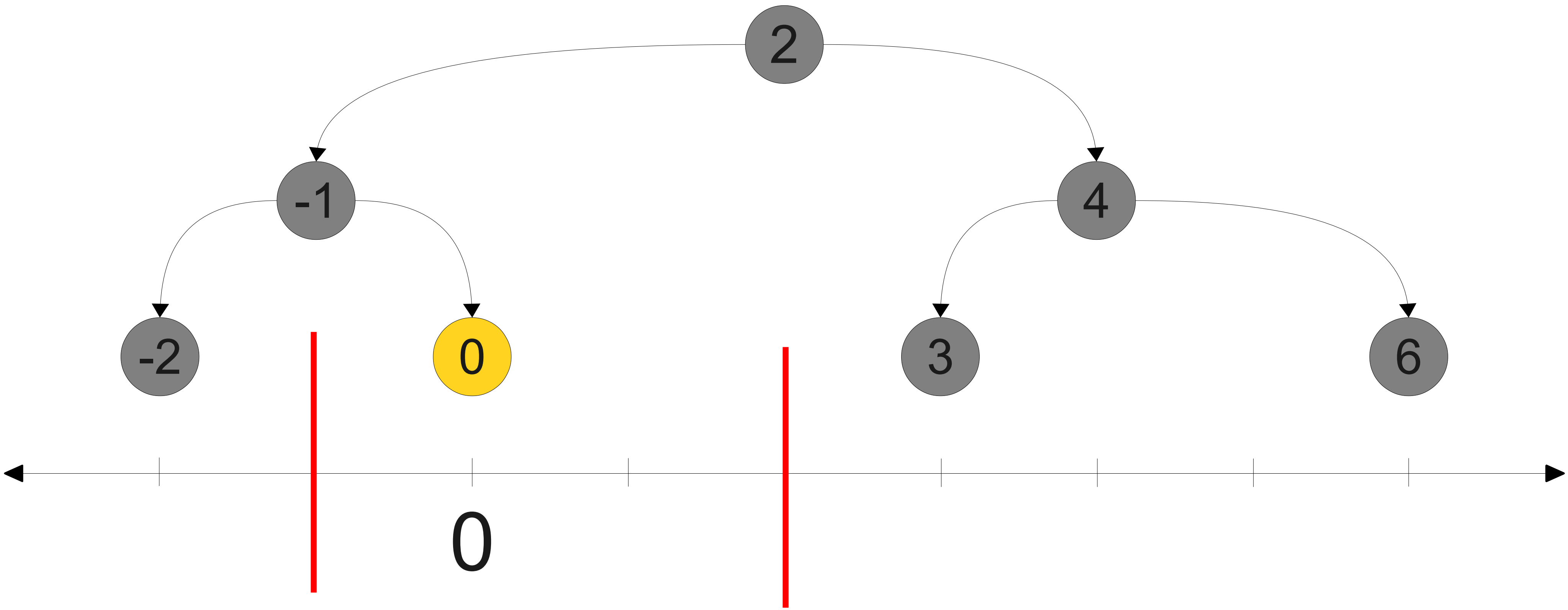
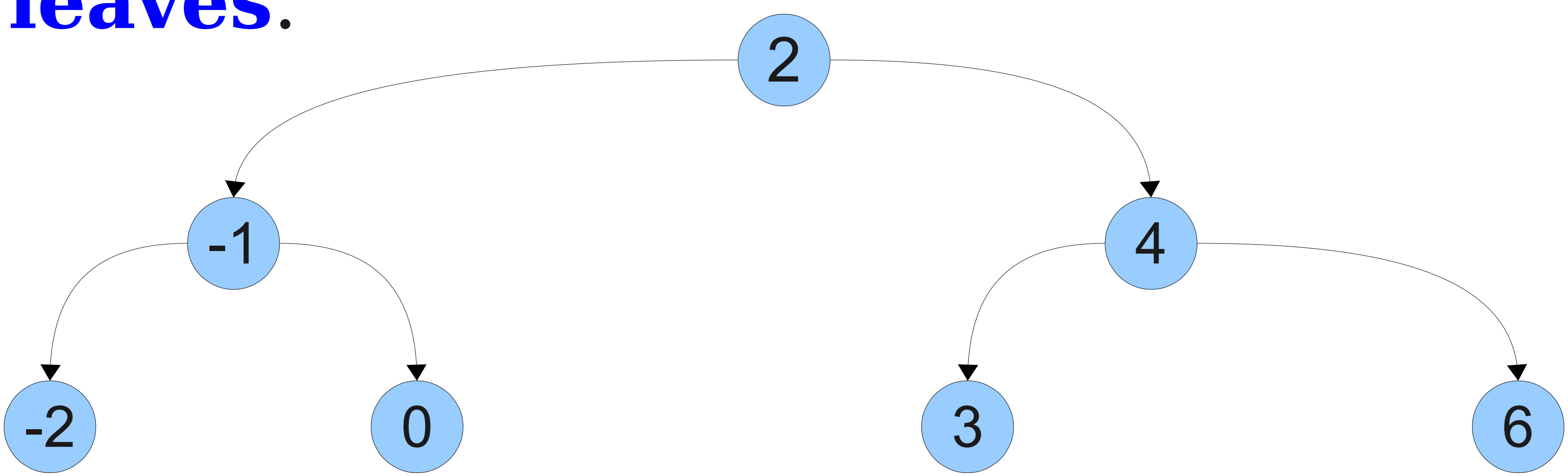
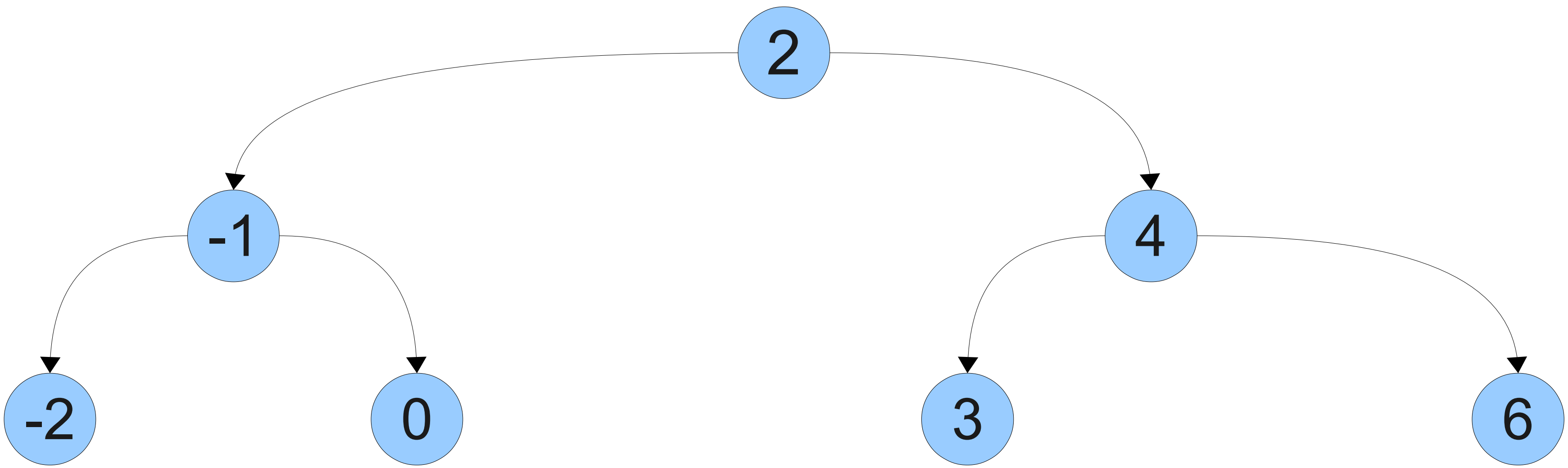


Tree Terminology

- As with a trie, a BST is a collection of **nodes**.
- The top node is called the **root node**.
- Nodes with no children are called **leaves**.



A Recursive View of BSTs



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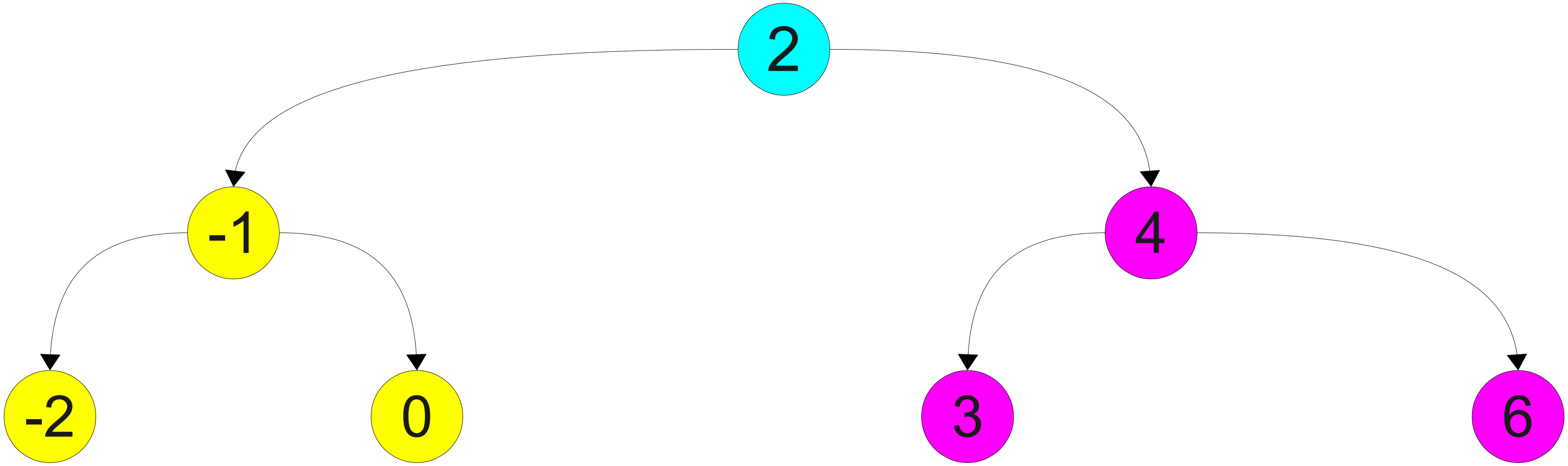

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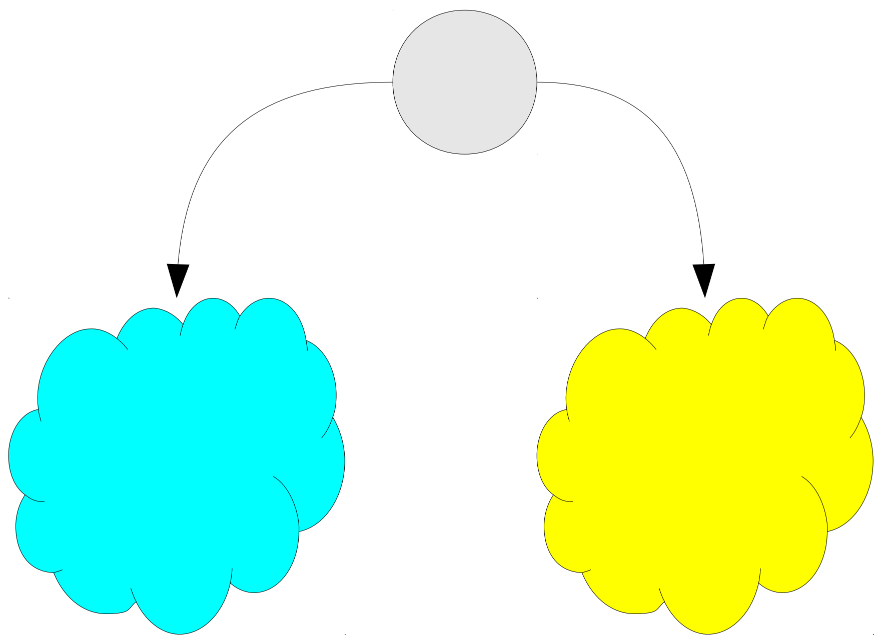
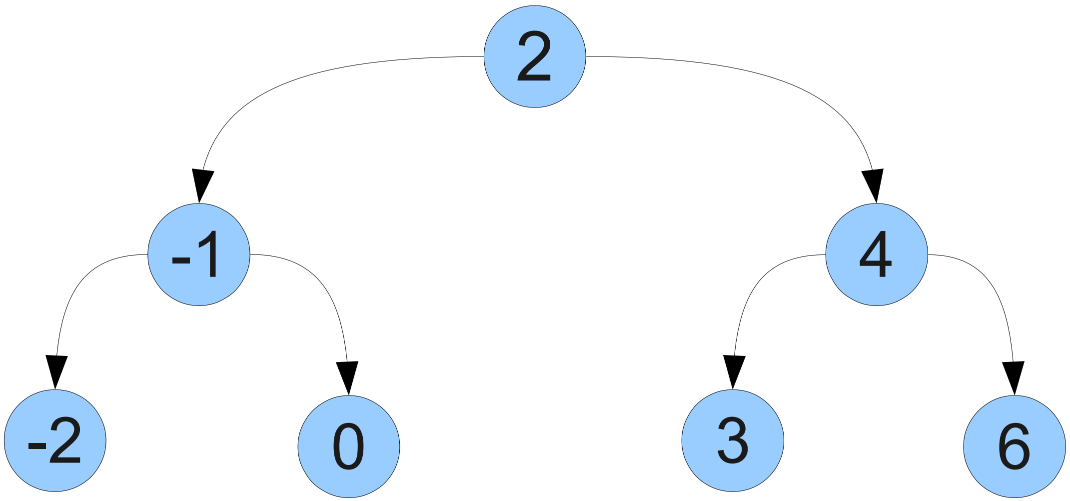


Implementing Lookups

Inserting into a BST



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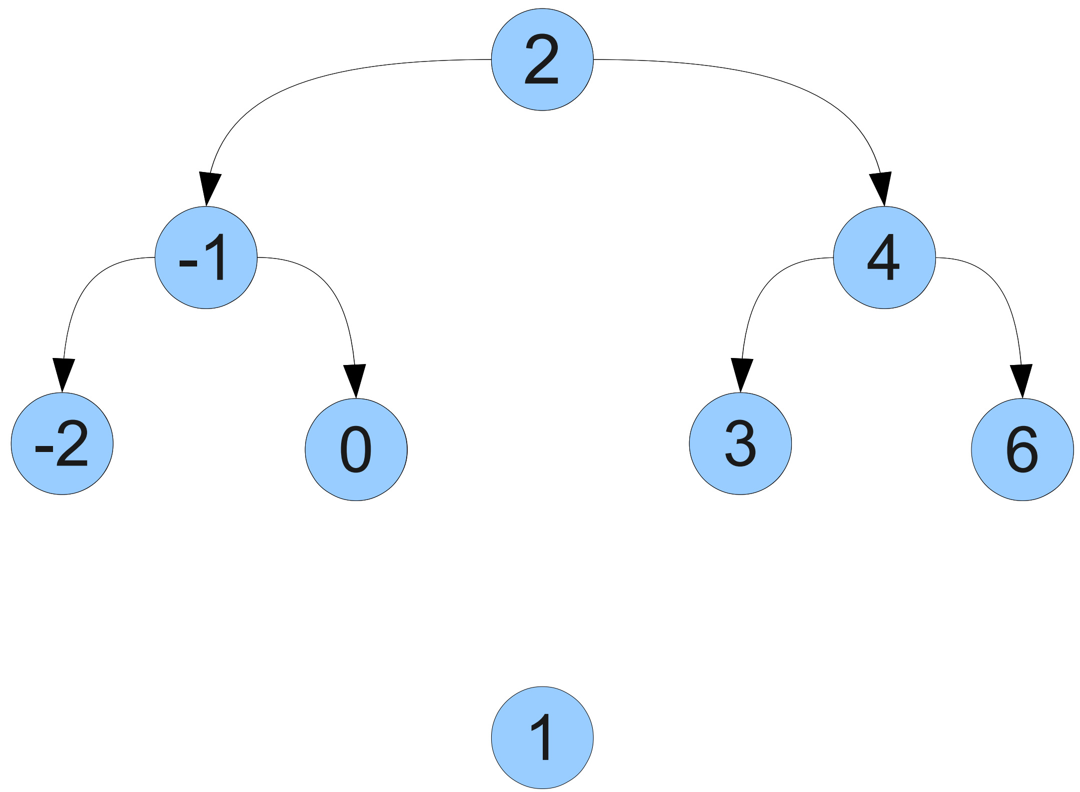

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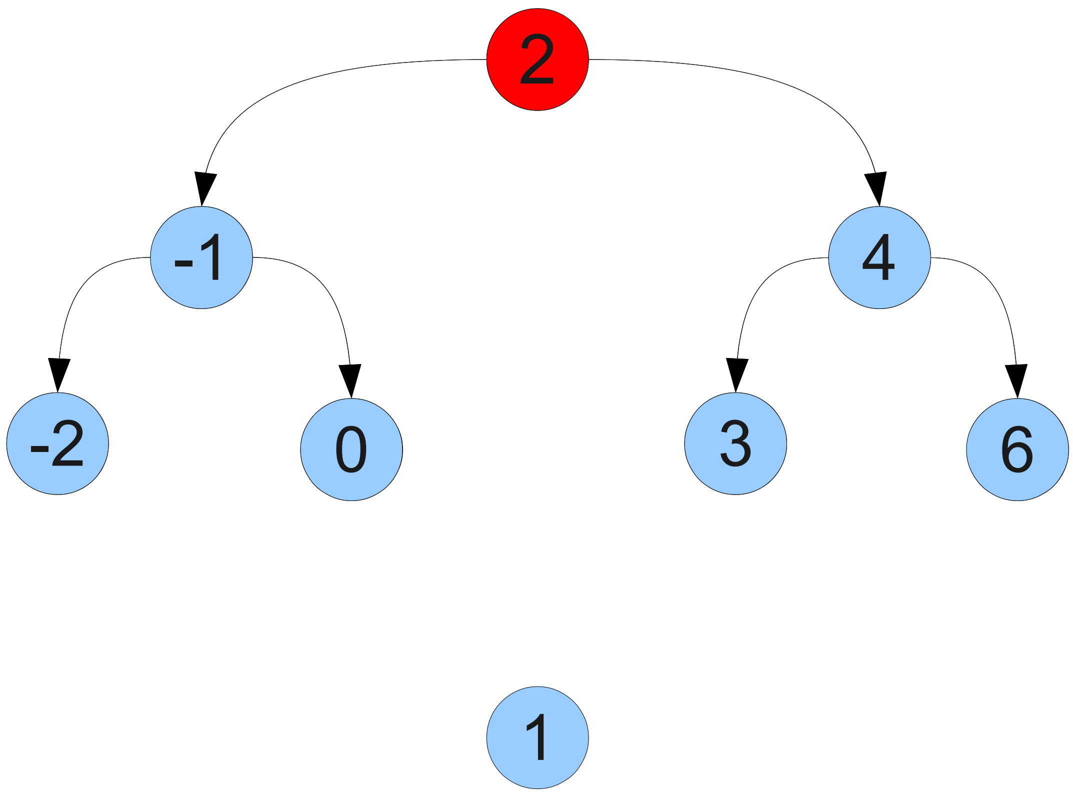

