# Regular Expressions

Problem Set Four is due using a late period in the box up front.

# Concatenation

- The **concatenation** of two languages $L_1$ and $L_2$ over the alphabet $\Sigma$ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \wedge x \in L_2 \}$$

- Intuitively, the set of all strings formed by concatenating some string from $L_1$ and some string from $L_2$.
- Conceptually similar to the Cartesian product of two sets, only with strings.

# Language Exponentiation

- We can define what it means to "exponentiate" a language as follows:
- $L^0 = \{ \varepsilon \}$
  - The set containing just the empty string.
  - Idea: Any string formed by concatenating zero strings together is the empty string.
- $L^{n+1} = LL^n$
  - Idea: Concatenating ($n$+1) strings together works by concatenating $n$ strings, then concatenating one more.

# The Kleene Closure

- An important operation on languages is the **Kleene Closure**, which is defined as

$$L^* = \bigcup_{i=0}^{\infty} L^i$$

- Mathematically:

$$w \in L^* \quad \textbf{iff} \quad \exists n \in \mathbb{N}.\ w \in L^n$$

- Intuitively, all possible ways of concatenating any number of copies of strings in $L$ together.

# Closure Properties

- The regular languages are closed under the following operations:
  - Complementation
  - Union
  - Intersection
  - Concatenation
  - Kleene closure

# Another View of Regular Languages

# Rethinking Regular Languages

- We currently have several tools for showing a language is regular.
  - Construct a DFA for it.
  - Construct an NFA for it.
  - Apply closure properties to existing languages.
- We have not spoken much of this last idea.

# Constructing Regular Languages

- **Idea**: Build up all regular languages as follows:
  - Start with a small set of simple languages we already know to be regular.
  - Using closure properties, combine these simple languages together to form more elaborate languages.
- *A bottom-up approach to the regular languages.*

# Regular Expressions

- **Regular expressions** are a family of descriptions that can be used to capture the regular languages.
- Often provide a compact and human-readable description of the language.
- Used as the basis for numerous software systems (Perl, **`flex`**, **`grep`**, etc.)

# Atomic Regular Expressions

- The regular expressions begin with three simple building blocks.
- The symbol **Ø** is a regular expression that represents the empty language Ø.
- The symbol **ε** is a regular expression that represents the language { ε }
  - ***This is not the same as Ø!***
- For any **a** ∈ Σ, the symbol **a** is a regular expression for the language { **a** }

# Compound Regular Expressions

- We can combine together existing regular expressions in four ways.
- If $R_1$ and $R_2$ are regular expressions, $\boldsymbol{R_1R_2}$ is a regular expression for the **concatenation** of the languages of $R_1$ and $R_2$.
- If $R_1$ and $R_2$ are regular expressions, $\boldsymbol{R_1 \mid R_2}$ is a regular expression for the **union** of the languages of $R_1$ and $R_2$.
- If $R$ is a regular expression, $\boldsymbol{R^*}$ is a regular expression for the **Kleene closure** of the language of $R$.
- If $R$ is a regular expression, $\boldsymbol{(R)}$ is a regular expression with the same meaning as $R$.

# Operator Precedence

- Regular expression operator precedence:

$$(R)$$

$$R*$$

$$R_1 R_2$$

$$R_1 \mid R_2$$

- So **`ab*c|d`** is parsed as **`((a(b*))c)|d`**

# Regular Expression Examples

- The regular expression **`trick|treat`** represents the regular language { **`trick`**, **`treat`** }
- The regular expression **`booo*`** represents the regular language { **`boo`**, **`booo`**, **`boooo`**, ... }
- The regular expression **`candy!(candy!)*`** represents the regular language { **`candy!`**, **`candy!candy!`**, **`candy!candy!candy!`**, ... }

# Regular Expressions, Formally

- The **language of a regular expression** is the language described by that regular expression.
- Formally:
  - $\mathscr{L}(\varepsilon) = \{\varepsilon\}$
  - $\mathscr{L}(\emptyset) = \emptyset$
  - $\mathscr{L}(\texttt{a}) = \{\texttt{a}\}$
  - $\mathscr{L}(R_1 R_2) = \mathscr{L}(R_1)\ \mathscr{L}(R_2)$
  - $\mathscr{L}(R_1 \mid R_2) = \mathscr{L}(R_1) \cup \mathscr{L}(R_2)$
  - $\mathscr{L}(R^*) = \mathscr{L}(R)^*$
  - $\mathscr{L}((R)) = \mathscr{L}(R)$

Worthwhile activity: Apply this recursive definition to

`a(b|c)((d))`

and see what you get.

# Regular Expressions are Awesome

- Let $\Sigma = \{$`0`, `1`$\}$
- Let $L = \{ w \in \Sigma^* \mid w$ contains `00` as a substring $\}$

# Regular Expressions are Awesome

- Let $\Sigma = \{$`0`, `1`$\}$
- Let $L = \{ w \in \Sigma^* \mid w$ contains `00` as a substring $\}$

**(0 | 1)*00(0 | 1)***

# Regular Expressions are Awesome

- Let $\Sigma = \{$`0`, `1`$\}$
- Let $L = \{ w \in \Sigma^* \mid w$ contains `00` as a substring $\}$

**(0 | 1)*00(0 | 1)***

# Regular Expressions are Awesome

- Let $\Sigma = \{$`0`, `1`$\}$
- Let $L = \{ w \in \Sigma^* \mid w$ contains `00` as a substring $\}$

**(0 | 1)*00(0 | 1)***

**11011100101**
**0000**
**1111101111001111**

# Regular Expressions are Awesome

- Let $\Sigma = \{0, 1\}$
- Let $L = \{ w \in \Sigma^* \mid w$ contains `00` as a substring $\}$

**(0 | 1)*00(0 | 1)***

**11011100101**
**0000**
**1111101111100011111**

# Regular Expressions are Awesome

- Let $\Sigma = \{0, 1\}$
- Let $L = \{ w \in \Sigma^* \mid w$ contains 00 as a substring $\}$

**Σ*00Σ***

**11011100101**
**0000**
**1111101111100011111**

# Regular Expressions are Awesome

- Let $\Sigma = \{0, 1\}$
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$

# Regular Expressions are Awesome

Let $\Sigma = \{0, 1\}$

Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$

# Regular Expressions are Awesome

Let $\Sigma = \{0, 1\}$

Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$

The length of a string $w$ is denoted $|w|$

# Regular Expressions are Awesome

- Let $\Sigma = \{0, 1\}$
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$

# Regular Expressions are Awesome

- Let $\Sigma = \{0, 1\}$
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$

$$\mathbf{\Sigma\Sigma\Sigma\Sigma}$$

# Regular Expressions are Awesome

- Let $\Sigma = \{0, 1\}$
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$

$$\Sigma\Sigma\Sigma\Sigma$$

# Regular Expressions are Awesome

- Let $\Sigma = \{0, 1\}$
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$

$$\Sigma\Sigma\Sigma\Sigma$$

**0000**
**1010**
**1111**
**1000**

# Regular Expressions are Awesome

- Let $\Sigma = \{0, 1\}$
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$

$$\Sigma\Sigma\Sigma\Sigma$$

$$0000$$
$$1010$$
$$1111$$
$$1000$$

# Regular Expressions are Awesome

- Let $\Sigma = \{0, 1\}$
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$

$$\Sigma^4$$

**0000**
**1010**
**1111**
**1000**

# Regular Expressions are Awesome

- Let $\Sigma = \{0, 1\}$
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$

$$\Sigma^4$$

**0000**
**1010**
**1111**
**1000**

# Regular Expressions are Awesome

- Let $\Sigma = \{$`0`, `1`$\}$
- Let $L = \{\ w \in \Sigma^* \mid w$ contains at most one `0` $\}$

# Regular Expressions are Awesome

- Let $\Sigma = \{$`0`, `1`$\}$
- Let $L = \{\ w \in \Sigma^* \mid w$ contains at most one `0` $\}$

**1*(0 | ε)1***

# Regular Expressions are Awesome

- Let $\Sigma = \{$`0`, `1`$\}$
- Let $L = \{\ w \in \Sigma^* \mid w$ contains at most one `0` $\}$

**1*(0 | ε)1***

# Regular Expressions are Awesome

- Let $\Sigma = \{$`0`, `1`$\}$
- Let $L = \{\ w \in \Sigma^* \mid w$ contains at most one `0` $\}$

**1*(0 | ε)1***

**11110111**
**111111**
**0111**
**0**

# Regular Expressions are Awesome

- Let $\Sigma = \{0, 1\}$
- Let $L = \{ w \in \Sigma^* \mid w$ contains at most one `0` $\}$

**1*(0 | ε)1***

**11110111**
**111111**
**0111**
**0**

# Regular Expressions are Awesome

- Let $\Sigma = \{0, 1\}$
- Let $L = \{ w \in \Sigma^* \mid w$ contains at most one `0` $\}$

**1*0?1***

**11110111**
**111111**
**0111**
**0**

# Regular Expressions are Awesome

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Regular expression for email addresses:

# Regular Expressions are Awesome

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Regular expression for email addresses:

**aa*(.aa*)*@aa*.aa*(.aa*)***

# Regular Expressions are Awesome

- Let Σ = { `a`, `.`, `@` }, where `a` represents "some letter."
- Regular expression for email addresses:

**aa*(.aa*)*@aa*.aa*(.aa*)***

**cs103@cs.stanford.edu**
**first.middle.last@mail.site.org**
**barack.obama@whitehouse.gov**

# Regular Expressions are Awesome

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Regular expression for email addresses:

**aa*(.aa*)*@aa*.aa*(.aa*)***

**cs103@cs.stanford.edu**
**first.middle.last@mail.site.org**
**barack.obama@whitehouse.gov**

# Regular Expressions are Awesome

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Regular expression for email addresses:

**aa*(.aa*)*@aa*.aa*(.aa*)***

**cs103@cs.stanford.edu**
**first.middle.last@mail.site.org**
**barack.obama@whitehouse.gov**

# Regular Expressions are Awesome

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Regular expression for email addresses:

**aa*(.aa*)*@aa*.aa*(.aa*)***

**cs103@cs.stanford.edu**
**first.middle.last@mail.site.org**
**barack.obama@whitehouse.gov**

# Regular Expressions are Awesome

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Regular expression for email addresses:

**$a^+$ (.aa*)*@aa*.aa*(.aa*)***

**cs103@cs.stanford.edu**
**first.middle.last@mail.site.org**
**barack.obama@whitehouse.gov**

# Regular Expressions are Awesome

- Let $\Sigma$ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Regular expression for email addresses:

$$\mathbf{a^+ \ (.a^+)^* \ @ \ a^+.a^+ \ (.a^+)^*}$$

**cs103@cs.stanford.edu**
**first.middle.last@mail.site.org**
**barack.obama@whitehouse.gov**

# Regular Expressions are Awesome

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Regular expression for email addresses:

$$\mathbf{a^+\ (.a^+)^*\ @\ \ a^+.a^+\ \ (.a^+)^*}$$

**cs103@cs.stanford.edu**
**first.middle.last@mail.site.org**
**barack.obama@whitehouse.gov**

# Regular Expressions are Awesome

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Regular expression for email addresses:

$$\mathbf{a^+\ (.a^+)^*\ @\quad a^+\quad (.a^+)^+}$$

**cs103@cs.stanford.edu**
**first.middle.last@mail.site.org**
**barack.obama@whitehouse.gov**

# Regular Expressions are Awesome

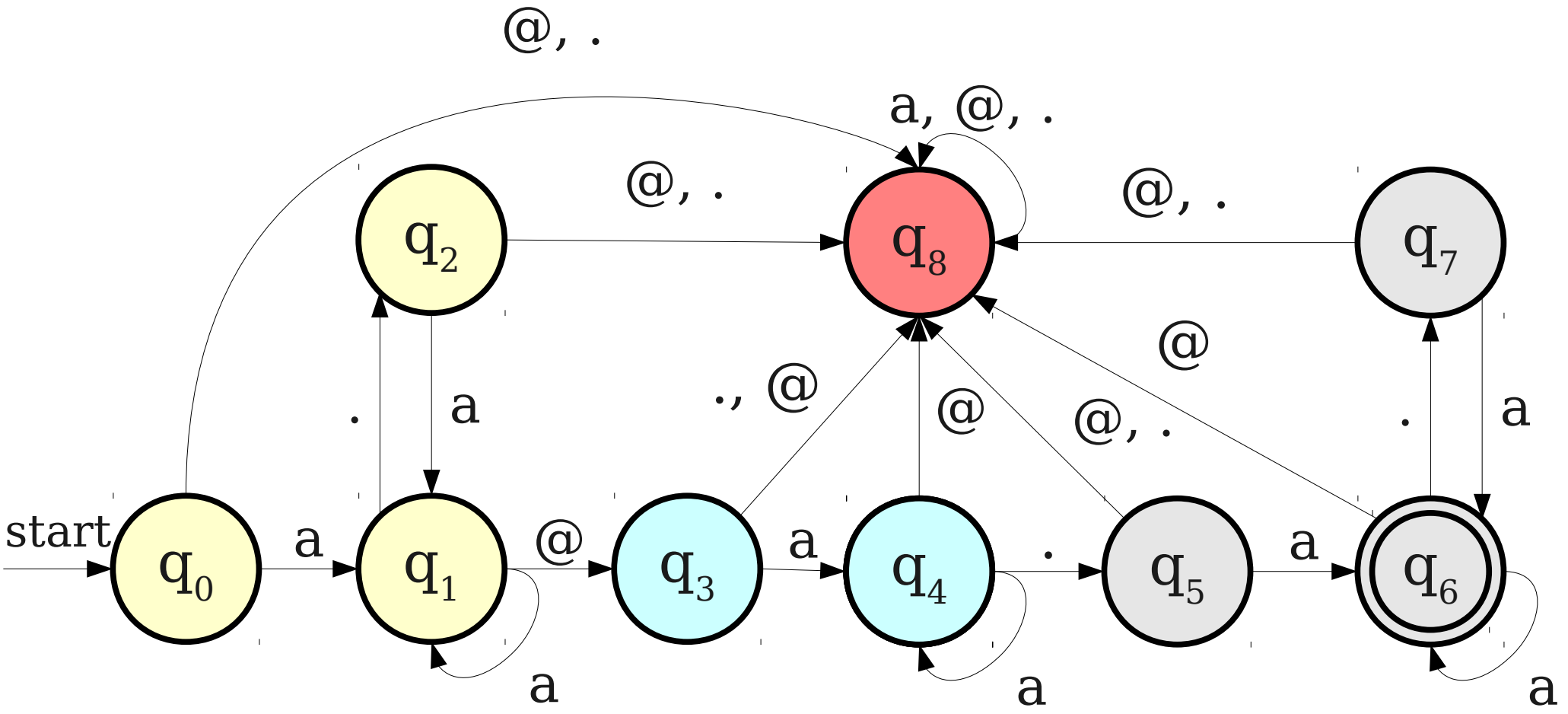
- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Regular expression for email addresses:

$$\mathbf{a^+(.a^+)^*@a^+(.a^+)^+}$$

**cs103@cs.stanford.edu**
**first.middle.last@mail.site.org**
**barack.obama@whitehouse.gov**

# Regular Expressions are Awesome

**$a^+(.a^+)^*@a^+(.a^+)^+$**



# Shorthand Summary

- $R^n$ is shorthand for $RR$ … $R$ ($n$ times).
- $\Sigma$ is shorthand for "any character in $\Sigma$."
- $R$? is shorthand for ($R$ | $\varepsilon$), meaning "zero or one copies of $R$."
- $R^+$ is shorthand for $RR^*$, meaning "one or more copies of $R$."

Break for Announcements!

# Midterm Logistics

- Midterm is tomorrow, October 29, from 7PM  - 10PM
- Room determined by last name:
  - A – G: Go to **Gates B01**
  - H – K: Go to **Gates B03**
  - L – P: Go to **200-002**
  - Q – V: Go to **420-041**
  - W – Z: Go to **Herrin T175**

# Your Questions

If you find that the function $f: A \to B$ is not surjective, have you proven that $|A| < |B|$? Or do you still need to do additional proof steps?

Problem Set 4 is due at 2:15PM with a late period. Please submit it ASAP!

When writing a logic statement, do you have to include the universal or existential quantifier for every variable that you state? I thought you had to, but this one from lecture doesn't:

$$\mathit{Tallest}(x) \to \forall y.\ (x \neq y \to \mathit{IsShorterThan}(y, x))$$

This example is a "sentence fragment" in first-order logic; without a definition of x, this isn't a valid statement. All variables need to be quantified.

"When writing first-order logic statements with quantifiers, which one out of the following would be correct?

$$\forall x\ P(x).\ \exists y.\ R(y)$$

or

$$\forall x.\ (P(x) \rightarrow \exists y.\ R(y))$$

If you find that the function $f: A \to B$ is not surjective, have you proven that $|A| < |B|$? Or do you still need to do additional proof steps?

$$f : \mathbb{N} \to \mathbb{N}$$

$$f(n) = 137$$

“What is the best thing to do to prepare for the exam between now and 7PM tomorrow?”

“Is there some mathematical automaton that can determine whether or not two first-order logical statements are equivalent?”

More on that later in the quarter...

# Back to Regular Expressions!

# The Power of Regular Expressions

***Theorem:*** If $R$ is a regular expression, then $\mathscr{L}(R)$ is regular.

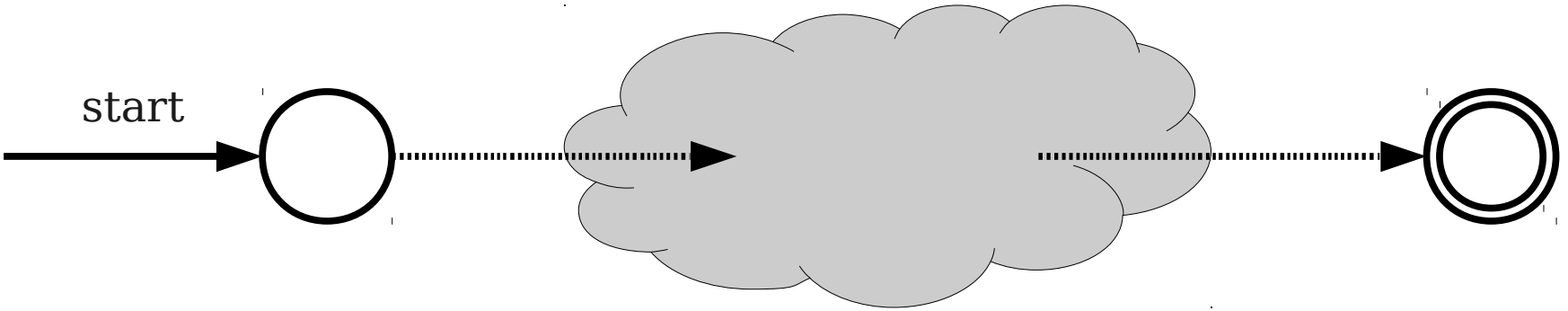
***Proof idea:*** Show how to convert a regular expression into an NFA.

# A Marvelous Construction

- The following theorem proves the language of any regular expression is regular:
- ***Theorem:*** For any regular expression $R$, there is an NFA $N$ such that

  $\mathscr{L}(R) = \mathscr{L}(N)$

  - $N$ has exactly one accepting state.
  - $N$ has no transitions into its start state.
  - $N$ has no transitions out of its accepting state.