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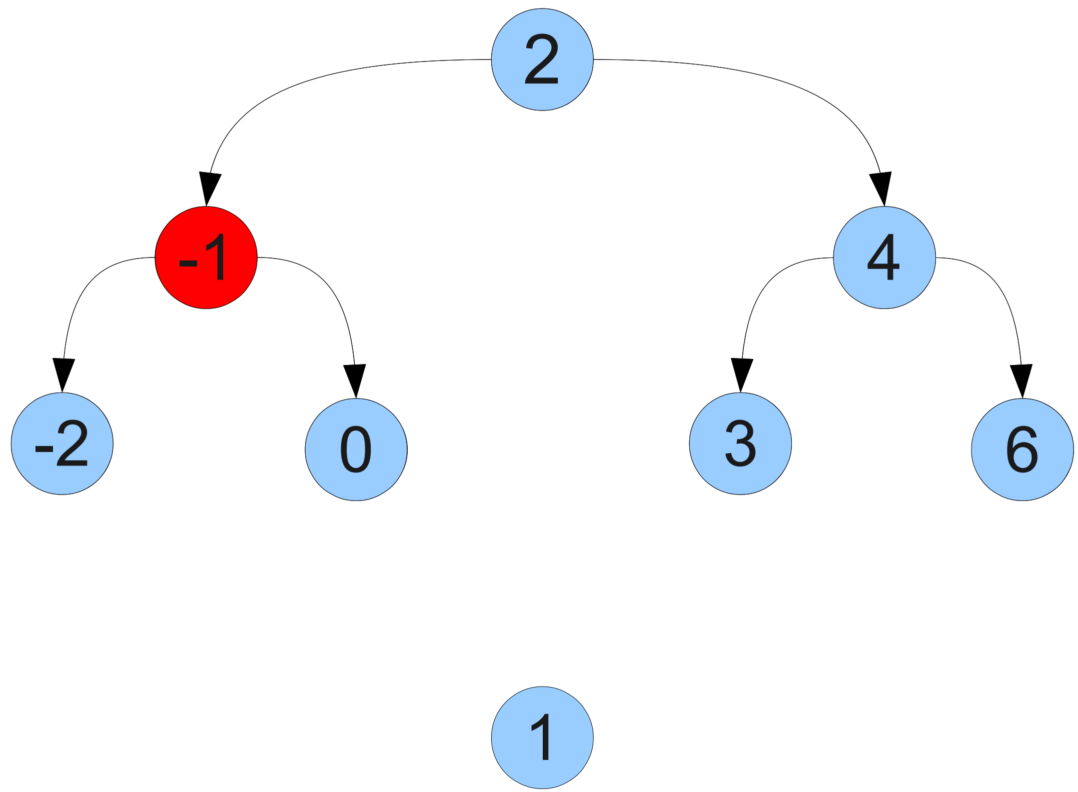

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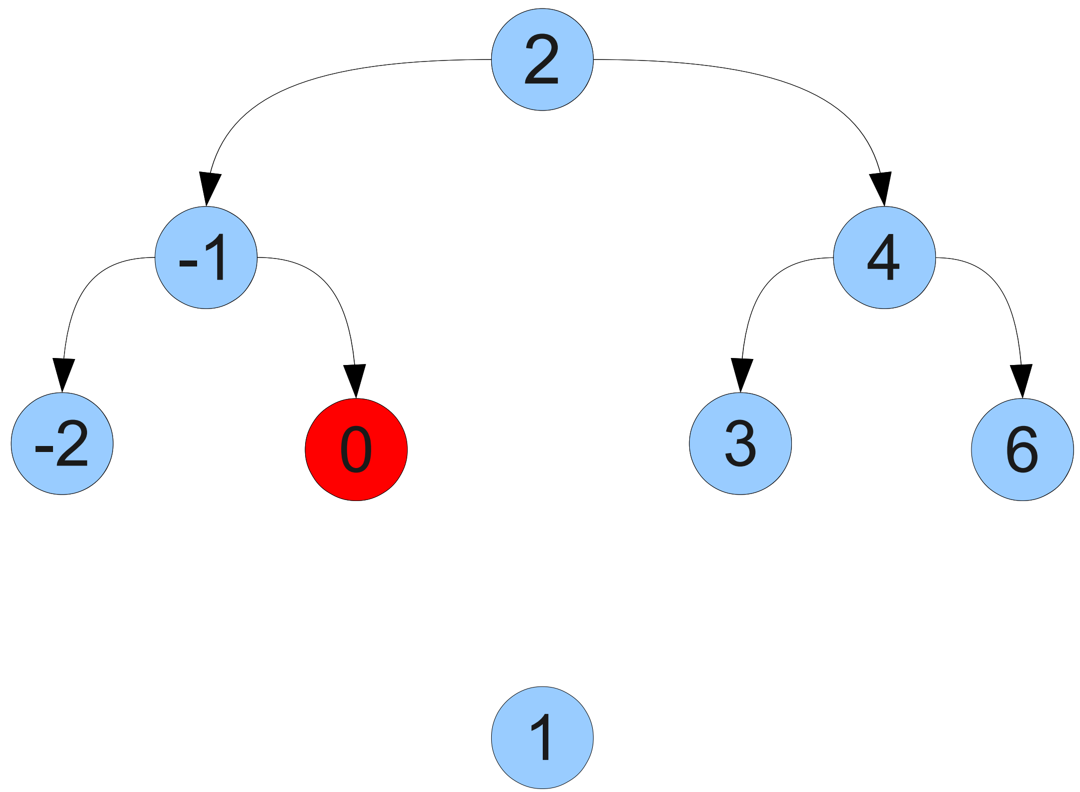

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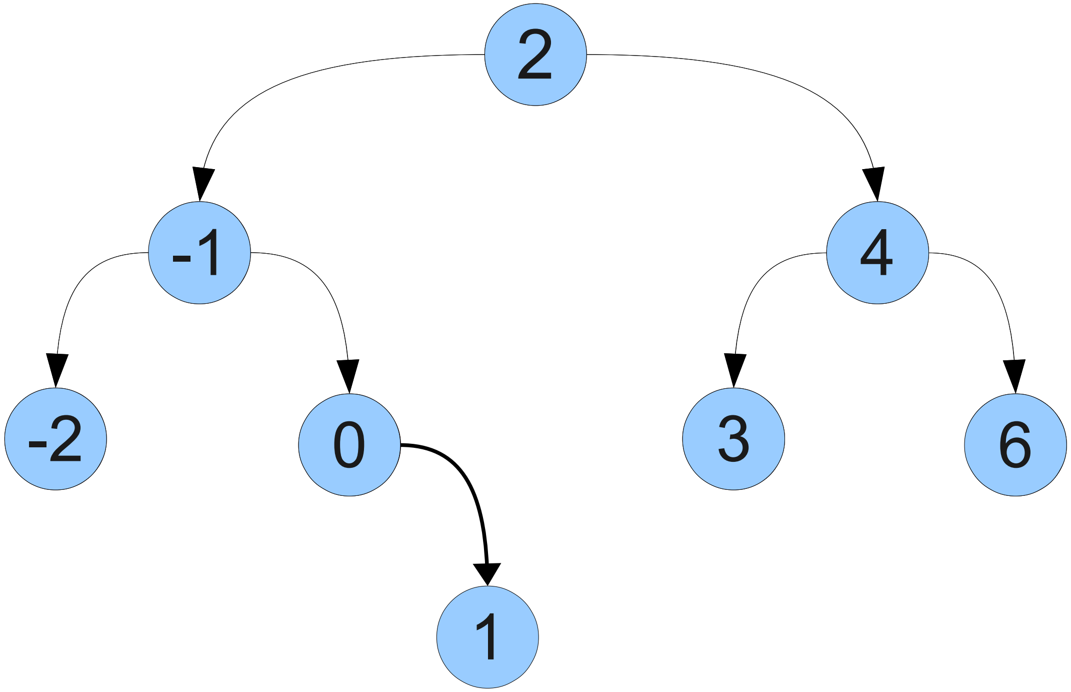
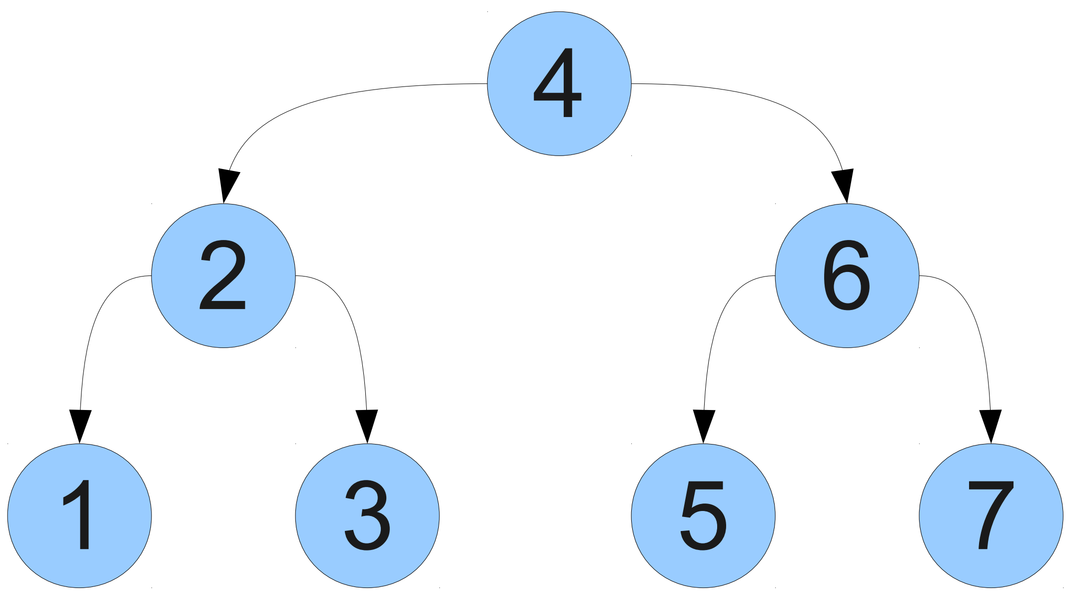


Let's Code it Up!

Insertion Order Matters

- Suppose we create a BST of numbers in this order:

4, 2, 1, 3, 6, 5, 7



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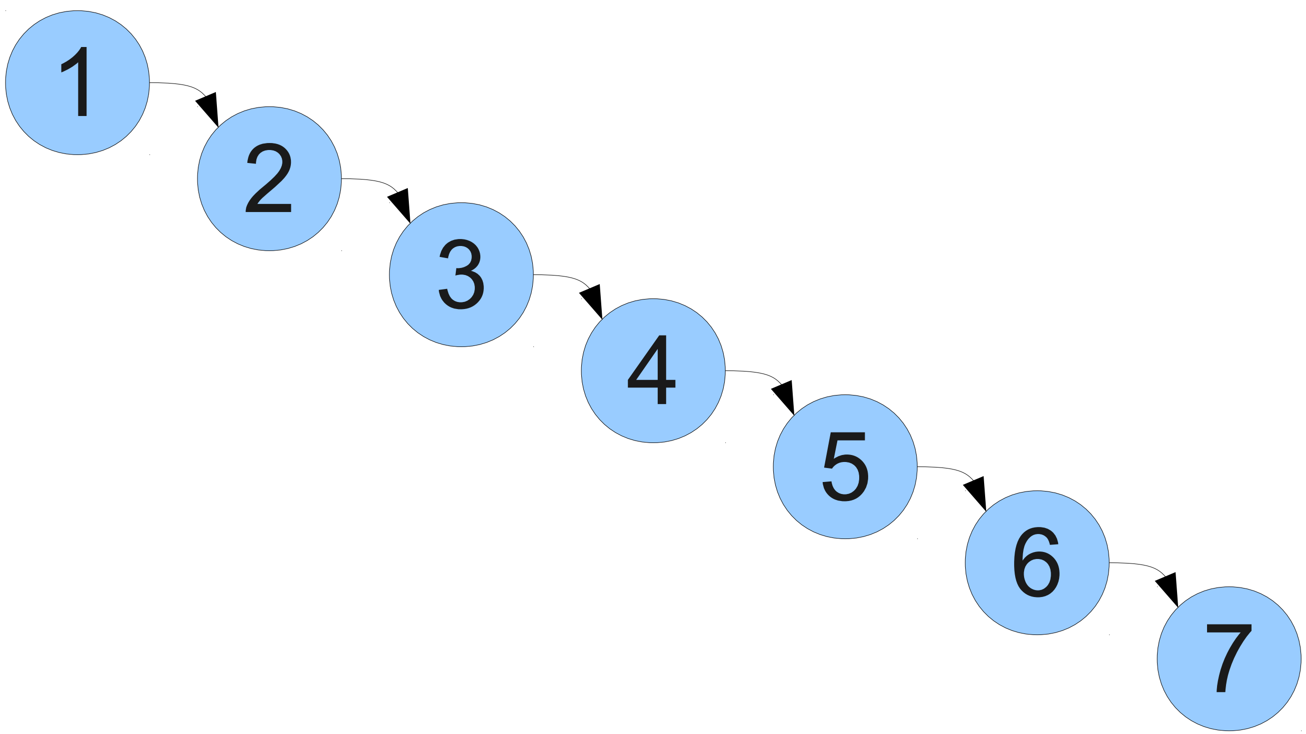
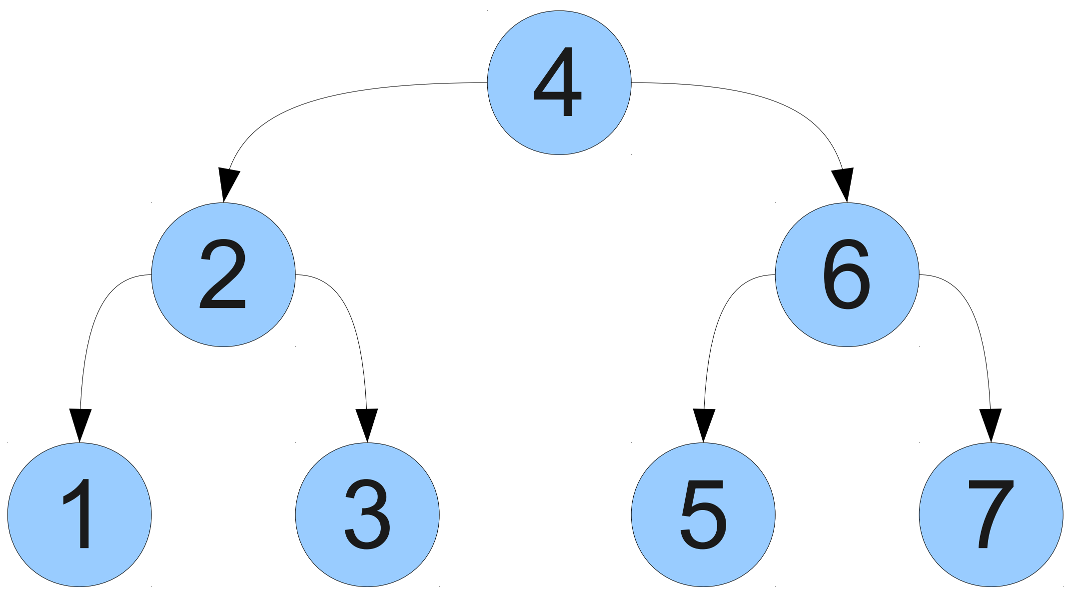


Tree Terminology

- The **height** of a tree is the number of nodes in the longest path from the root to a leaf.



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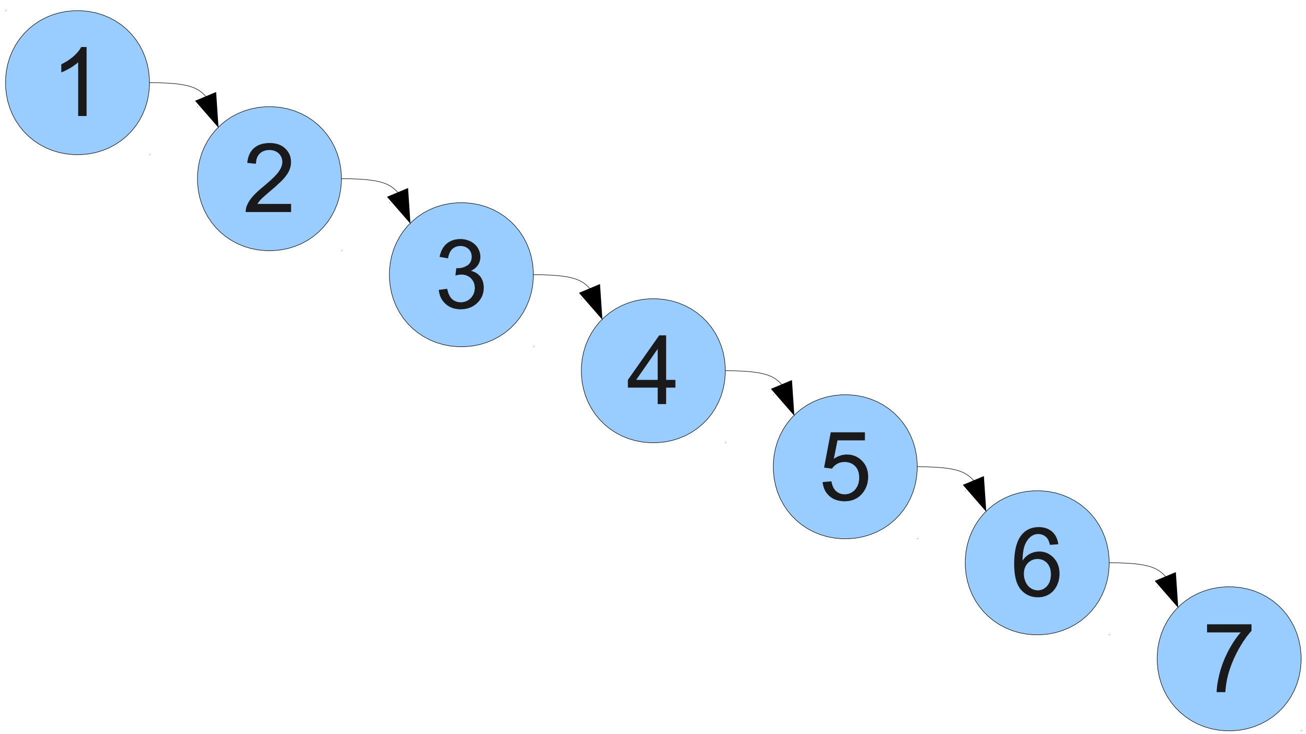


Efficiency of Insertion

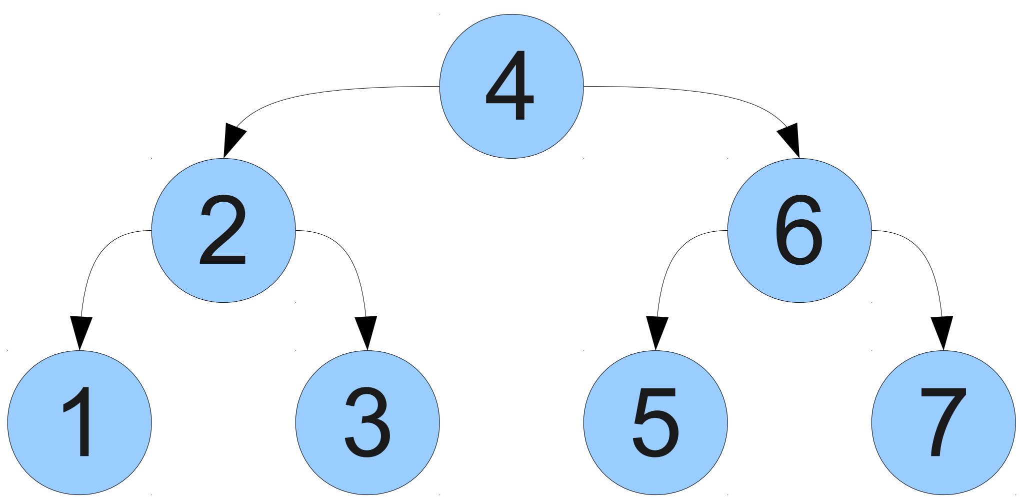
- What is the big-O complexity of adding a node to a tree?
- Depends on the height of a tree!
- Worst-case: have to take the longest path down to find where the node goes.
- Time is $O(h)$, where h is the height of the tree.