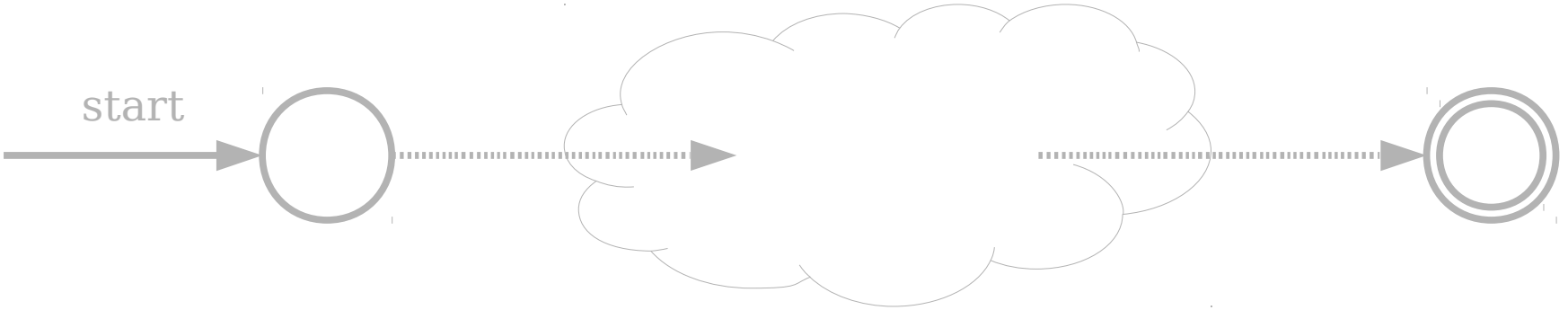
# A Marvelous Construction

The following theorem proves the language of any regular expression is regular:

***Theorem:*** For any regular expression $R$, there is an NFA $N$ such that

$$\mathscr{L}(R) = \mathscr{L}(N)$$

- $N$ has exactly one accepting state.
- $N$ has no transitions into its start state.
- $N$ has no transitions out of its accepting state.



# A Marvelous Construction

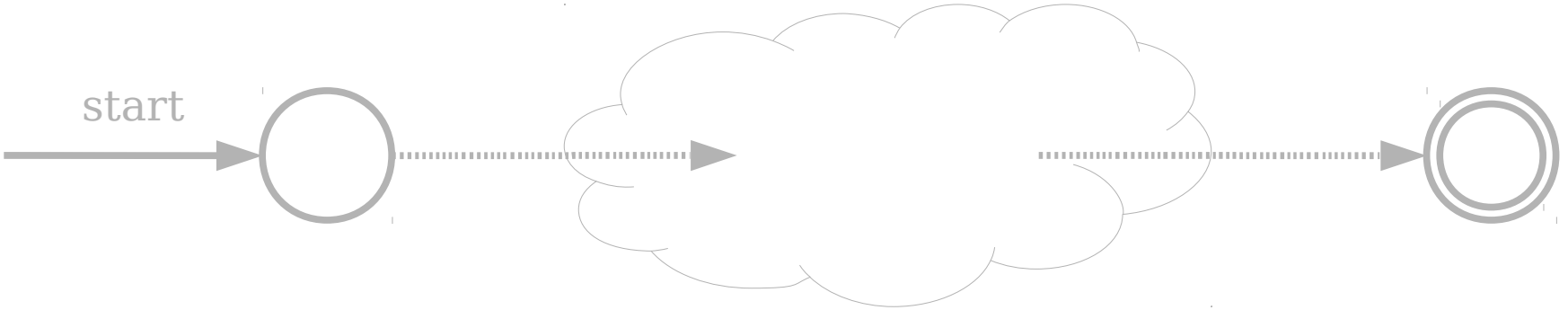
The following theorem p
any regular expression i

***Theorem:*** For any regu
is an NFA $N$ such that

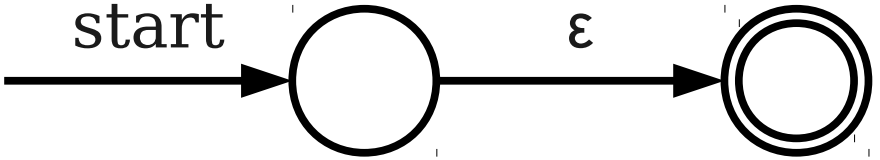
$$\mathscr{L}(R) = \mathscr{L}(N)$$

These are stronger requirements than are necessary for a normal NFA. We enforce these rules to simplify the construction.

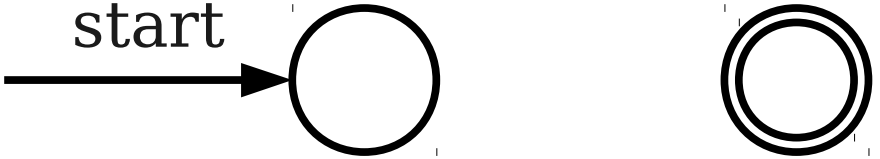
- $N$ has exactly one accepting state.
- $N$ has no transitions into its start state.
- $N$ has no transitions out of its accepting state.



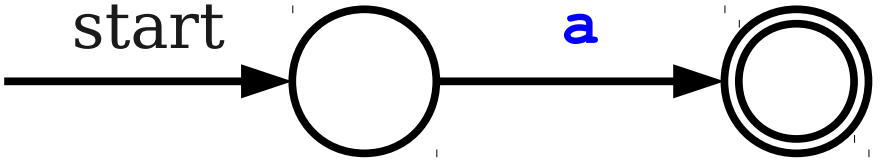
# Base Cases




Automaton for ε



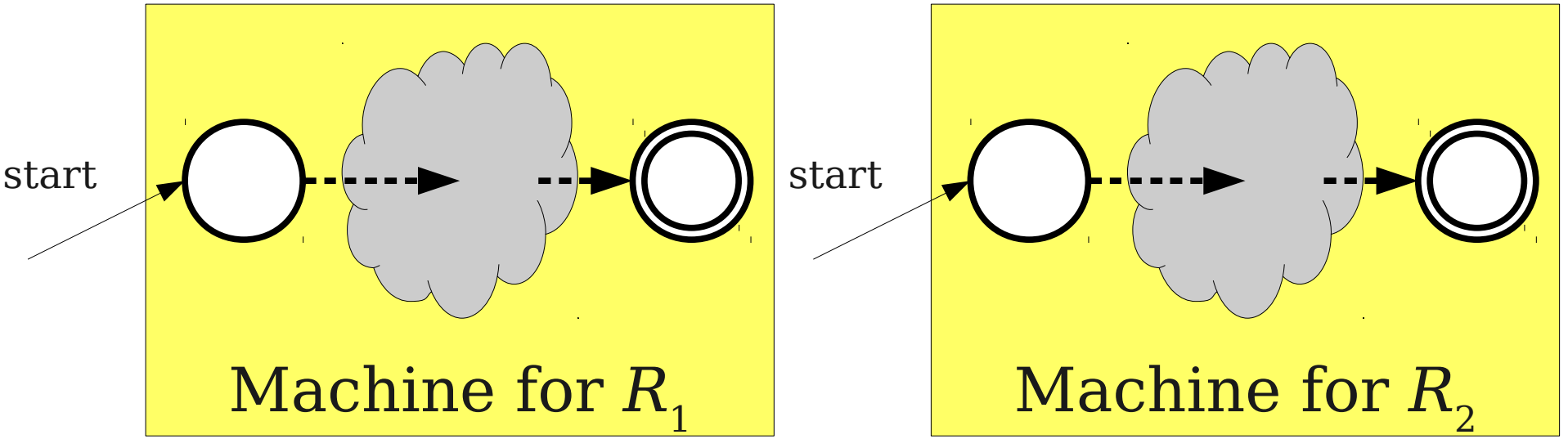

Automaton for Ø



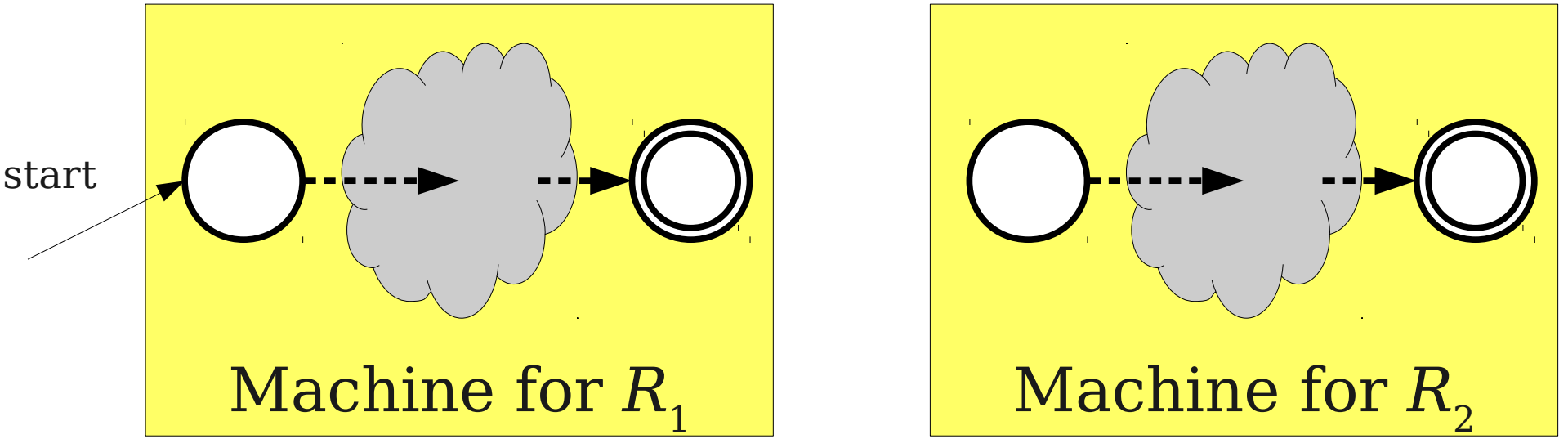

Automaton for single character **a**

# Construction for $R_1 R_2$

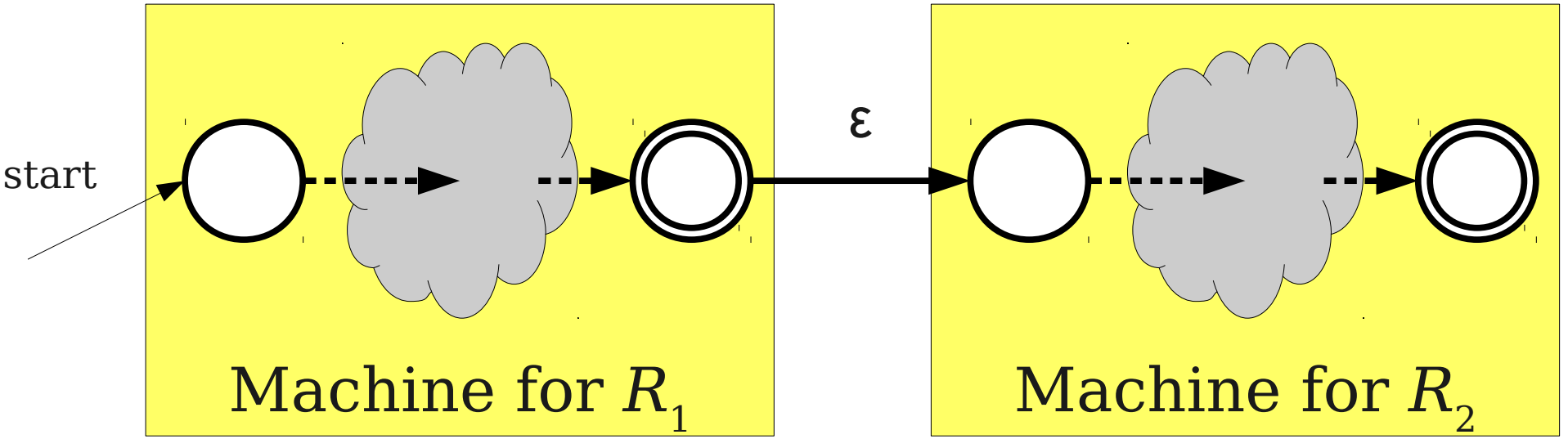