Tree Heights

- What are the maximum and minimum heights of a tree with n nodes?
- Maximum height: all nodes in a chain. Height is $O(n)$.

Tree Heights

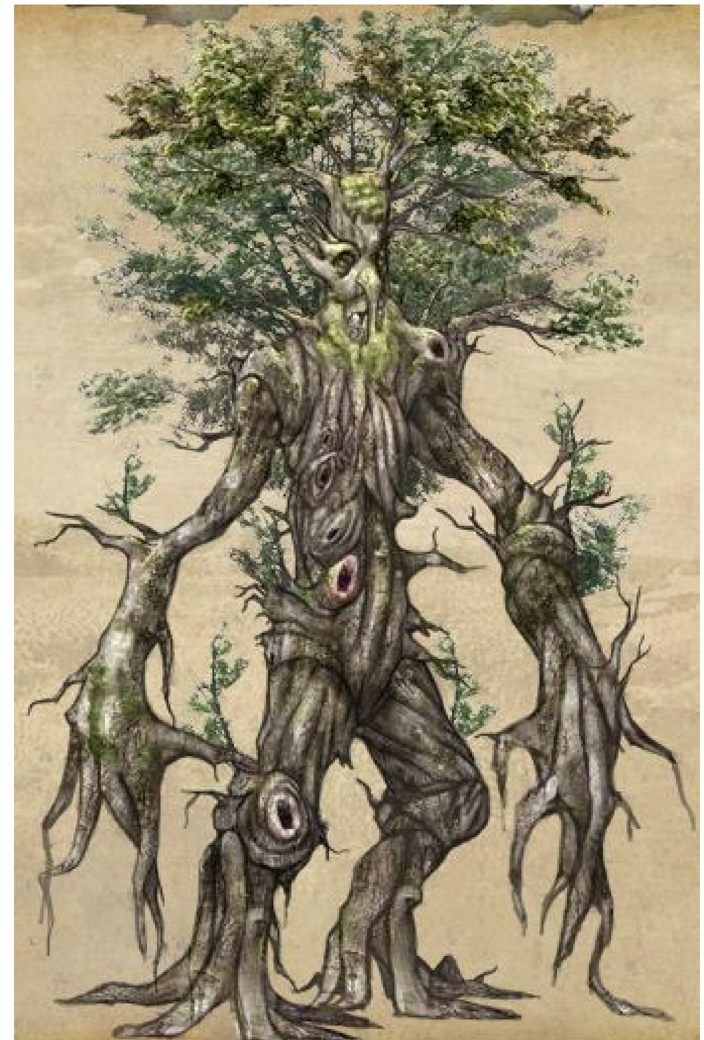
- What are the maximum and minimum heights of a tree with *n* nodes?
- Maximum height: all nodes in a chain. Height is O(*n*).
- Minimum height: Tree is as complete as possible. Height is O(log *n*).



Keeping the Height Low

- There are many modifications of the binary search tree designed to keep the height of the tree low (usually O(log n)).
- A **self-balancing binary search tree** is a binary search tree that automatically adjusts itself to keep the height low.
- Details next time.

Walking Trees



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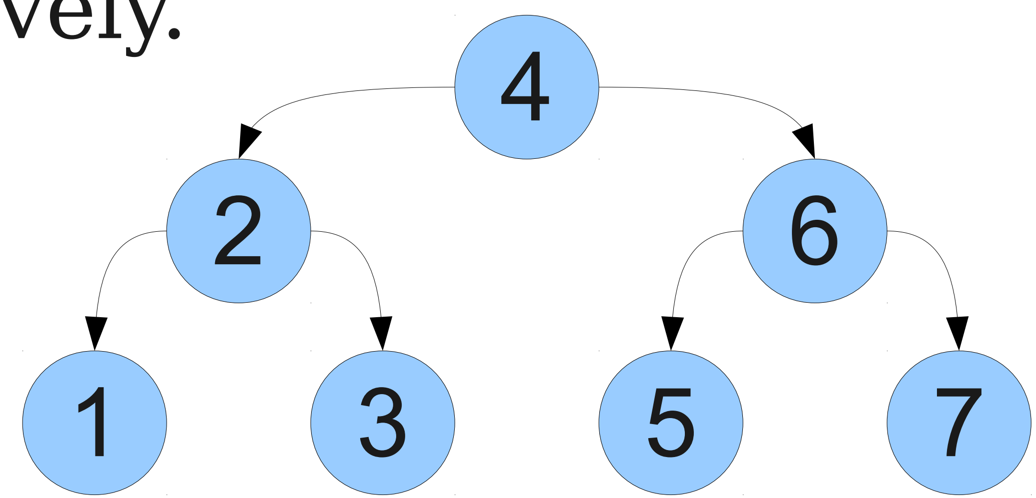
Walking a BST

- One advantage of a BST is that elements are stored in sorted order.
- We can iterate over the elements of a BST in sorted order by walking the tree recursively.



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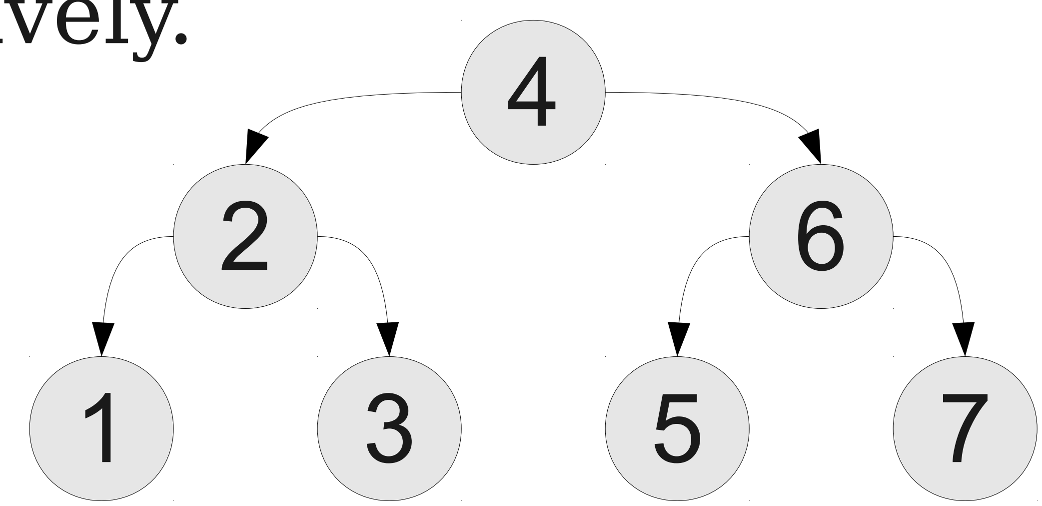

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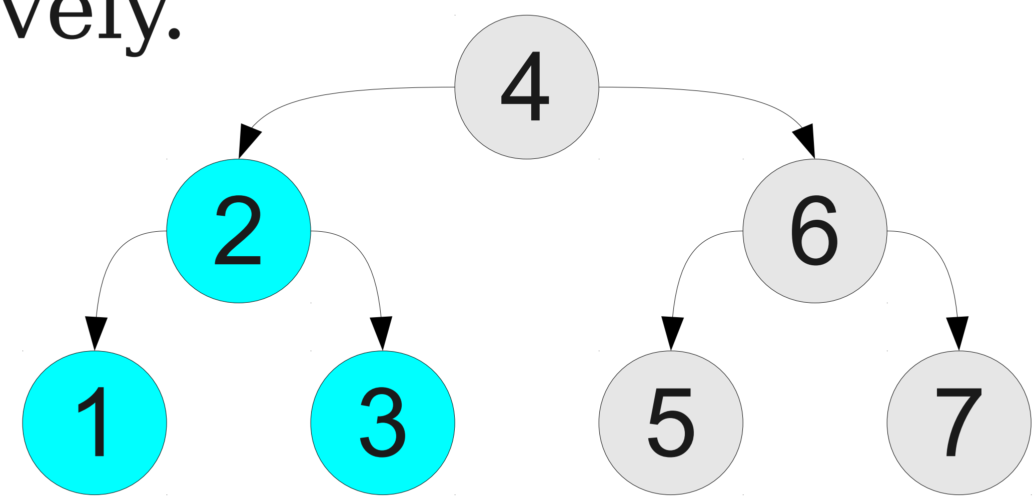

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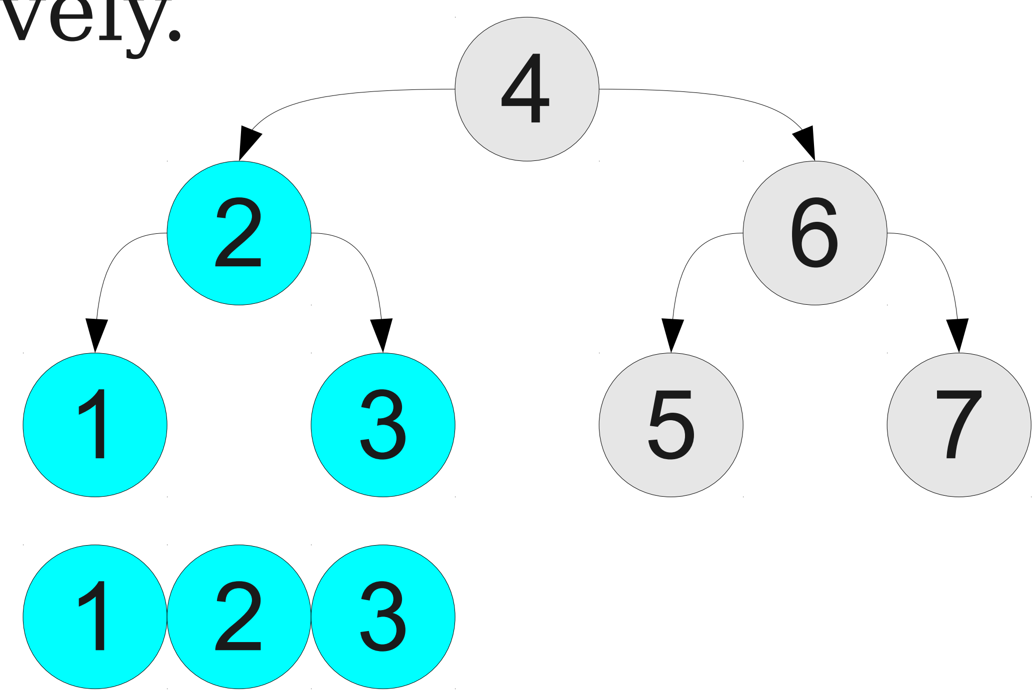

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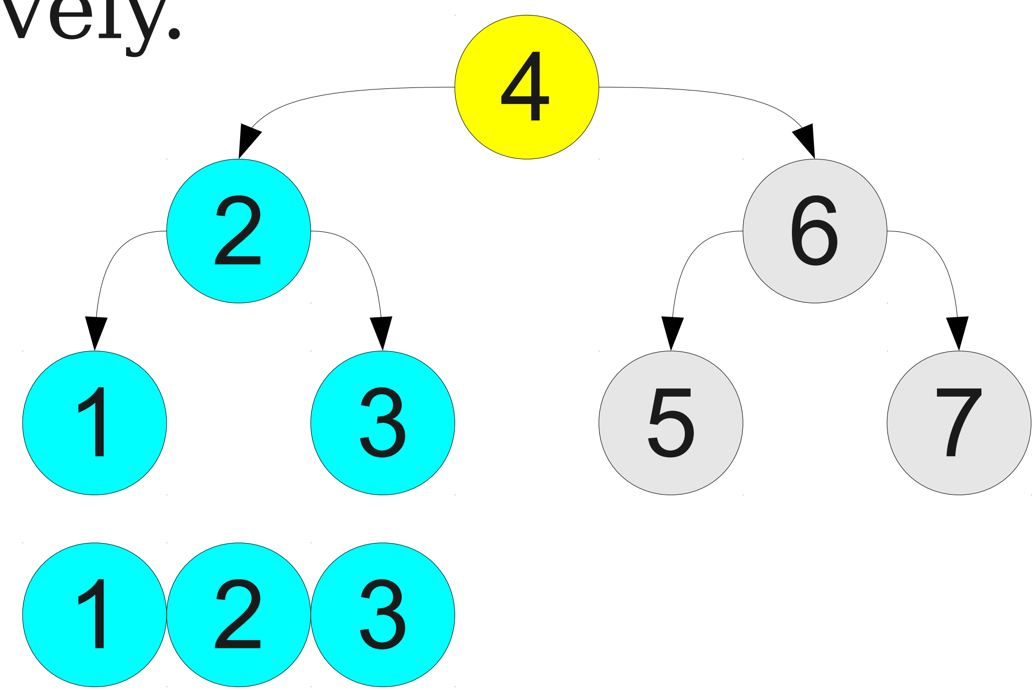

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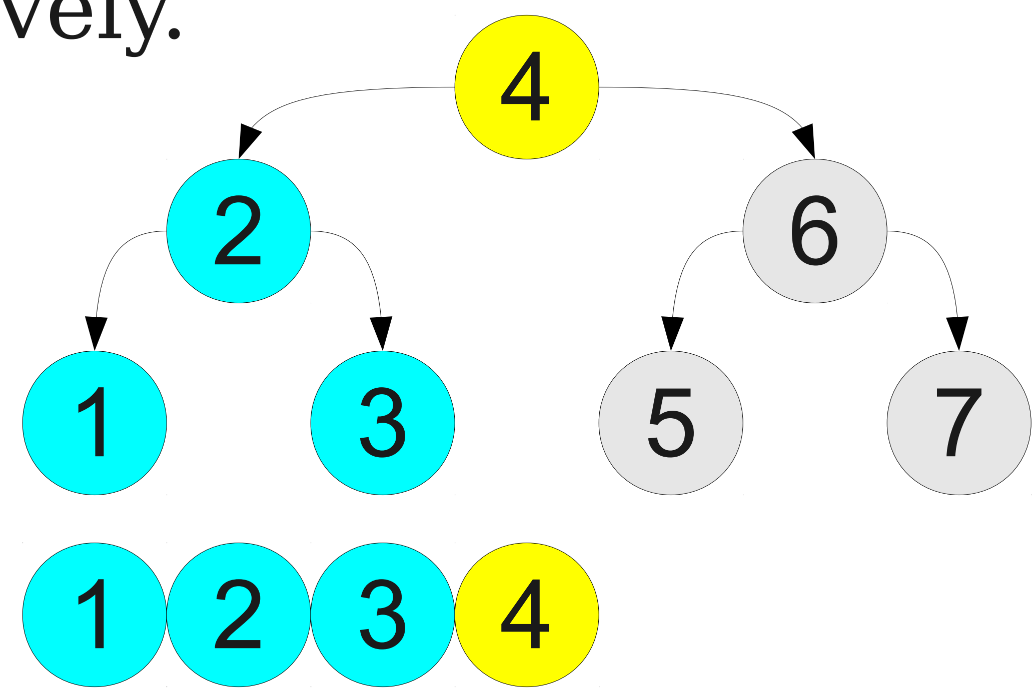

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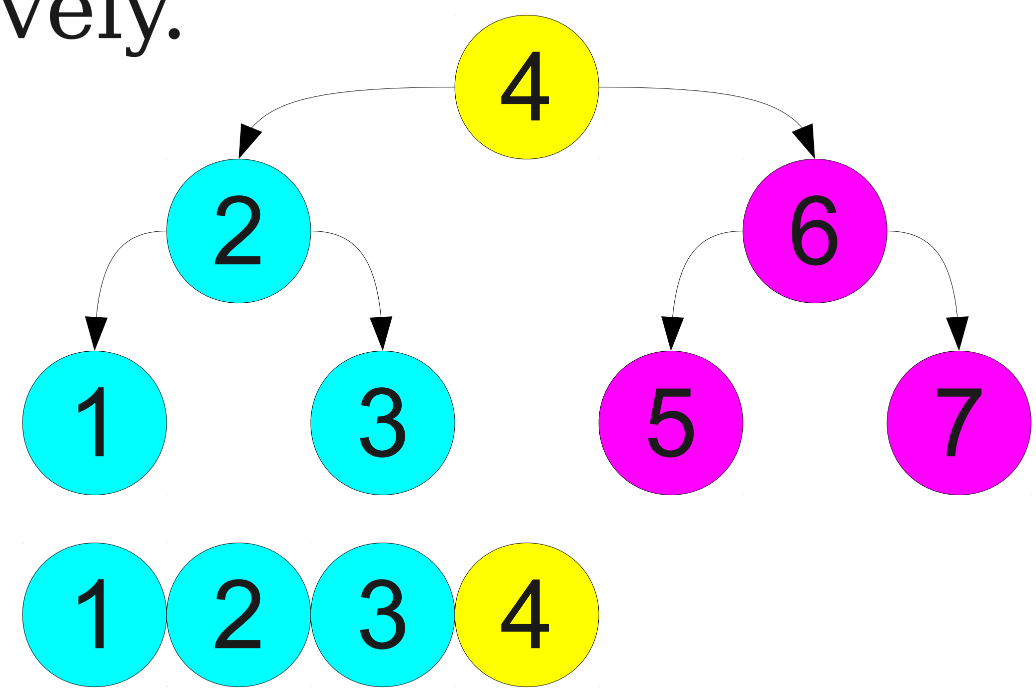

Walking a BST

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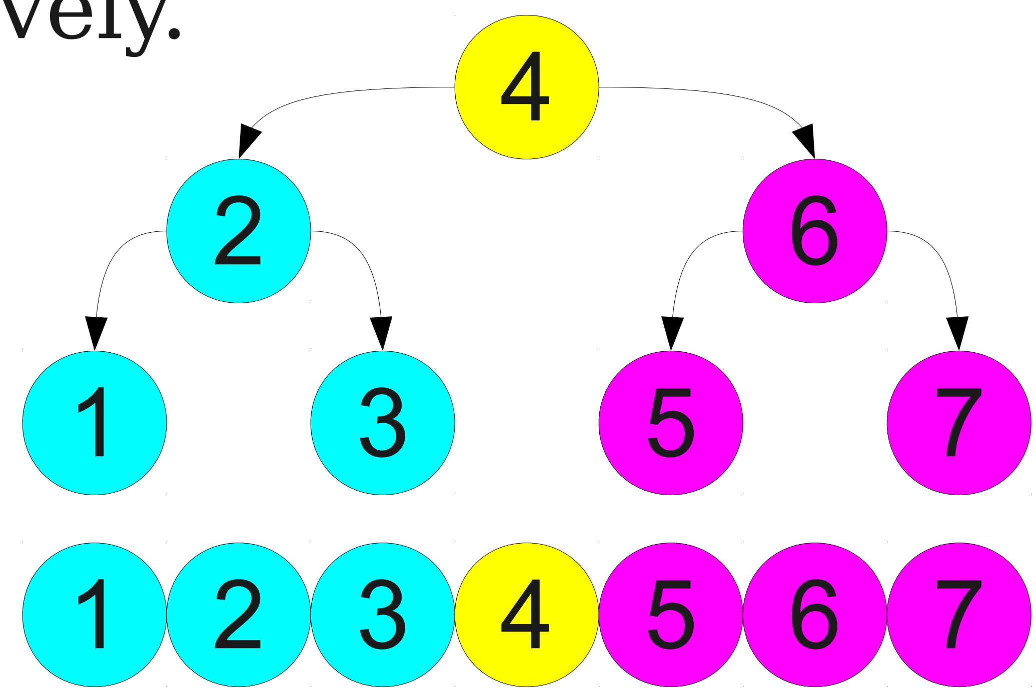
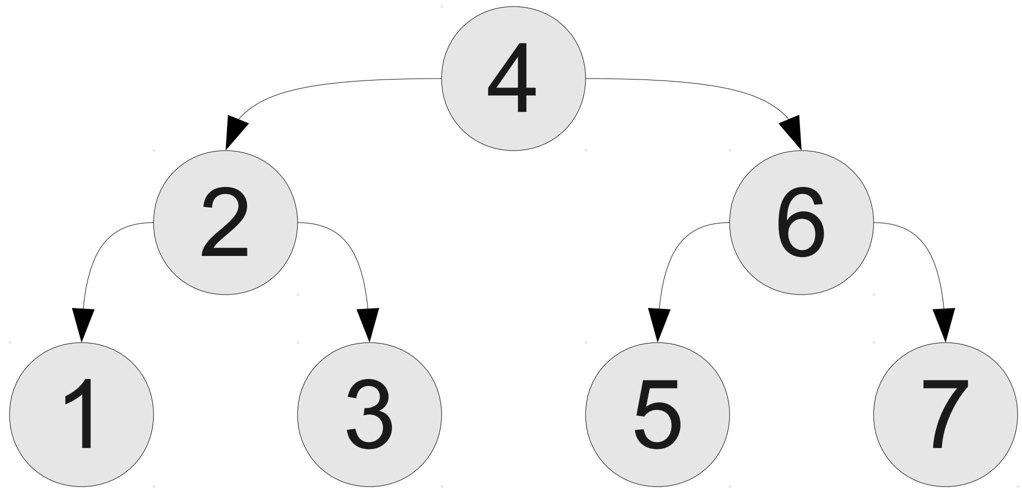


Tree Traversals

- There are three general types of tree traversals:
- **Preorder**: Visit the node, then visit the children.
- **Inorder**: Visit the left child, then the node, then the right child.
- **Postorder**: Visit the children, then visit the node.