# Construction for $R_1R_2$



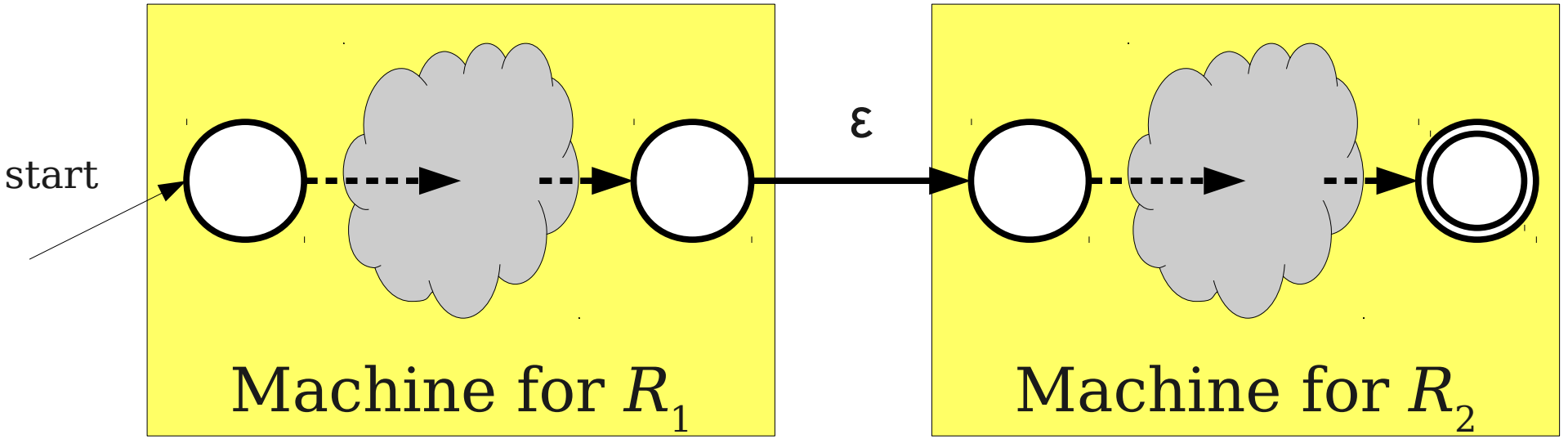
# Construction for $R_1 R_2$



# Construction for $R_1R_2$

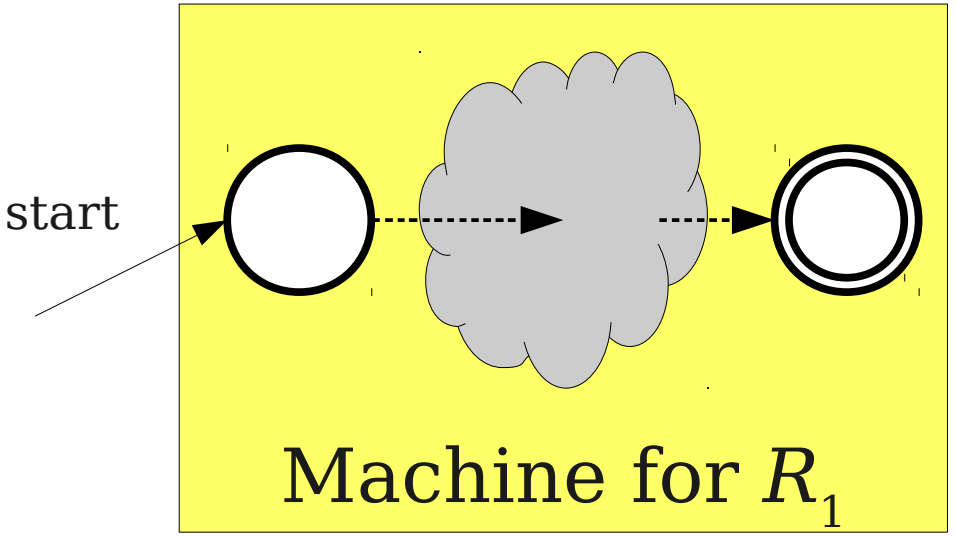


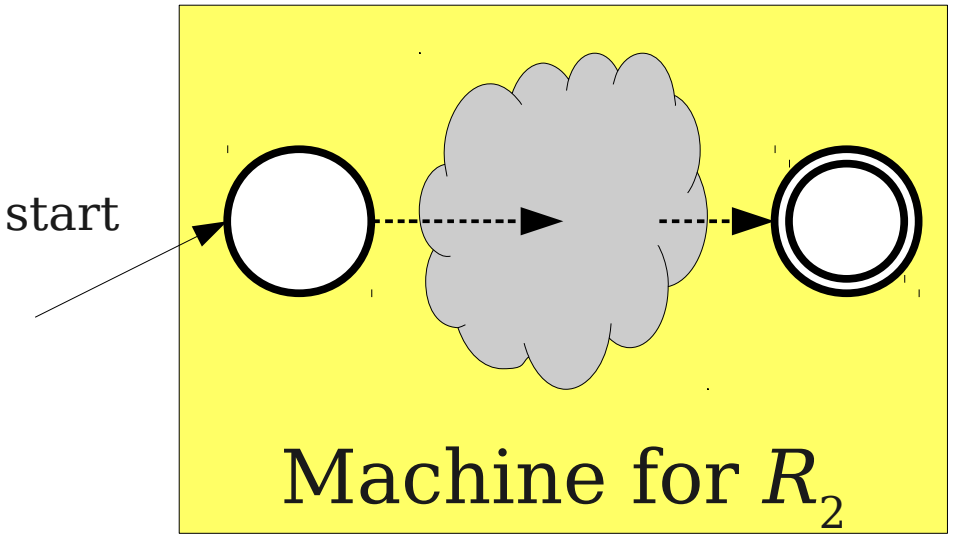
# Construction for $R_1R_2$



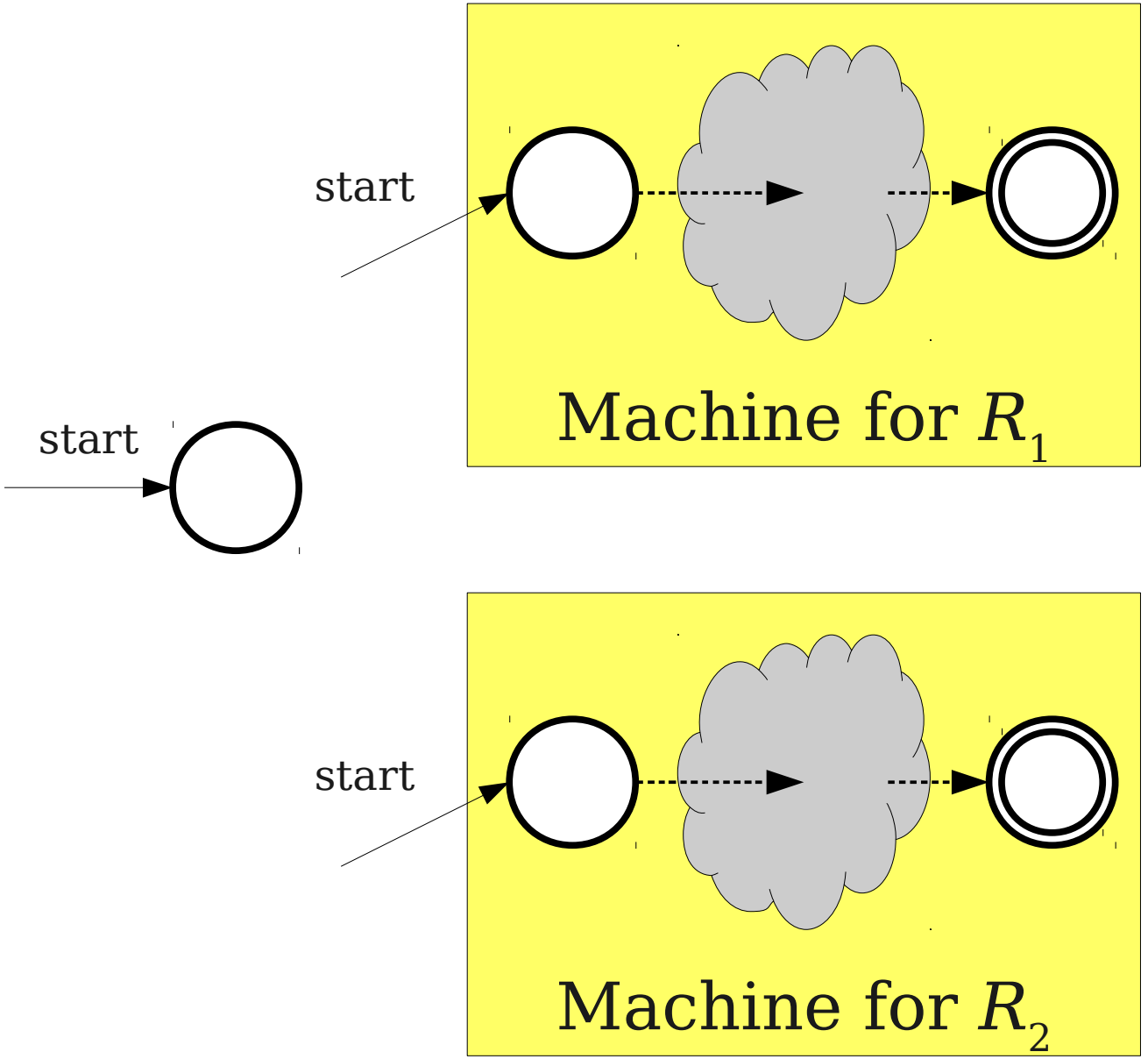
# Construction for $R_1 \mid R_2$

# Construction for $R_1 \mid R_2$

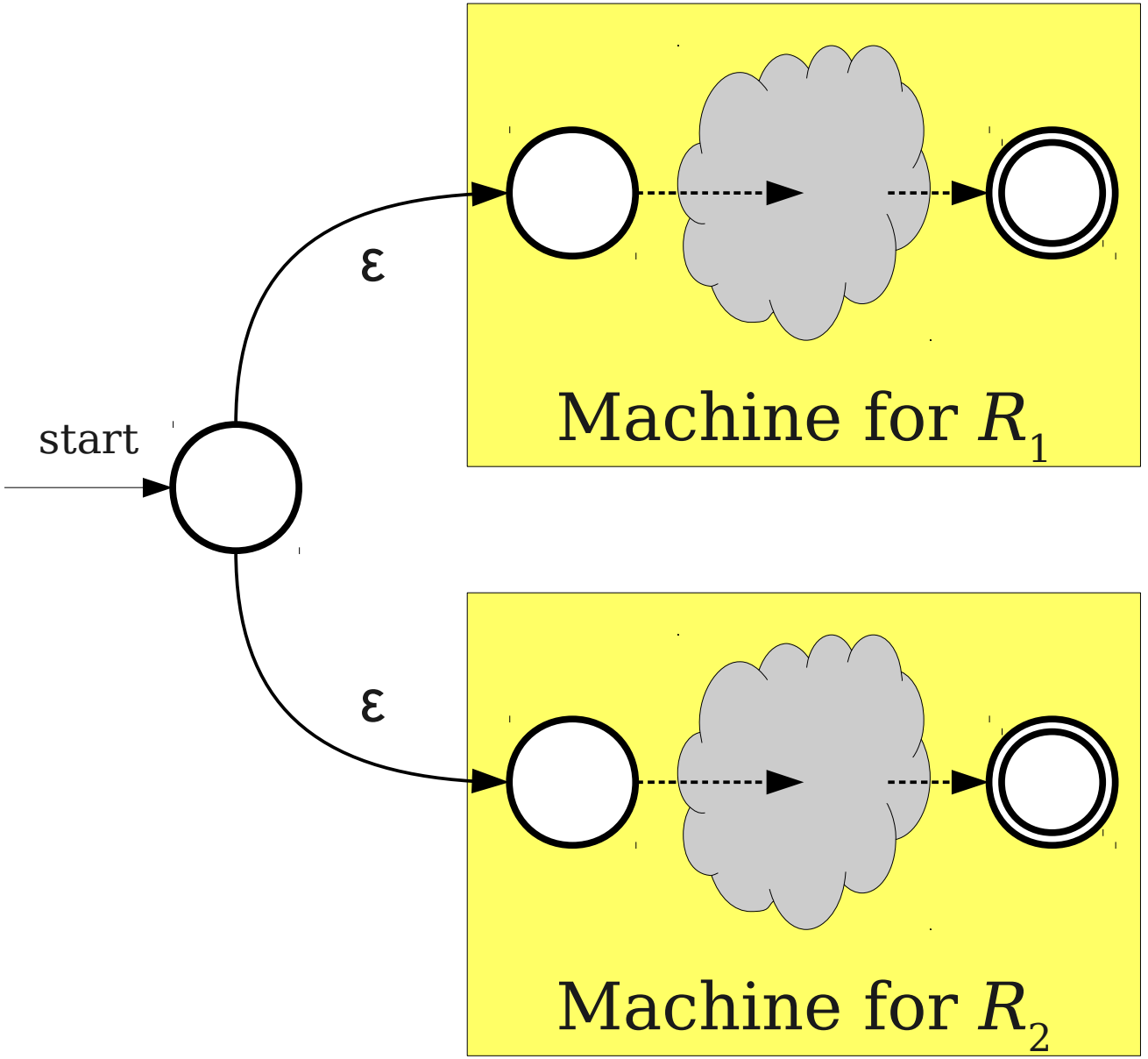


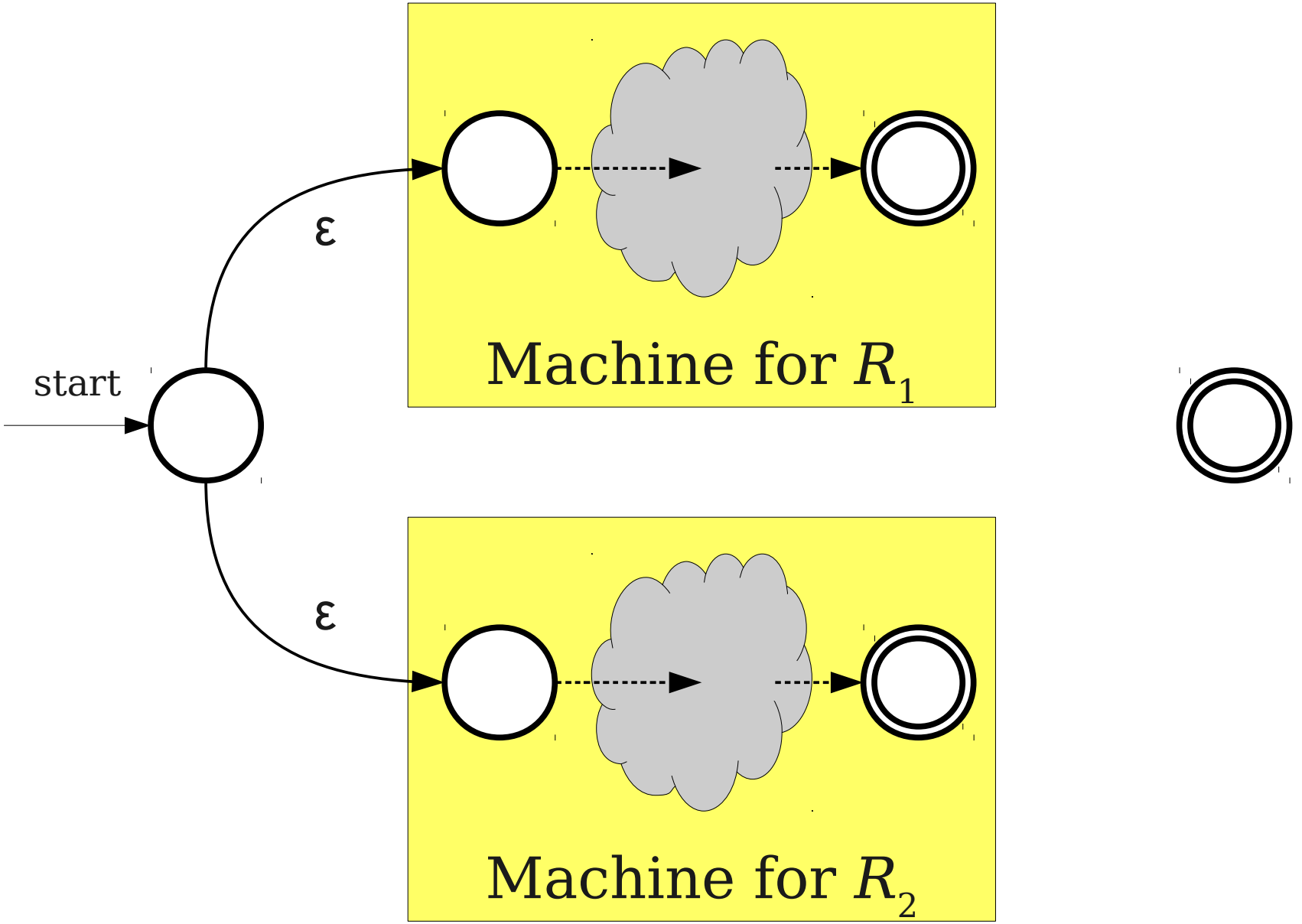
# Construction for $R_1 \mid R_2$



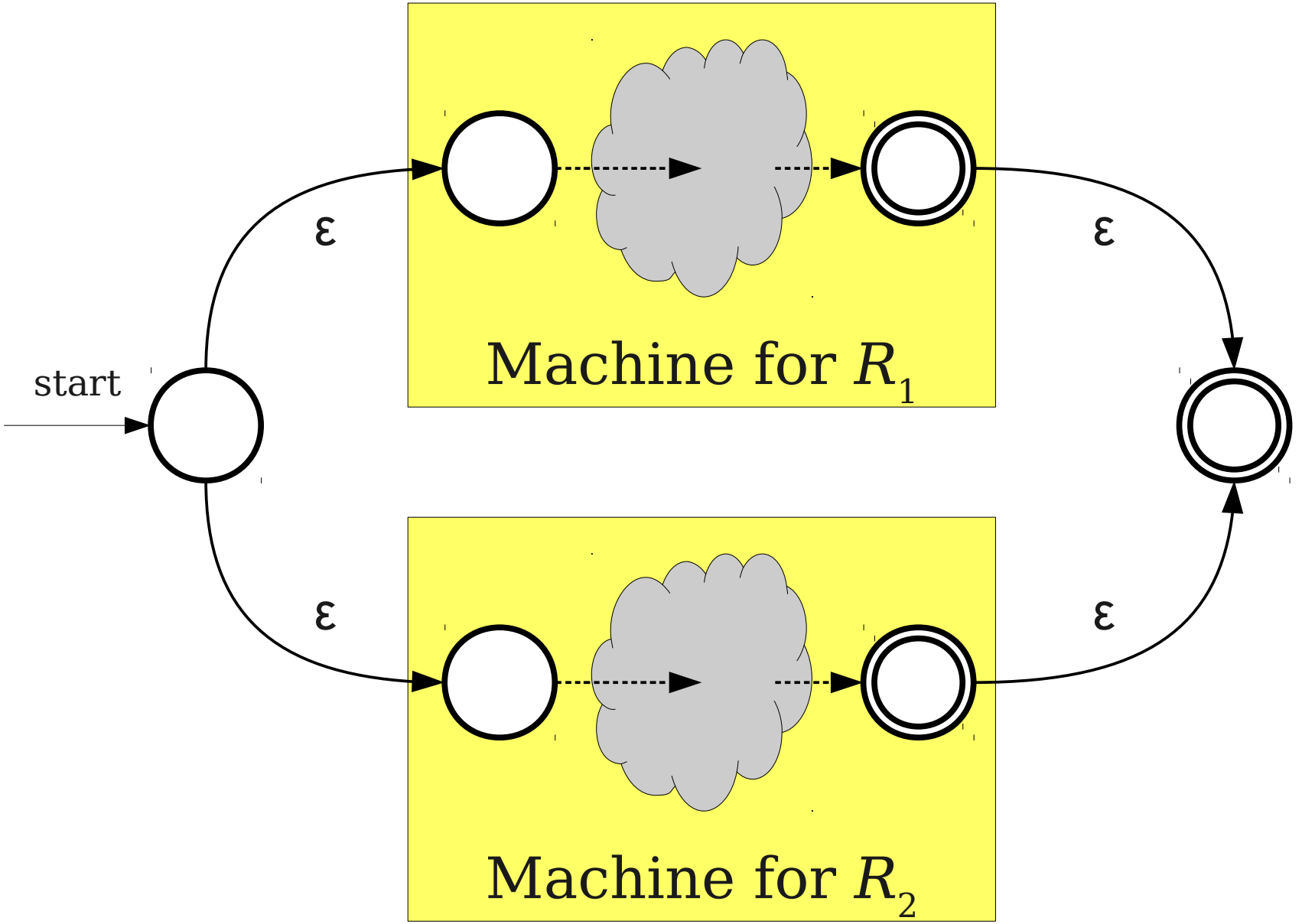
# Construction for $R_1 \mid R_2$



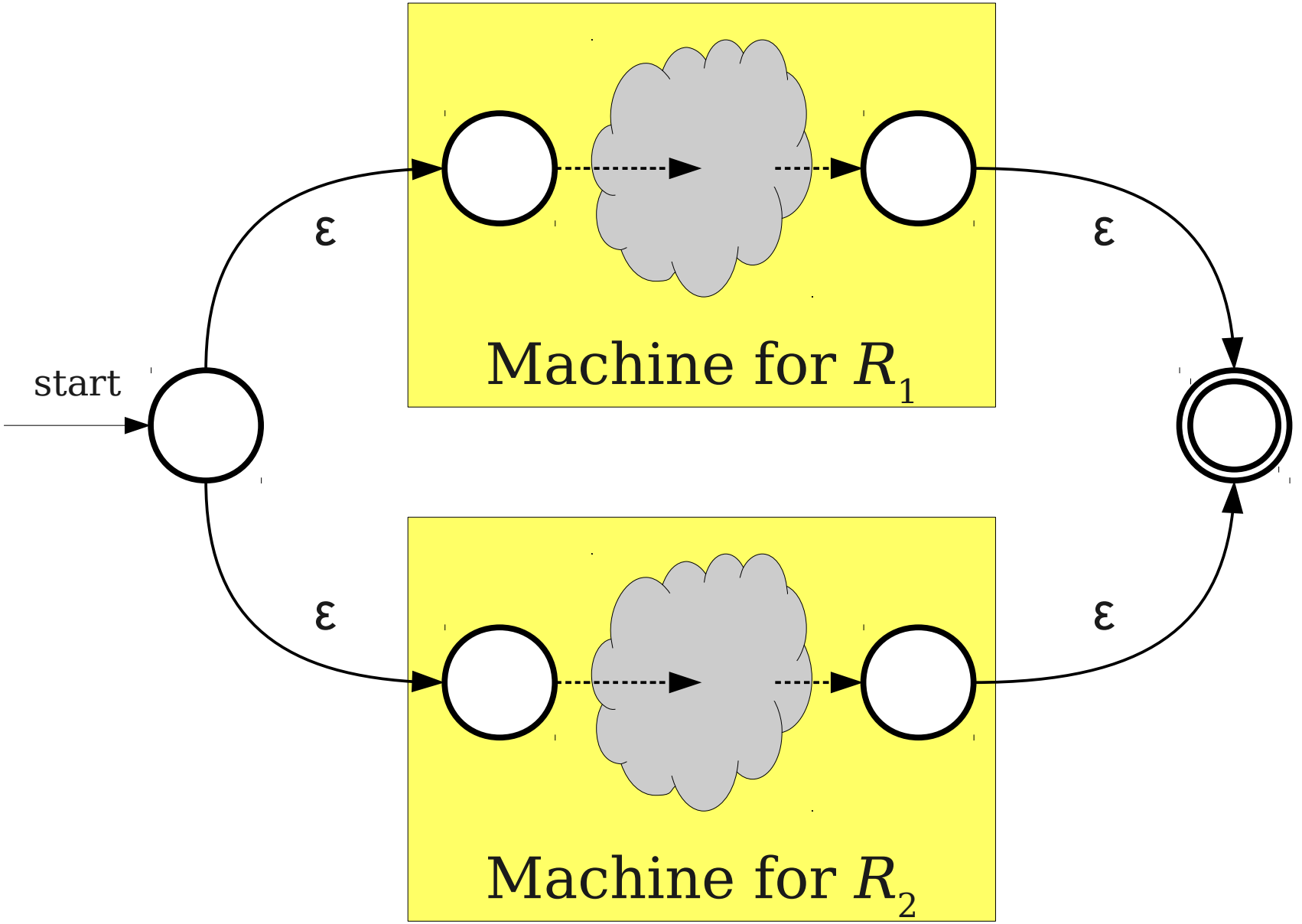