Walking a Tree



Inorder	1	2	3	4	5	6	7
Preorder	4	2	1	3	6	5	7
Postorder	1	3	2	5	7	6	4

Getting Rid of Trees

http://www.tigersheds.com/garden-resources/image.axd?picture=2010%2F6%2Fdeforestation1.jpg

Freeing a Tree

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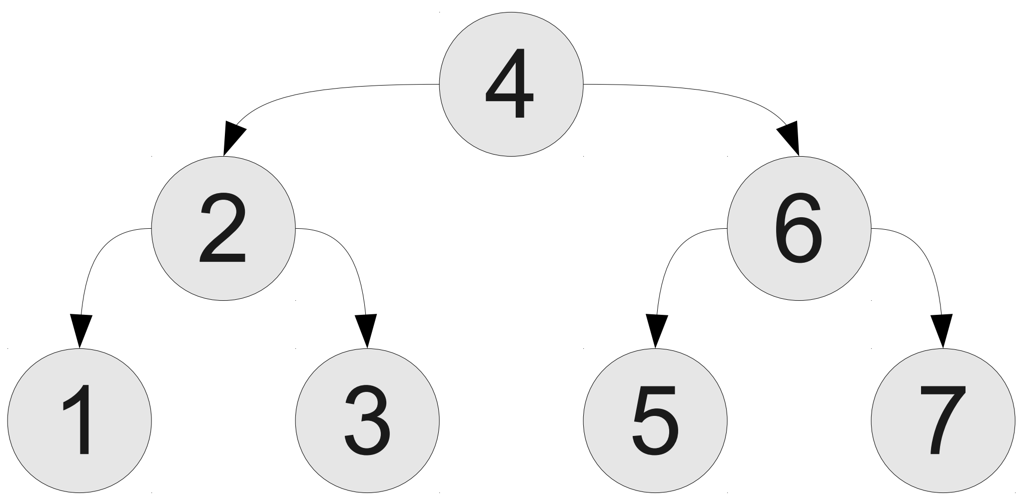

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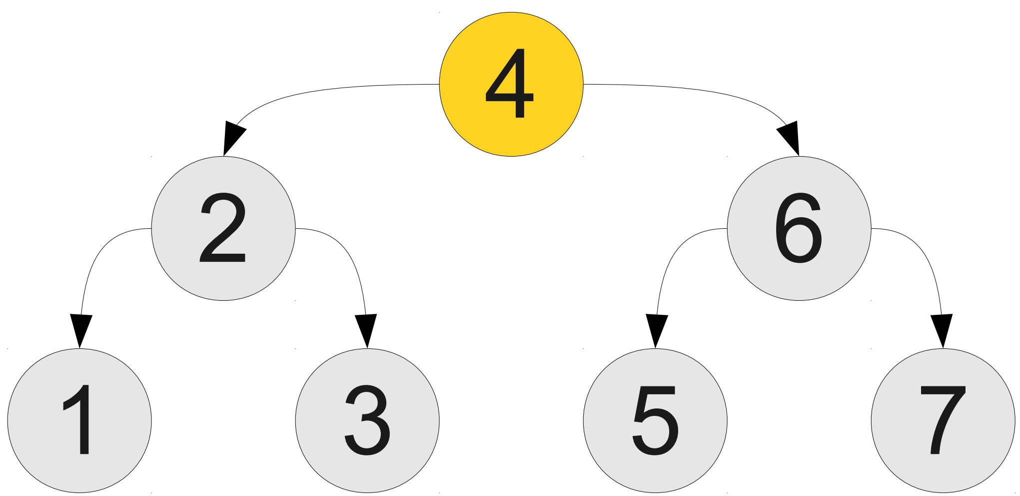

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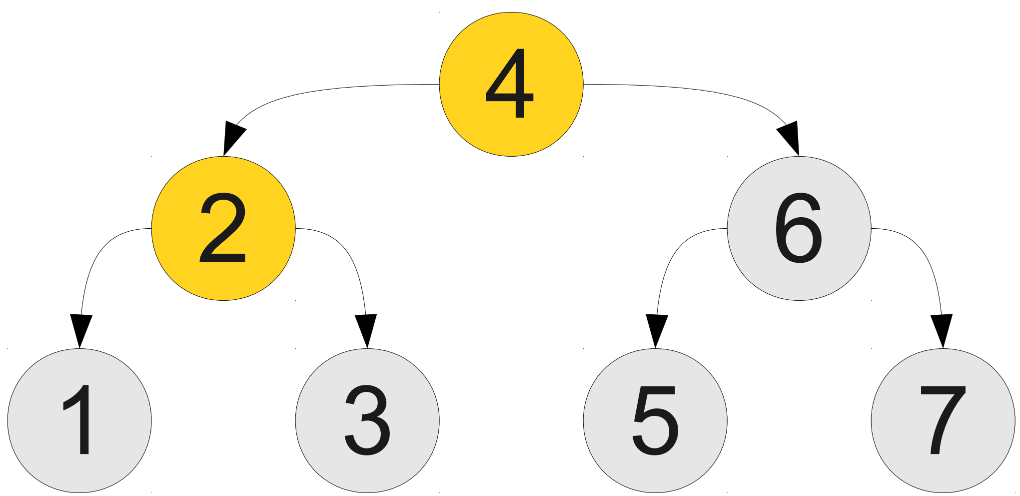

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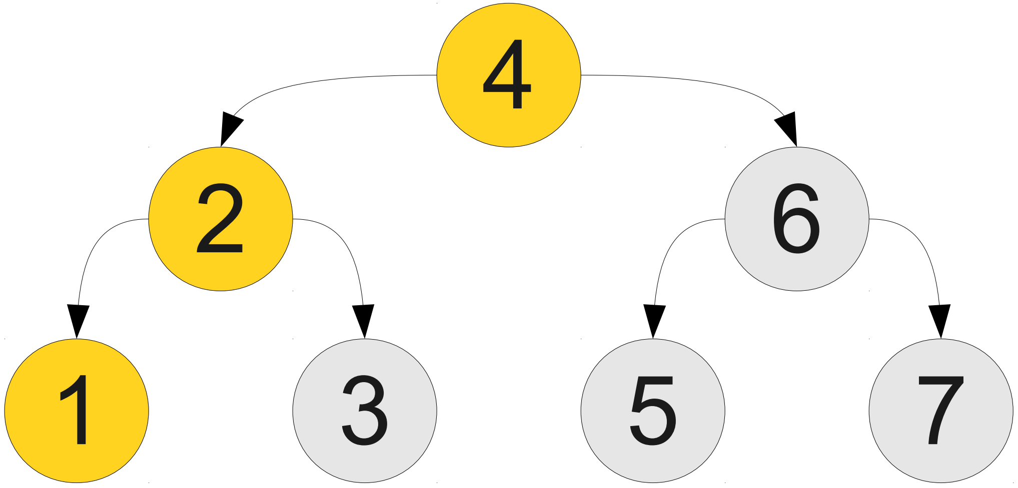

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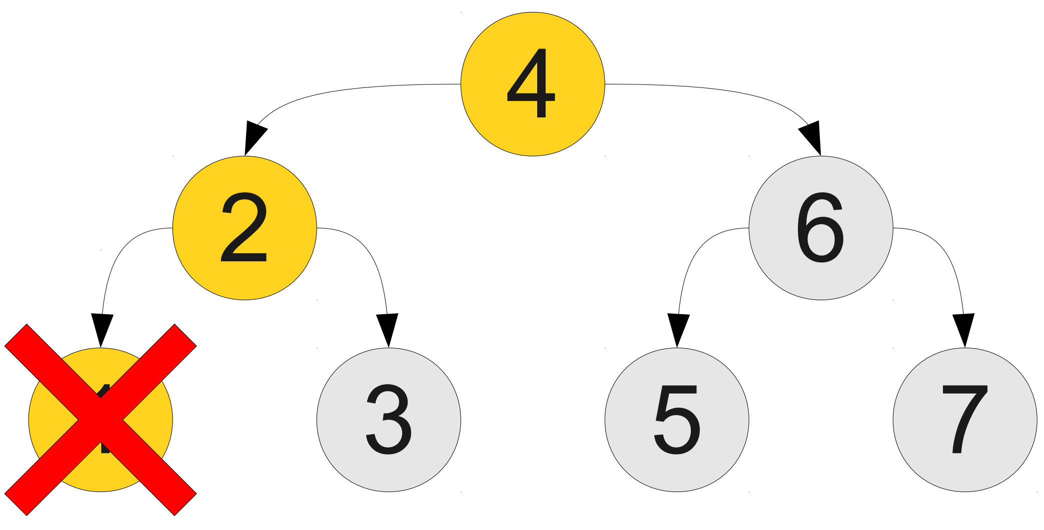

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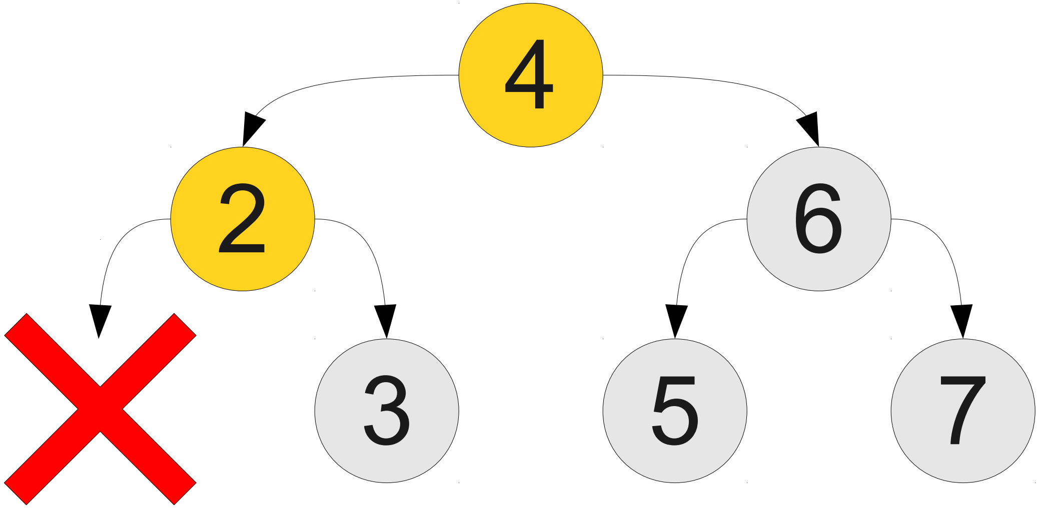

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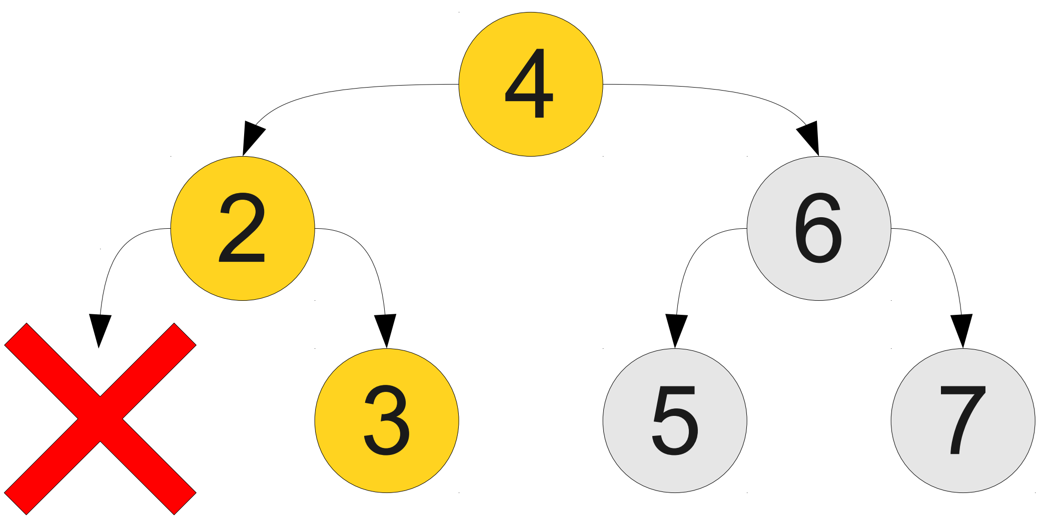

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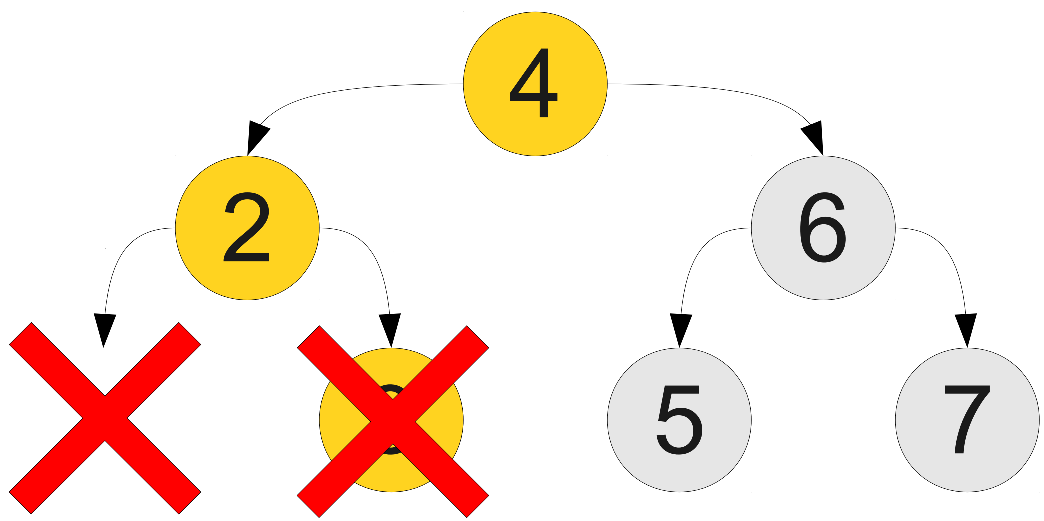

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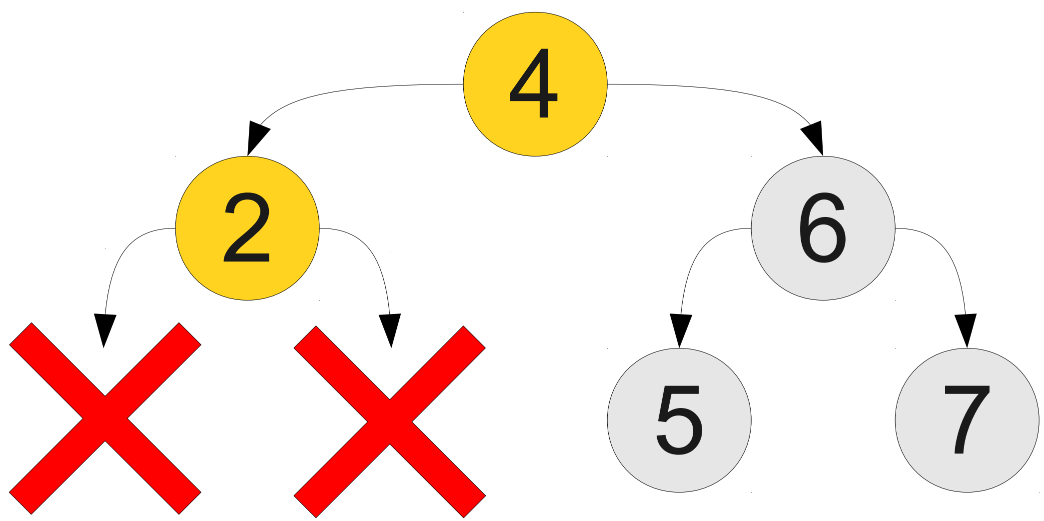

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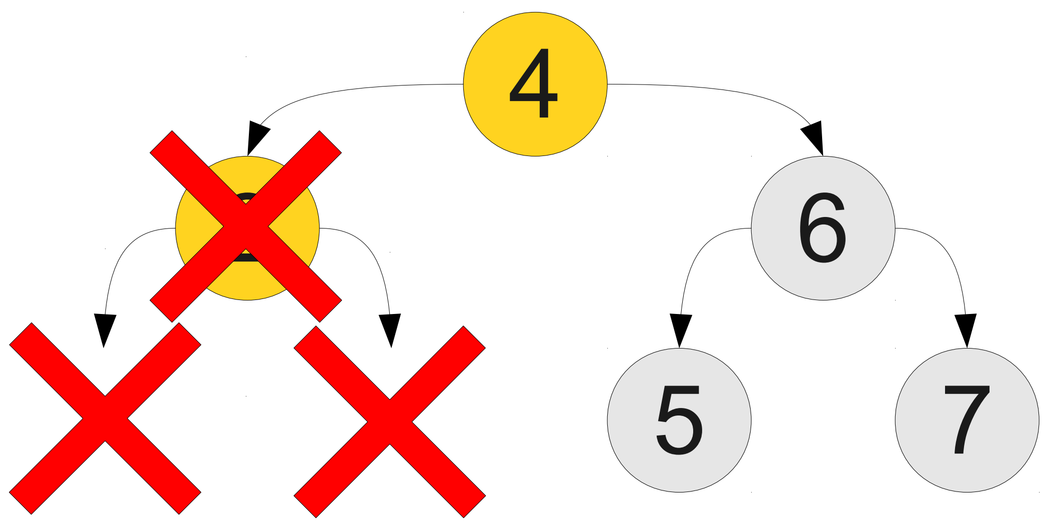