# Construction for $R_1 \mid R_2$



# Construction for $R_1 \mid R_2$

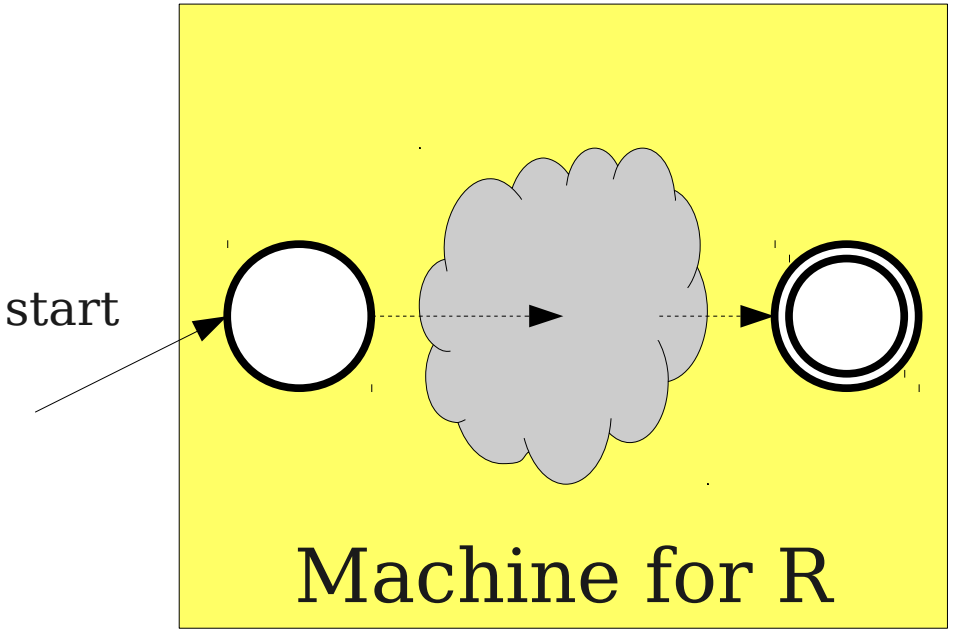


# Construction for $R_1 \mid R_2$

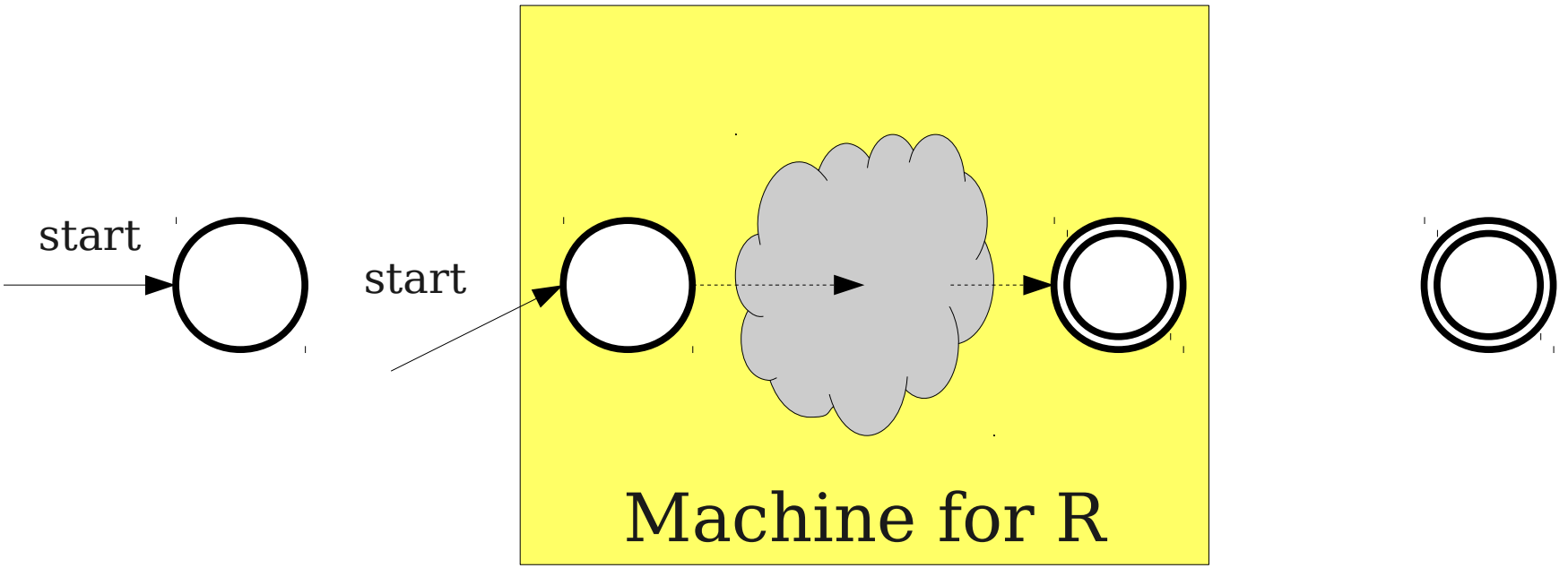


# Construction for $R^*$

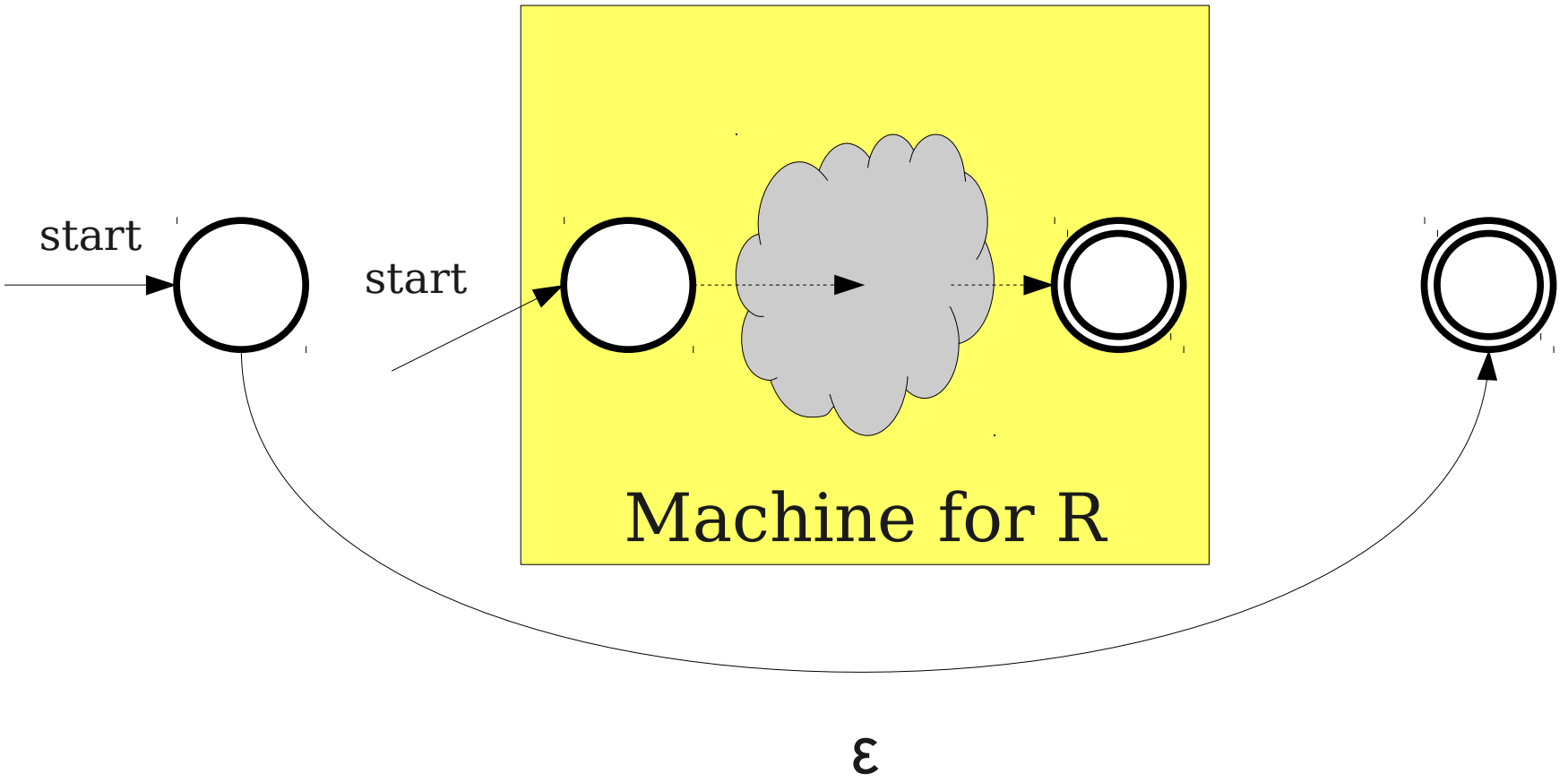