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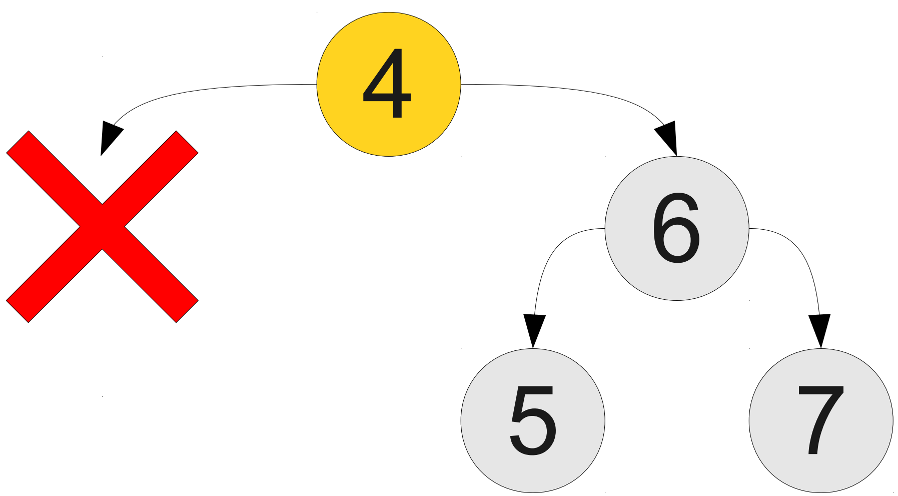

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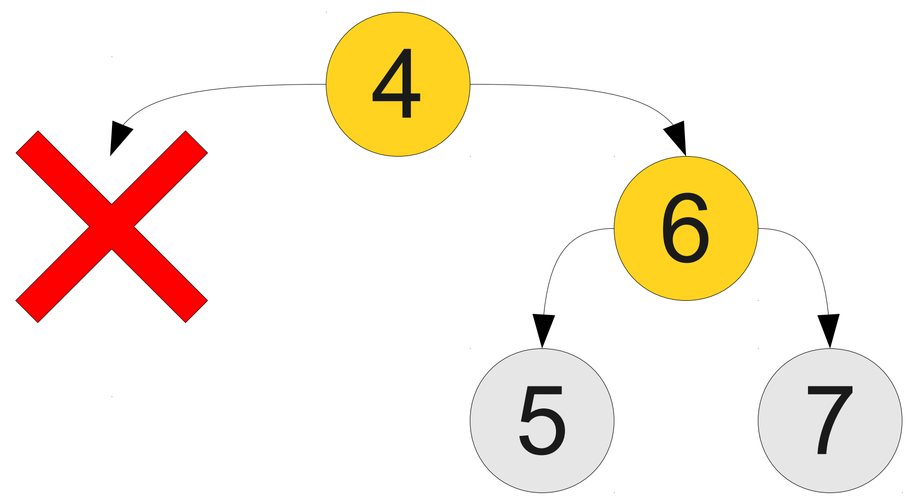

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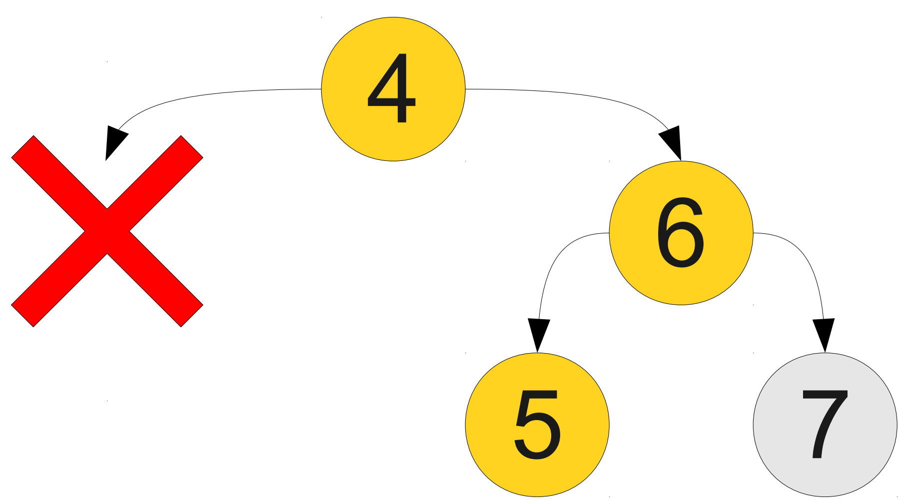

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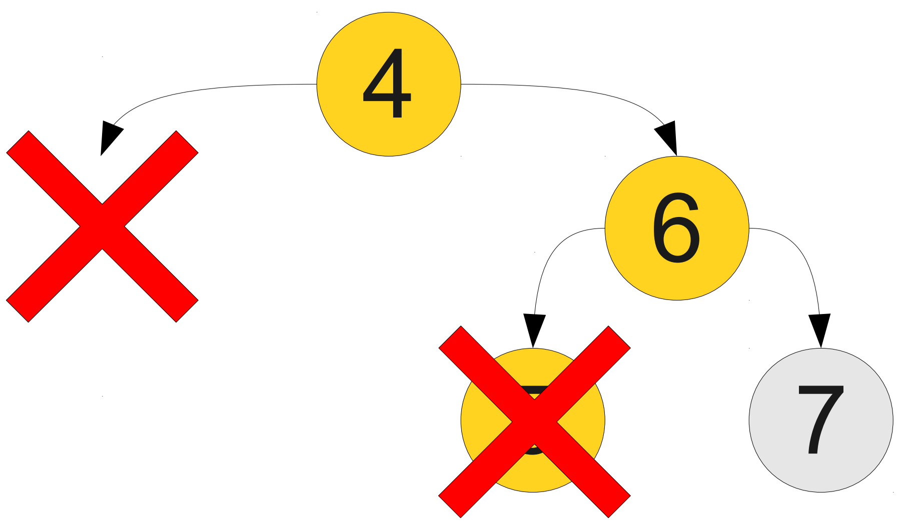

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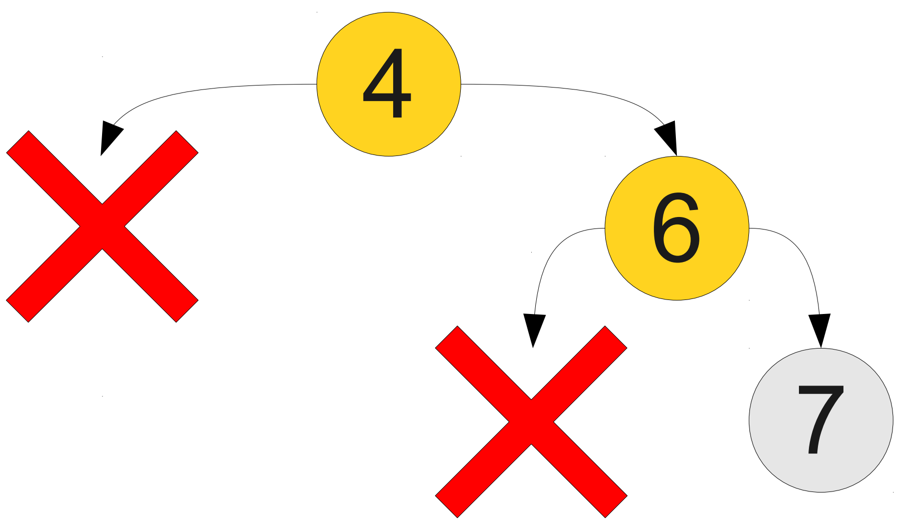

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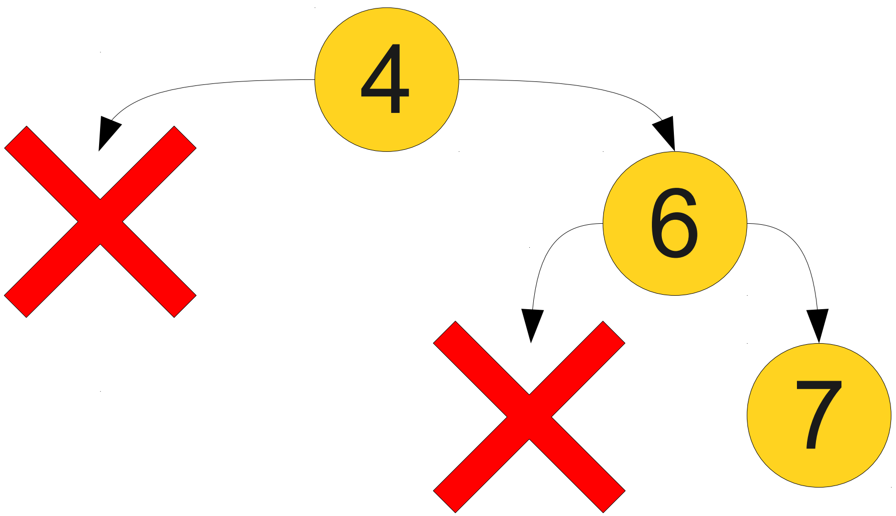

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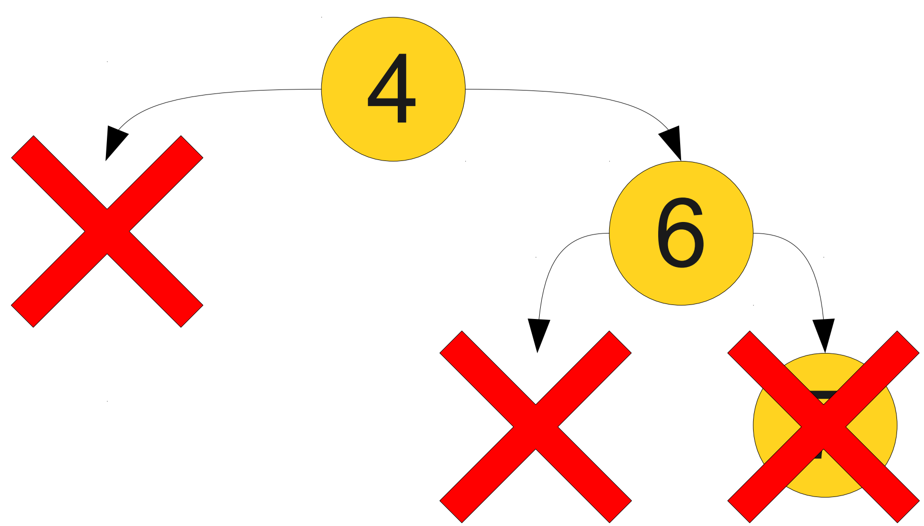

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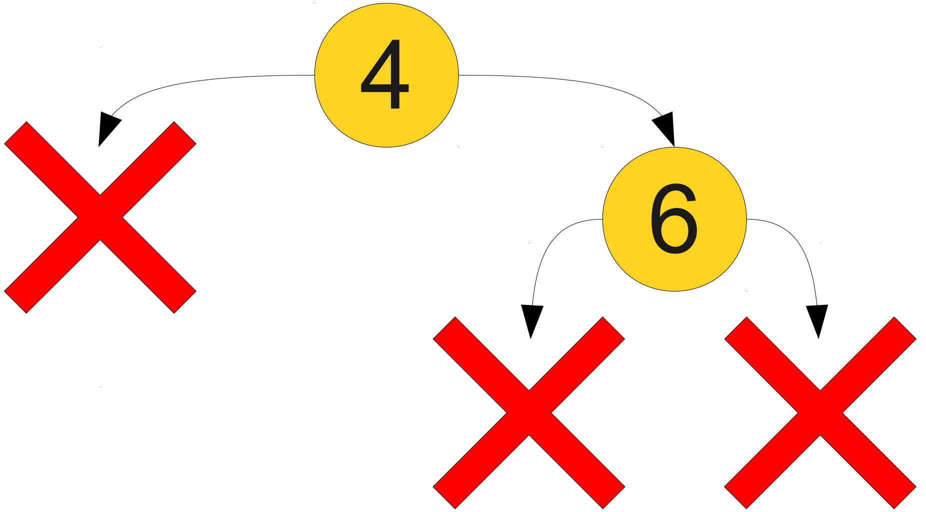

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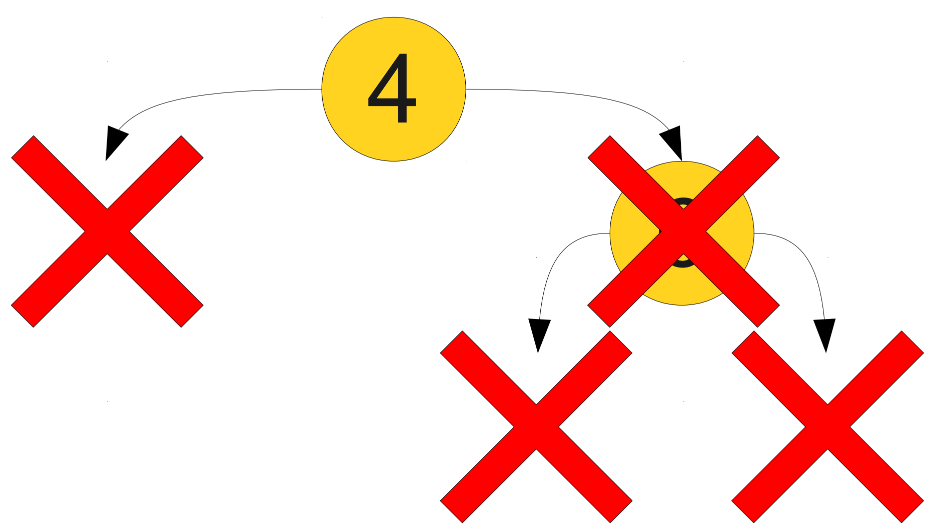

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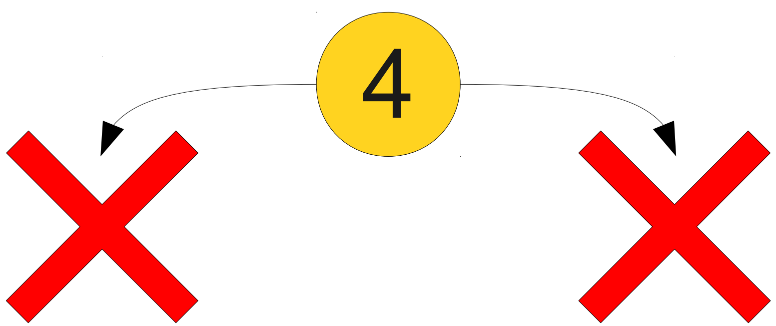

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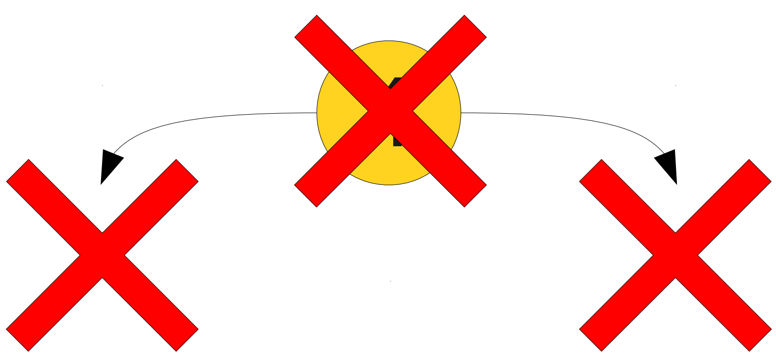

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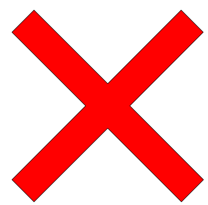

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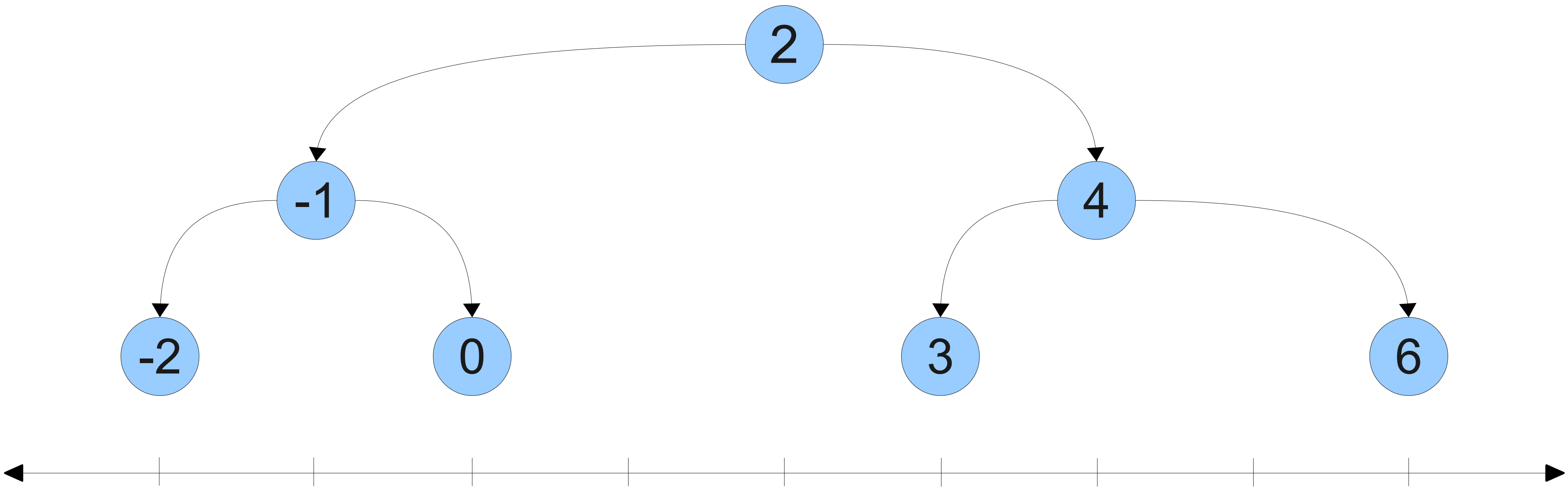
- This is done as follows:
  - **Base case**: There is nothing to delete in an empty tree.
  - **Recursive step**: Delete both subtrees, then delete the current node.
- What kind of tree traversal is this?

Range Searches

- We can use BSTs to do **range searches**, in which we find all values in the BST within some range.
- For example:
  - If values in a BST are dates, can find all events that occurred within some time window.
  - If values in a BST are samples of a random variable, can find everything within one and two standard deviations above the mean.

The Intuition



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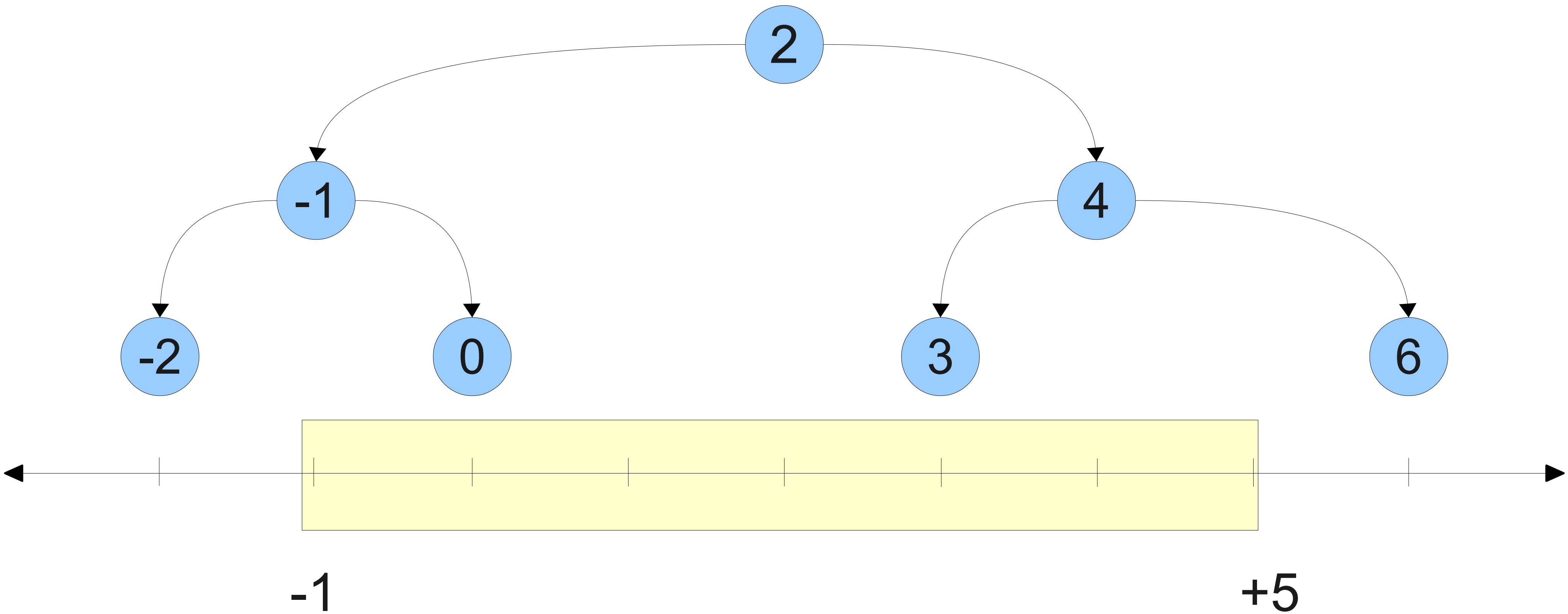

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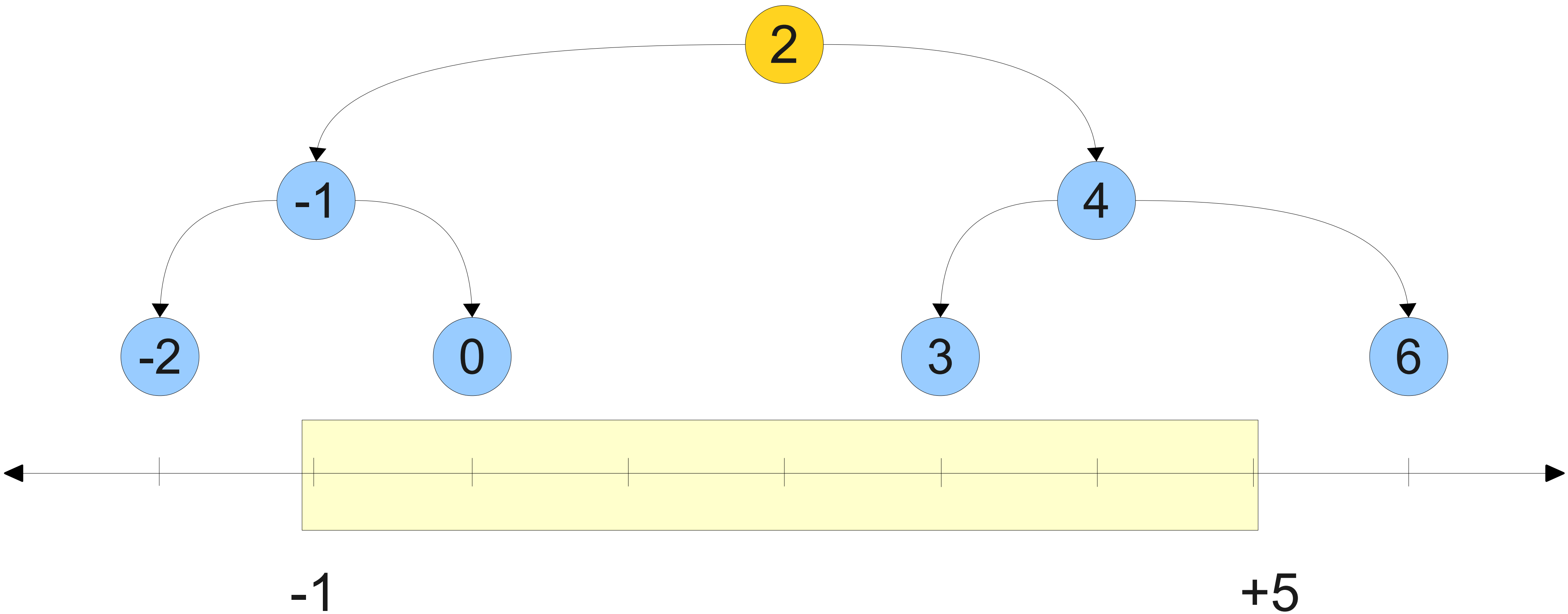

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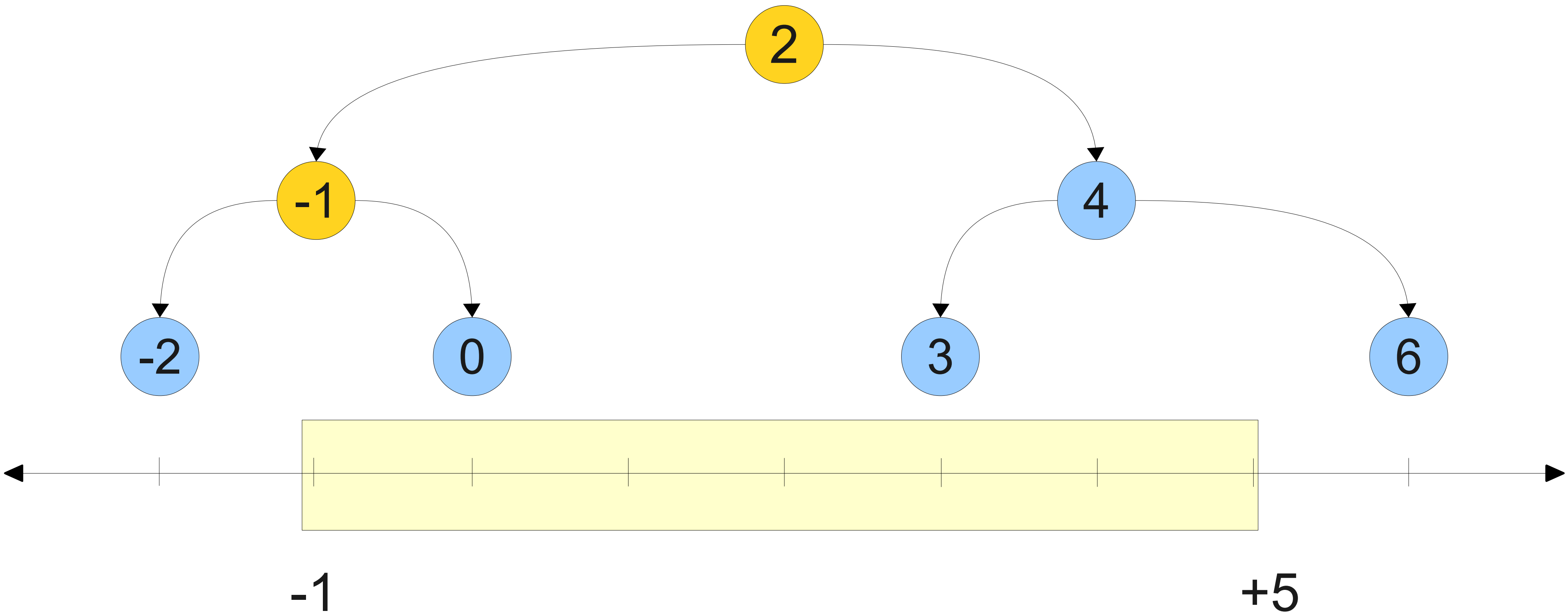

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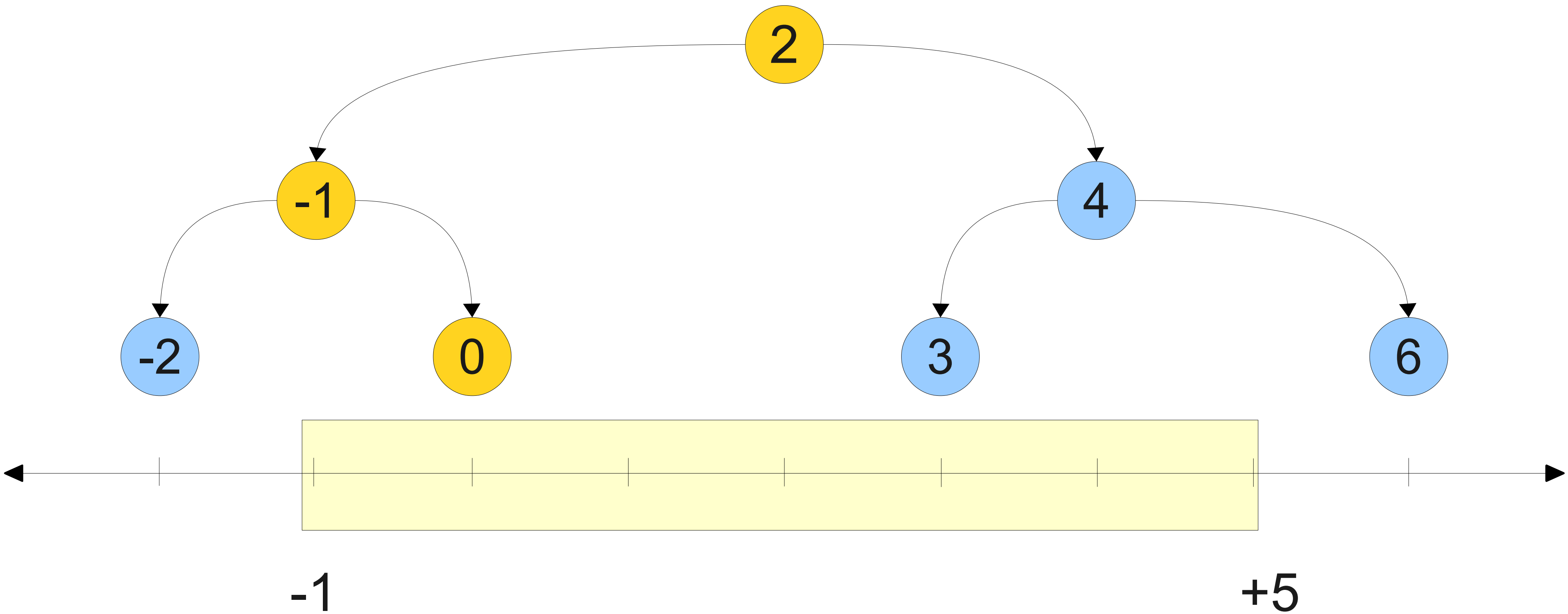

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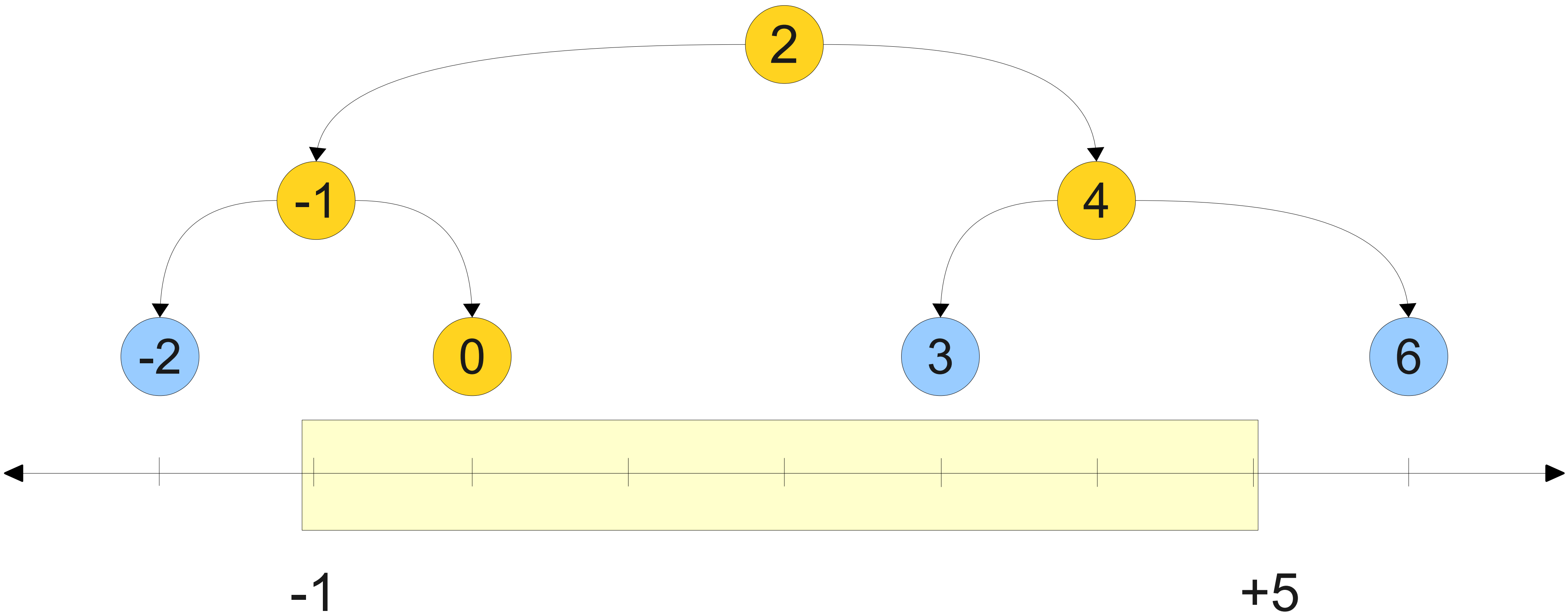

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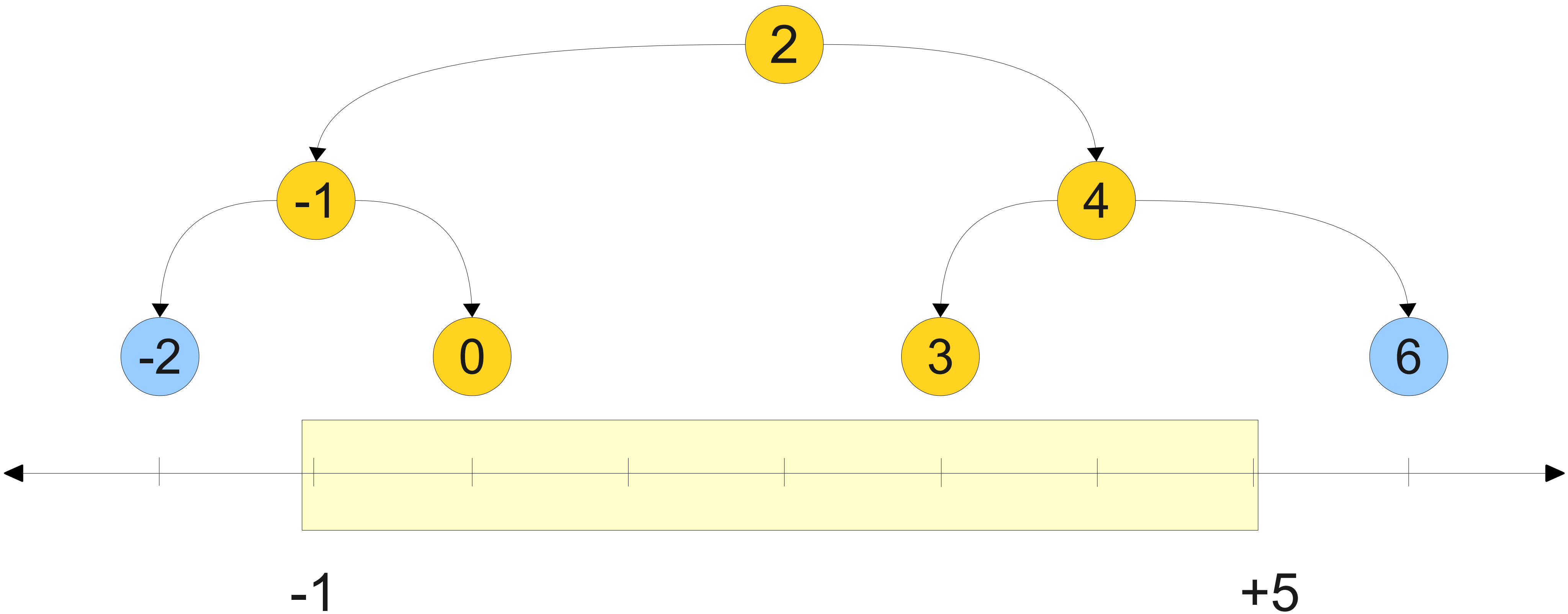

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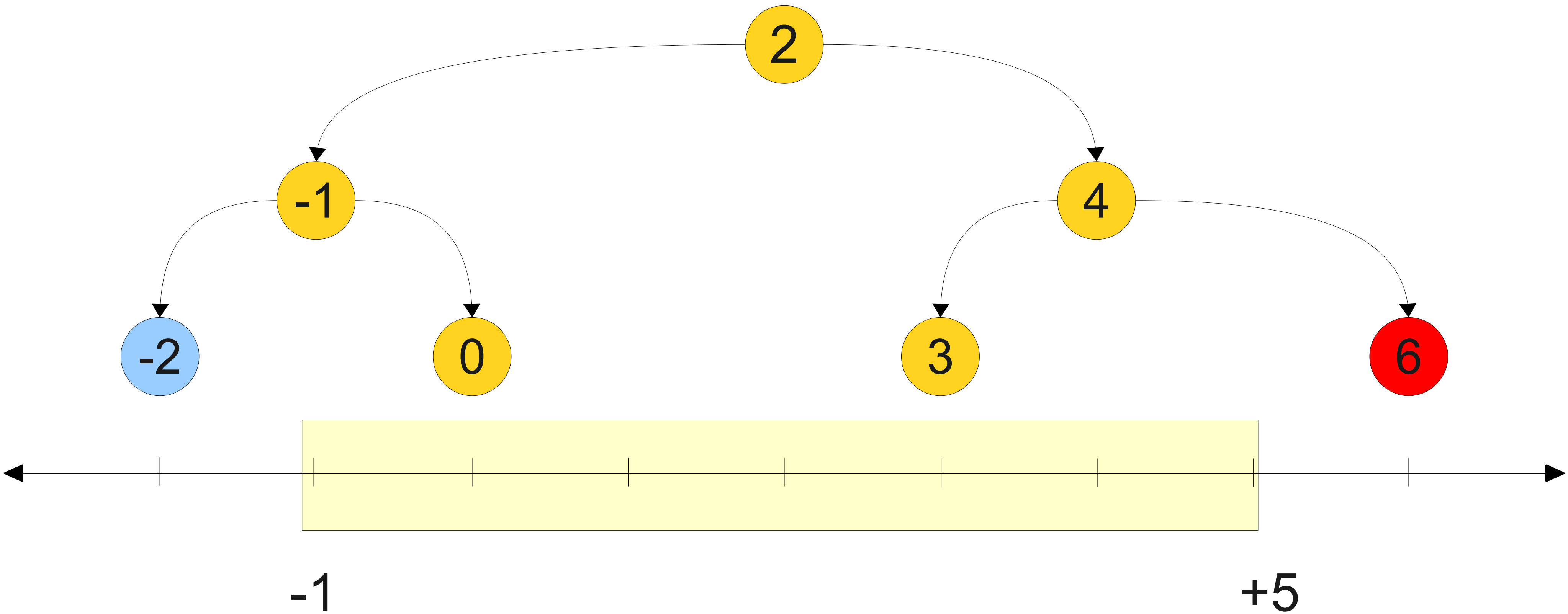
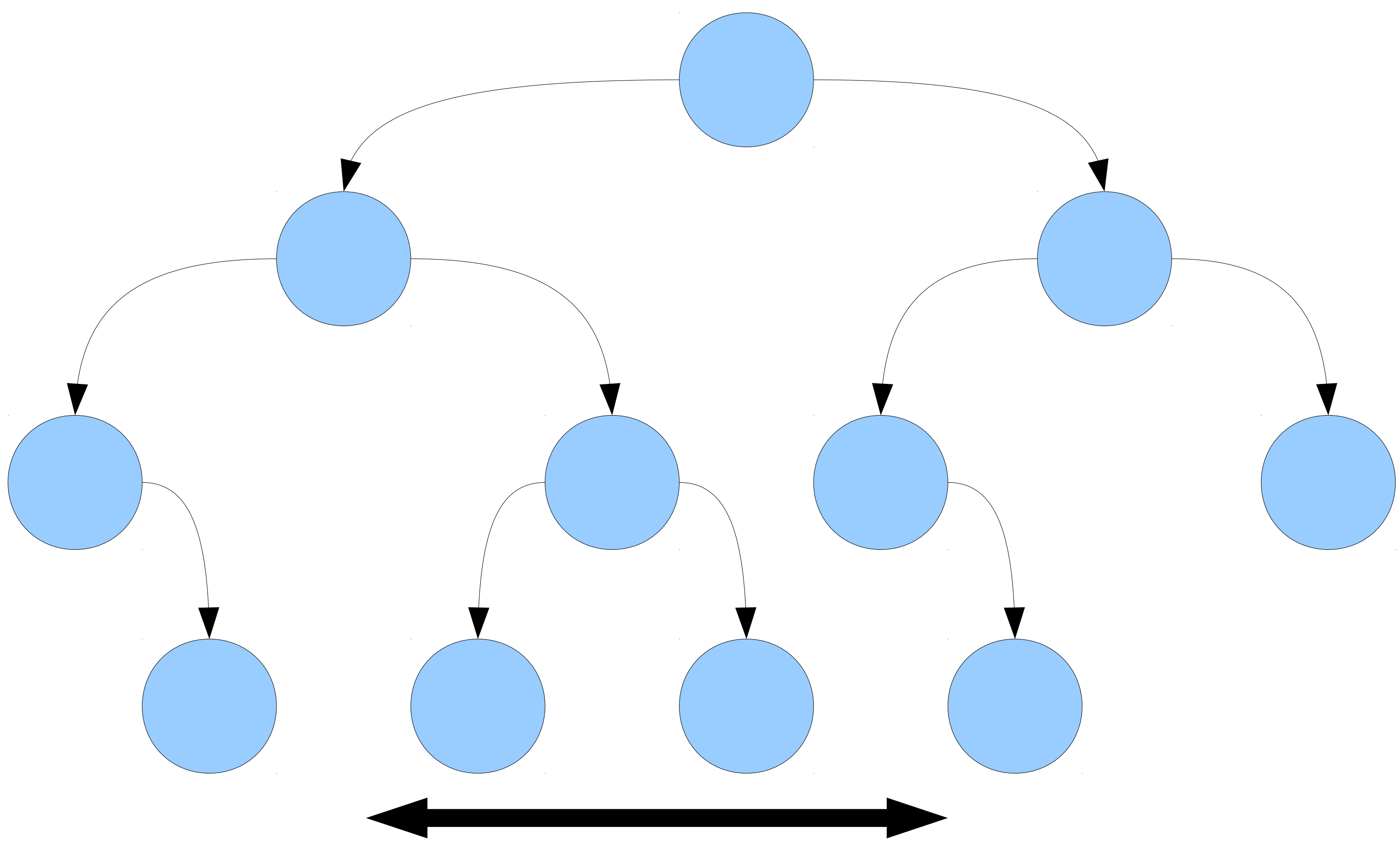


The Logic

- **Base case:**
  - The empty tree has no nodes within any range.
- **Recursive step:**
  - If this node is below the lower bound, recursively search the right subtree.
  - If this node is above the upper bound, recursively search the left subtree.
  - If this node is within bounds:
    - Search the left subtree.
    - Add this node to the output.
    - Search the right subtree.

Complexity of Range Searches

- How do we get a runtime for a range search?
- Depends on how many nodes we find.



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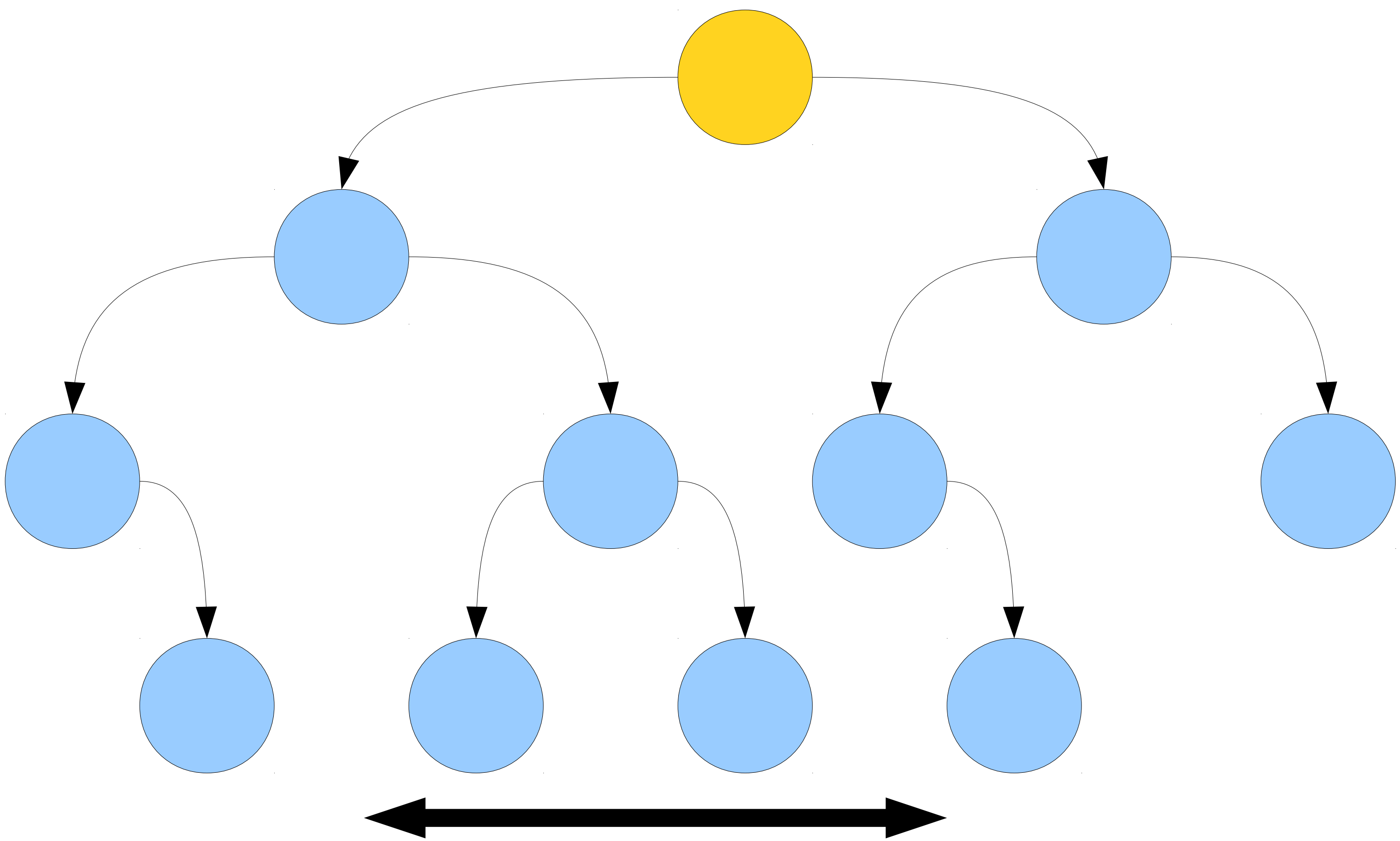

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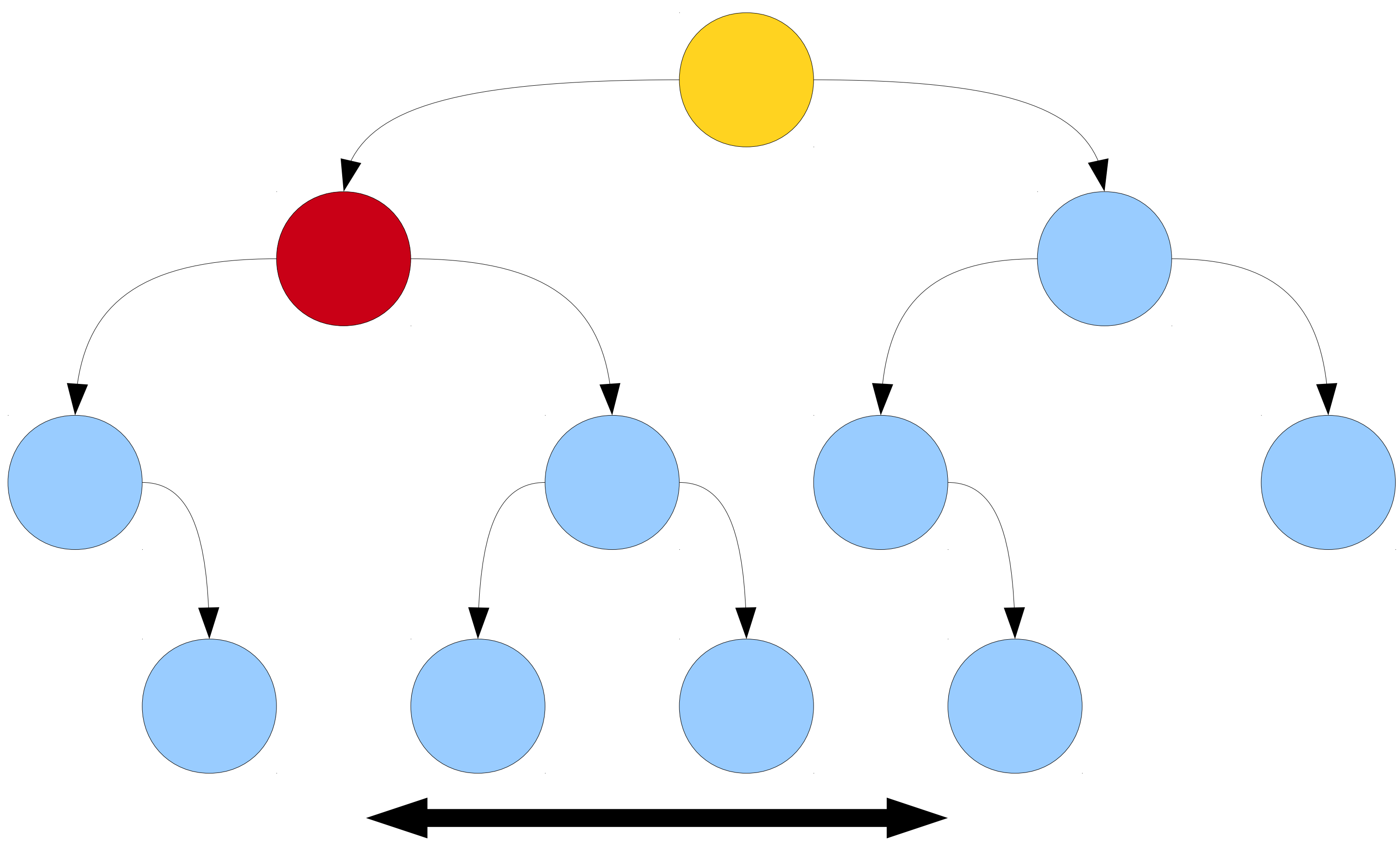

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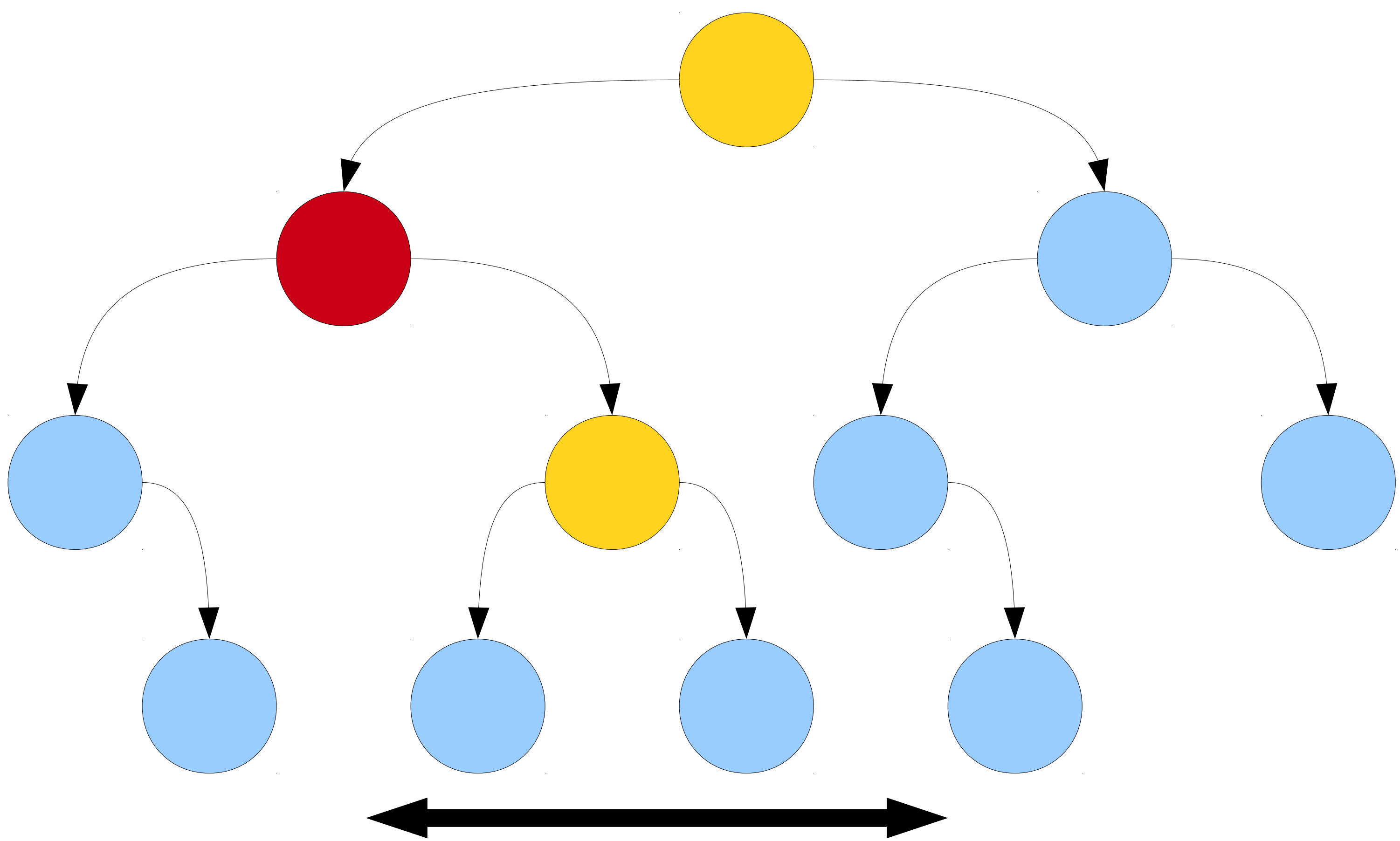

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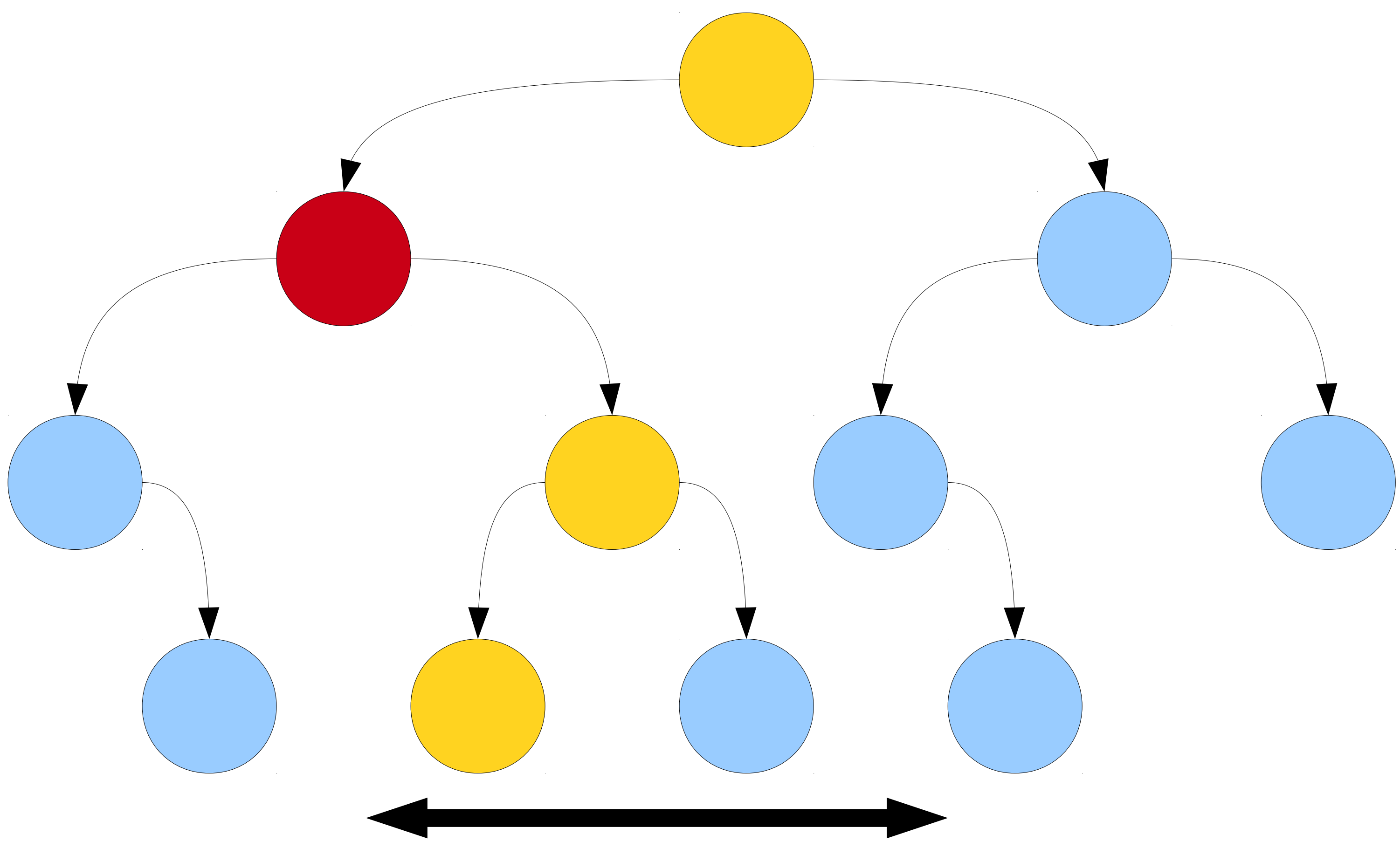

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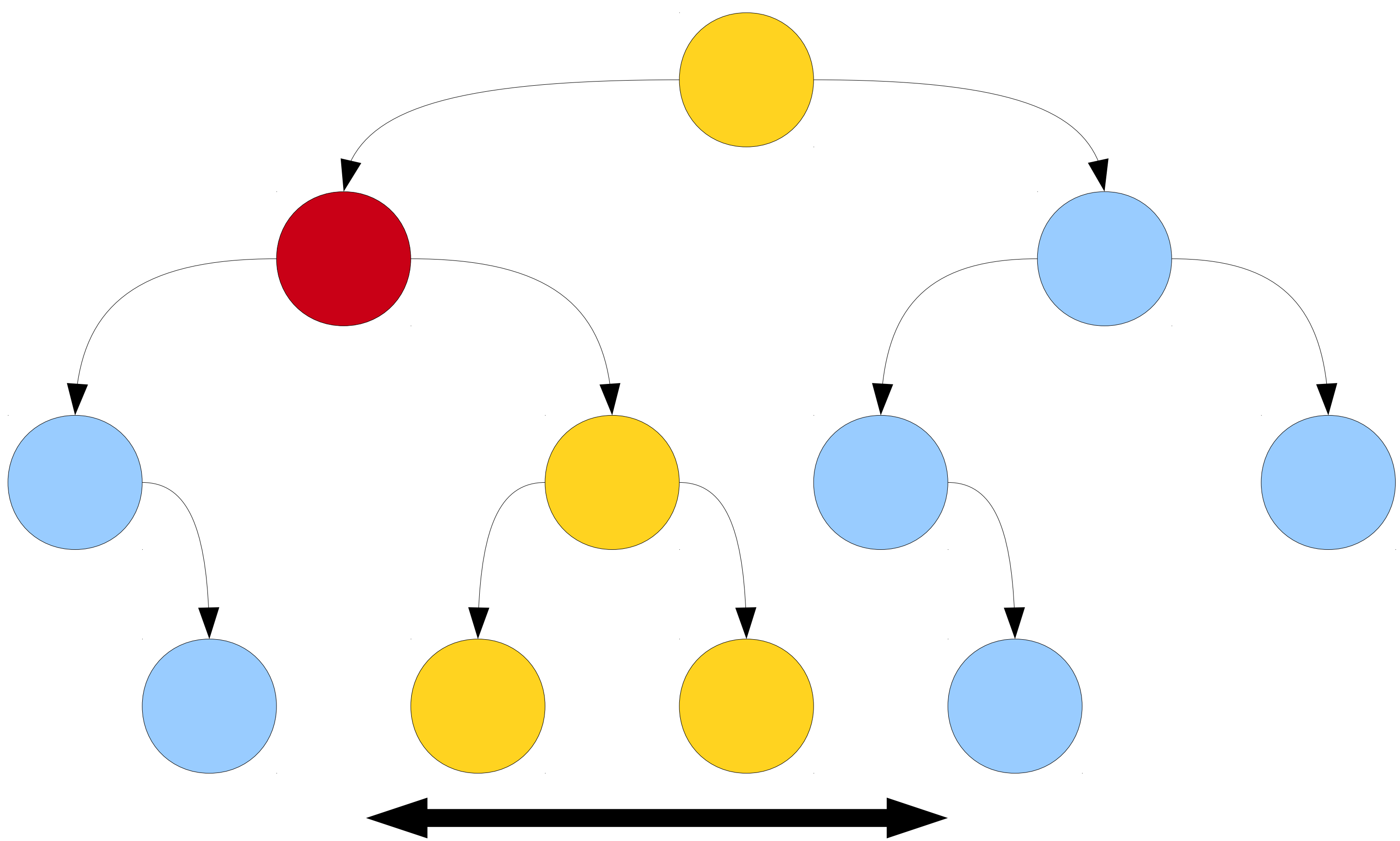

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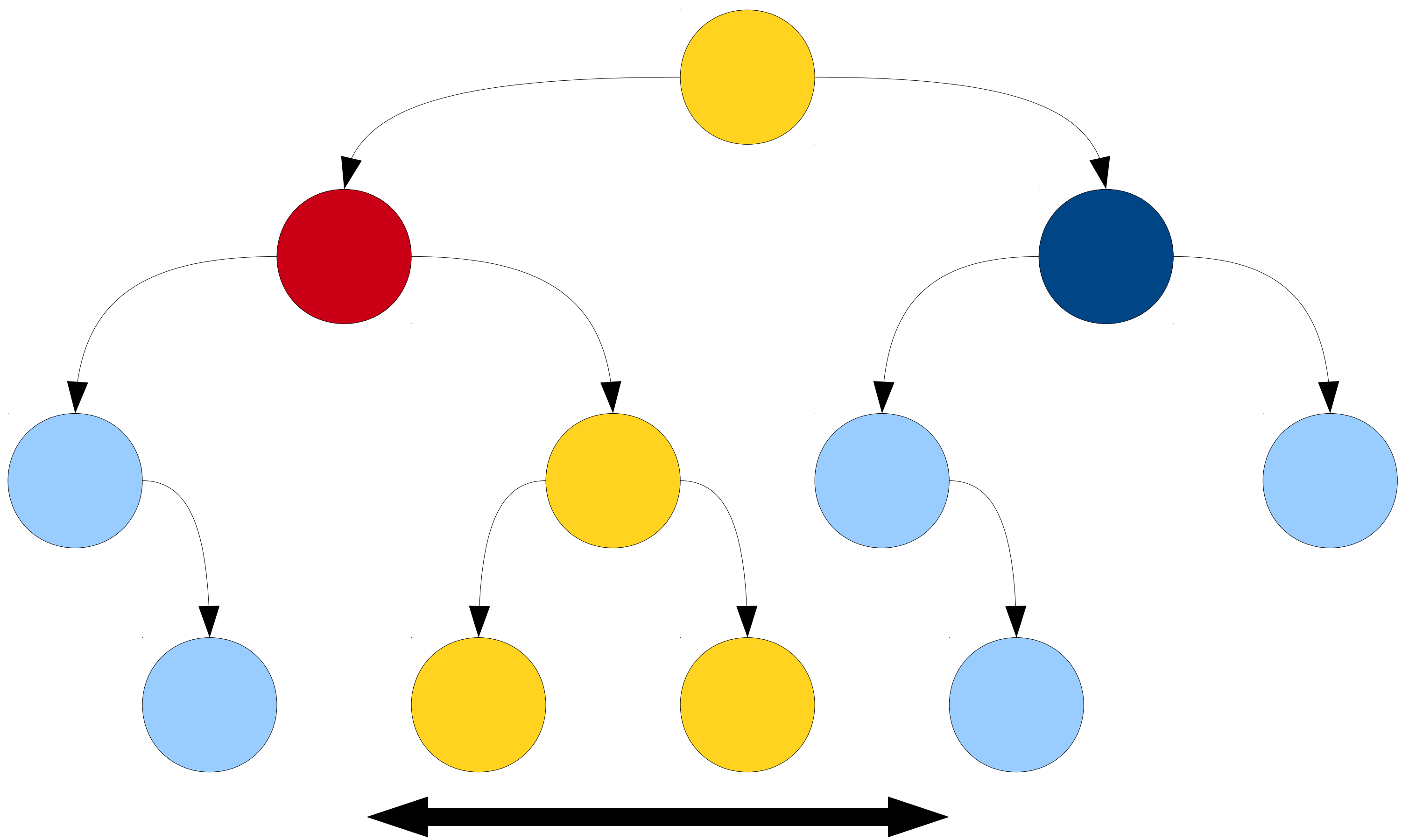

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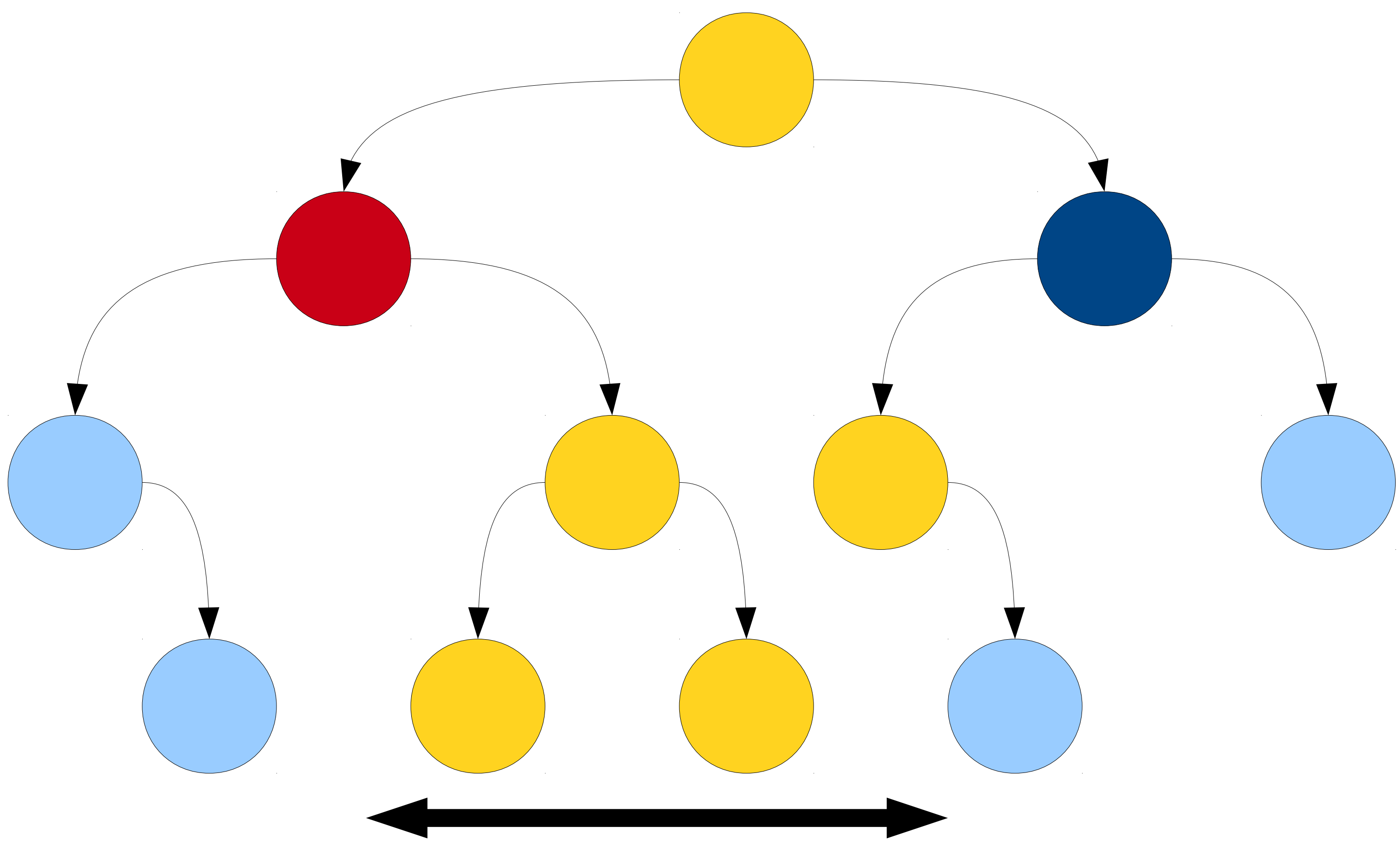

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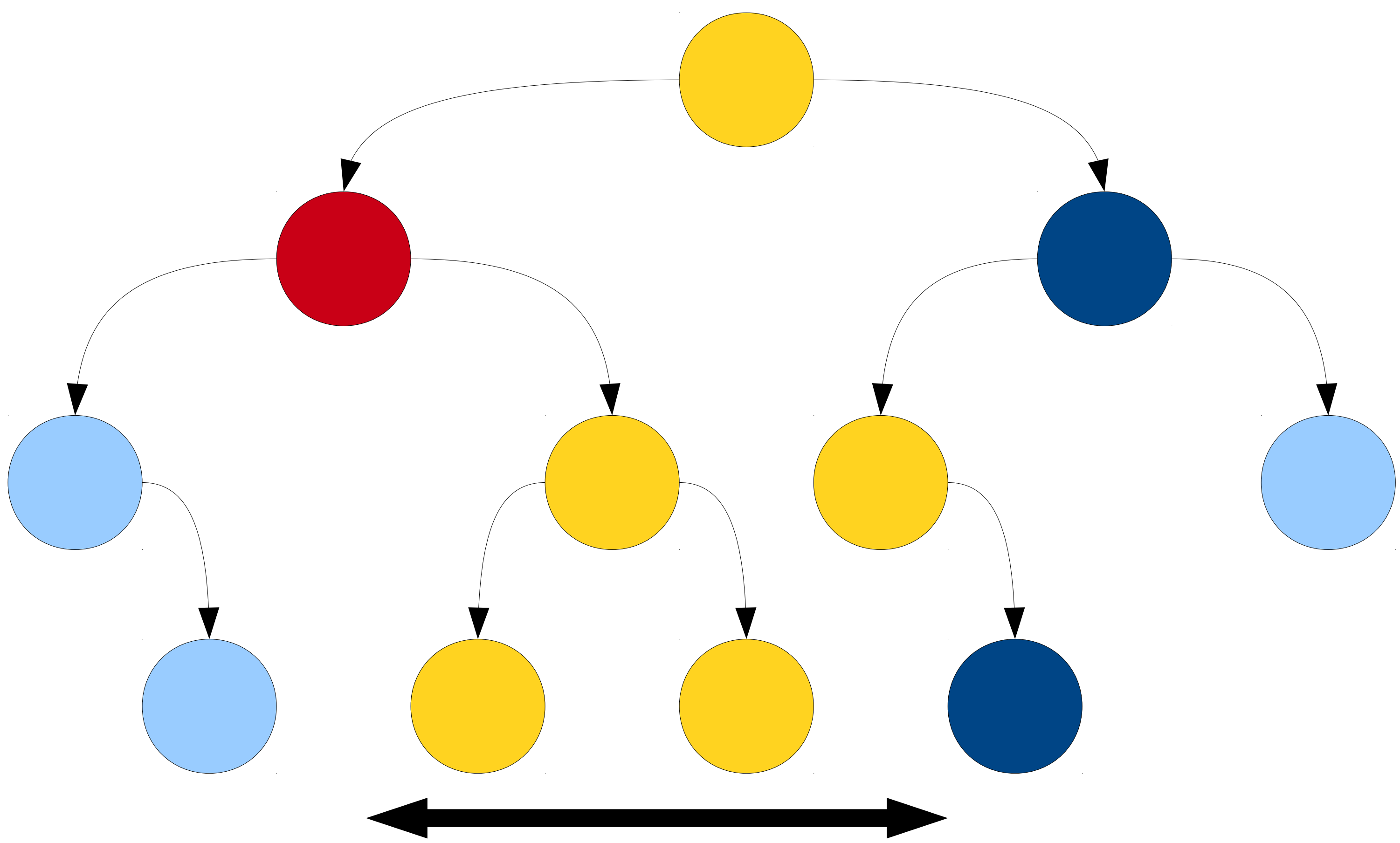

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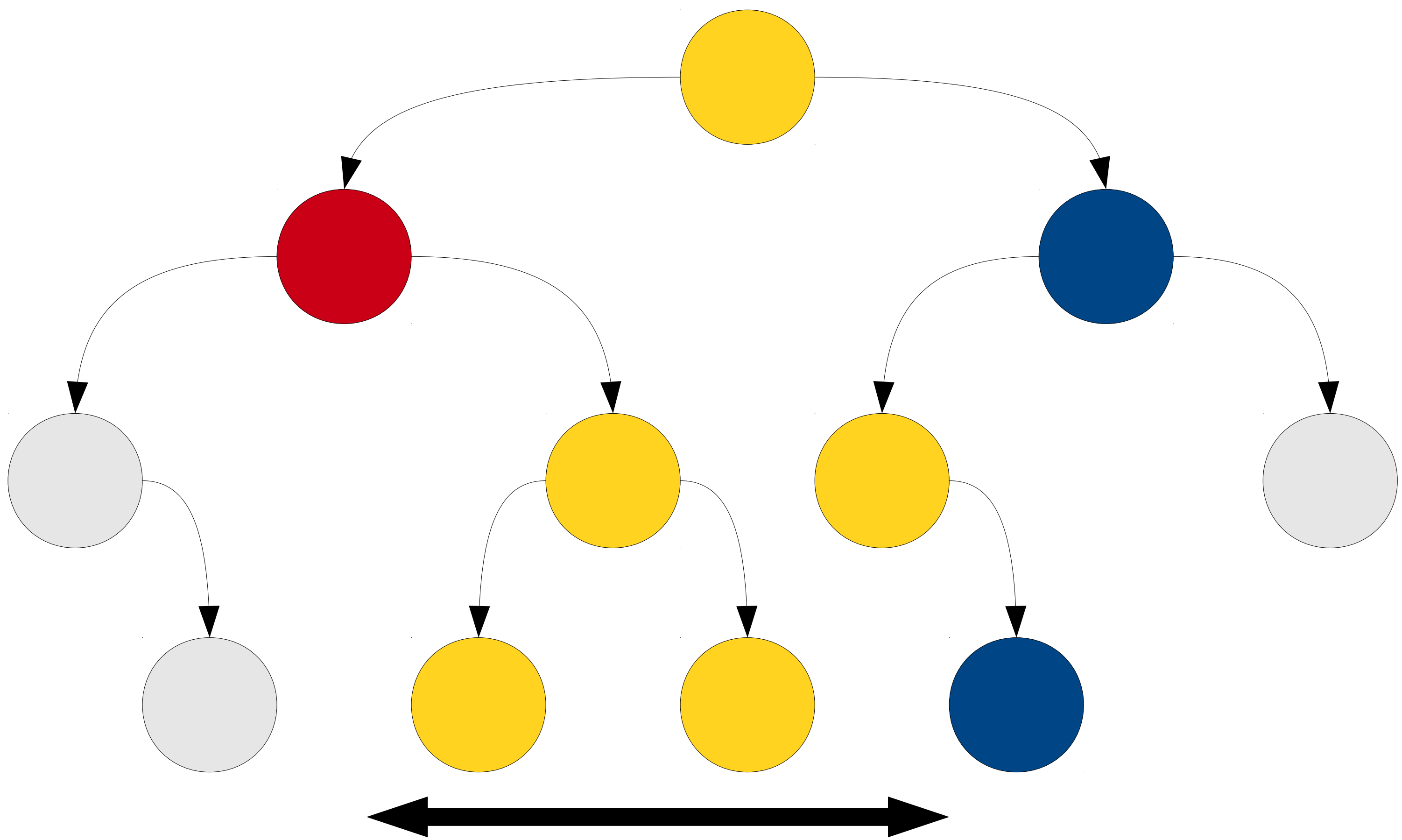

Complexity of Range Searches

- How do we get a runtime for a range search?
- Depends on how many nodes we find.
- If there are k nodes within the range, we do at least $O(k)$ work finding them.
- In addition, we have two "border sets" of nodes that are immediately outside that range. Each set has size $O(h)$, where h is the height of the tree.
- Total work done is $O(k + h)$.
- This is an **output-sensitive algorithm**.

Next Time

- **Fun With Data Structures.**
  - Balanced binary search trees.
  - Ternary search trees.
  - DAWGs.