# Construction for $R^*$



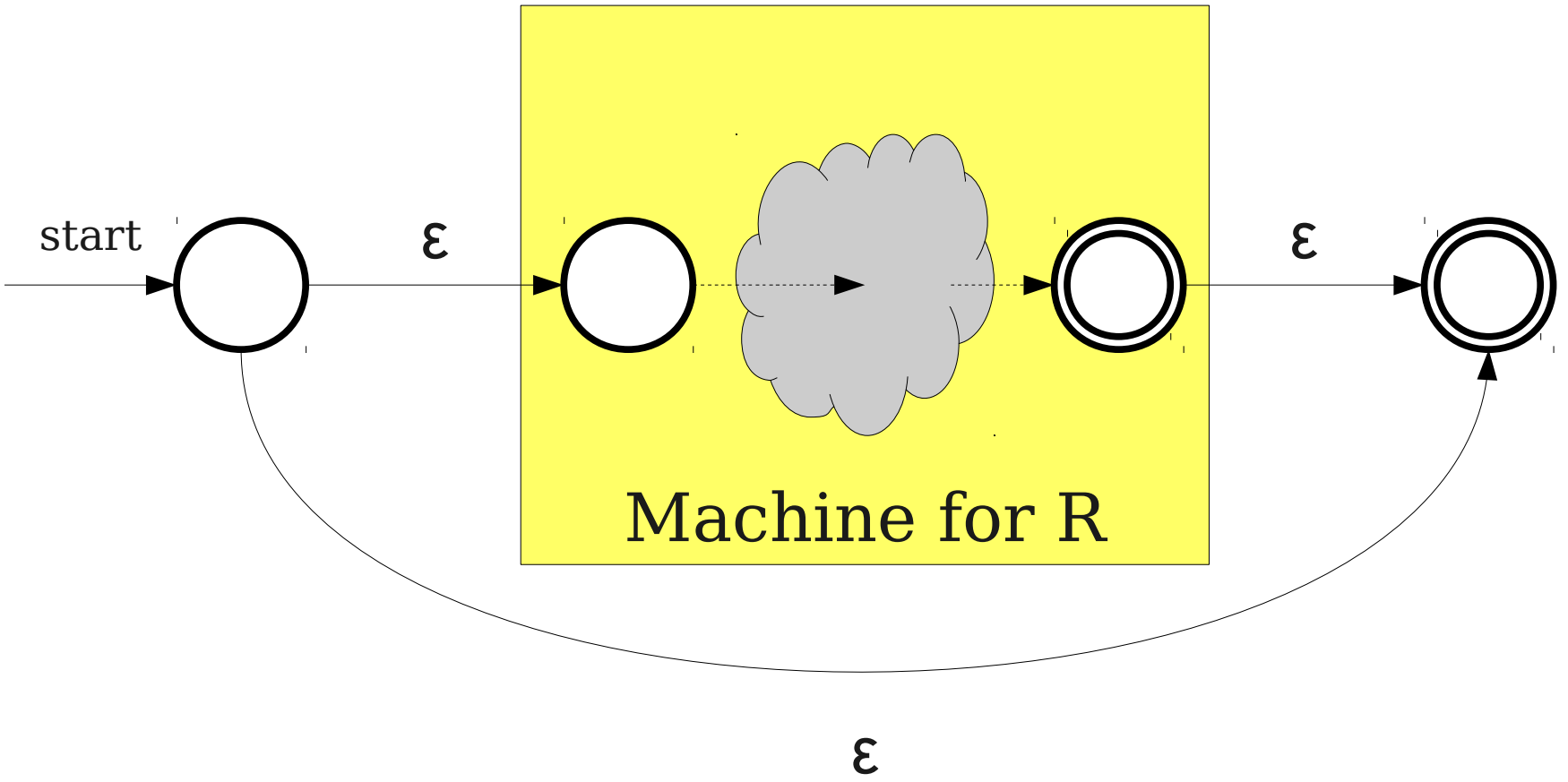
# Construction for $R^*$



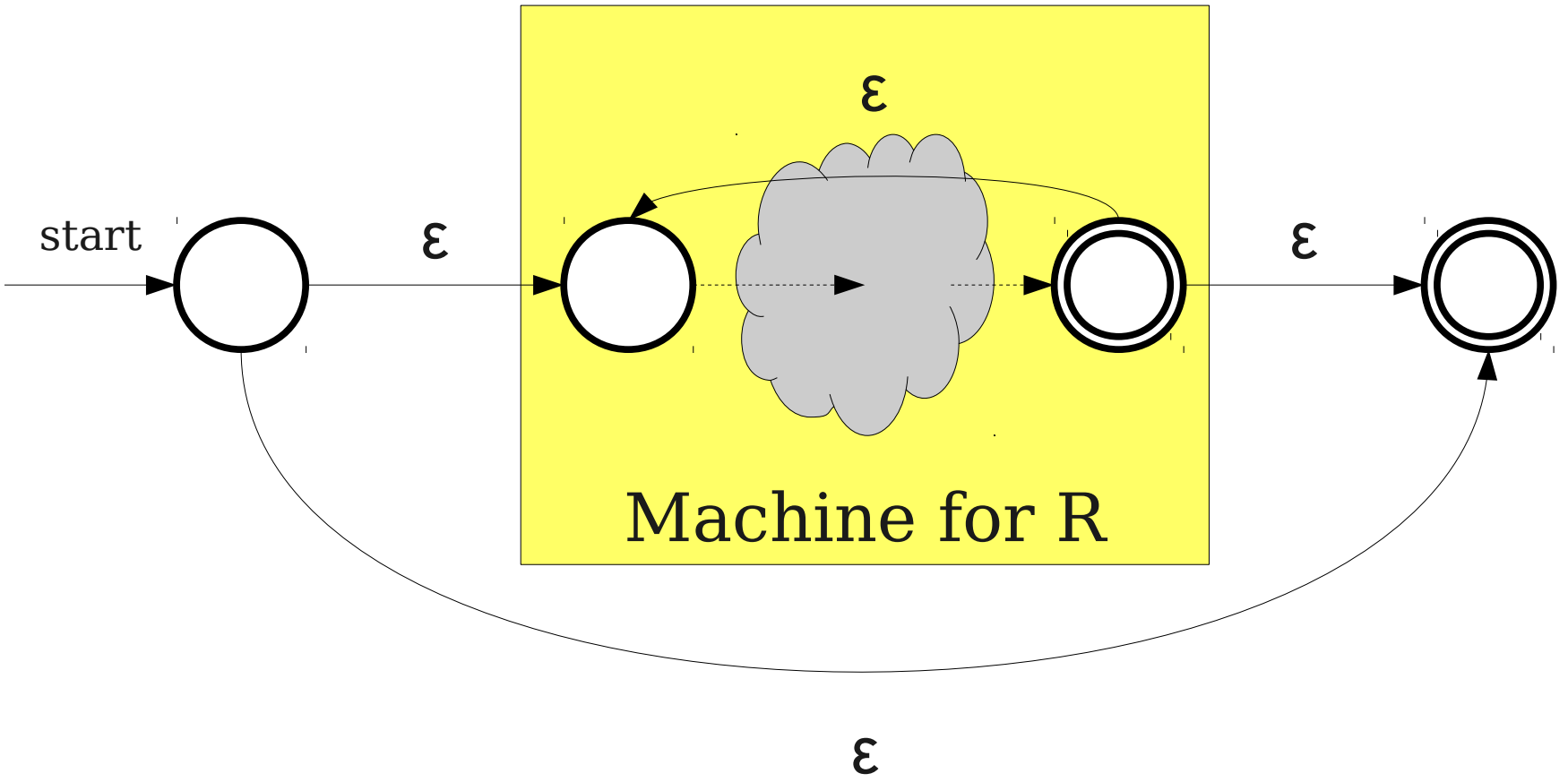
# Construction for $R^*$



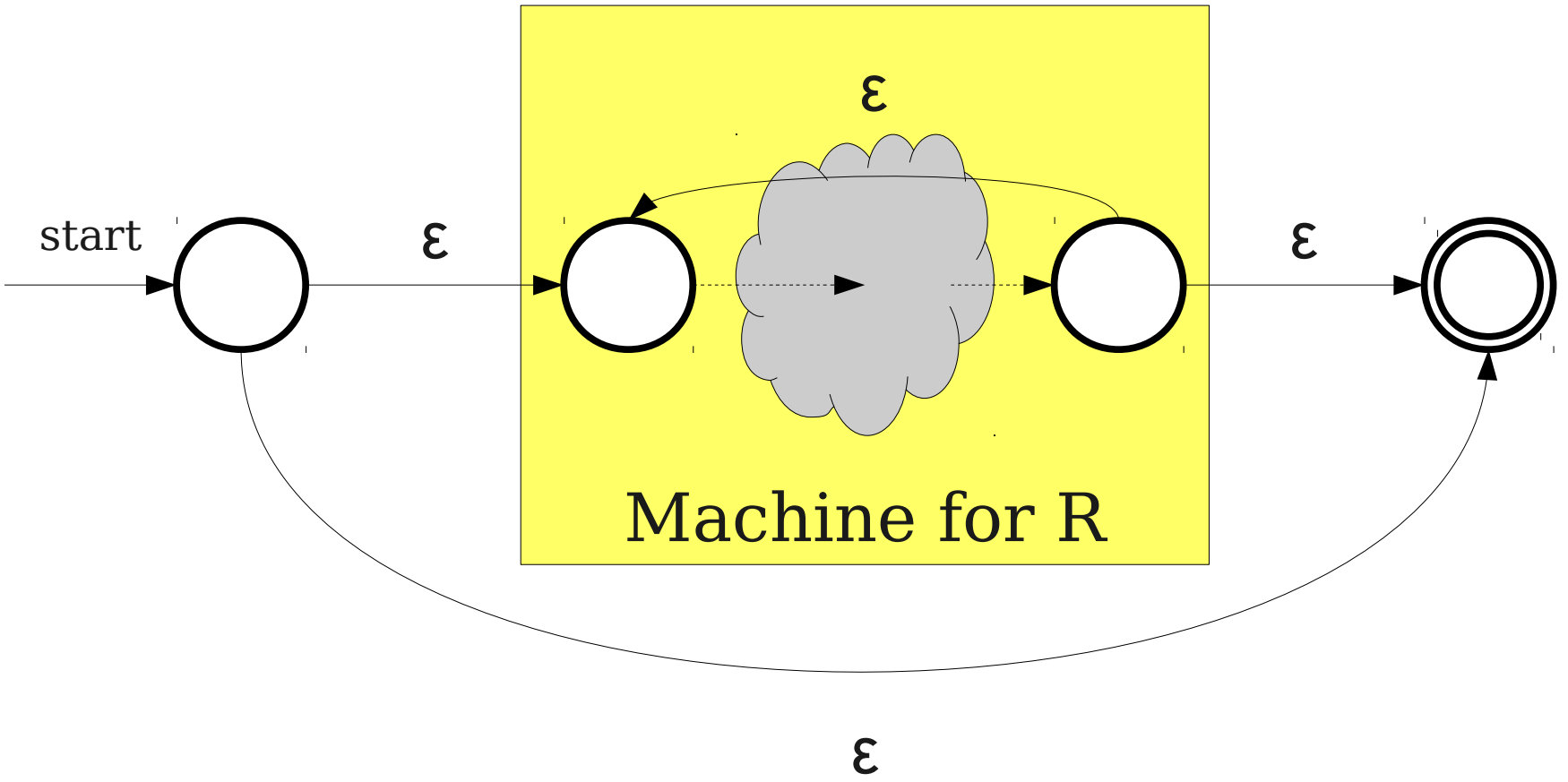
# Construction for $R^*$



# Construction for $R^*$



# Construction for $R^*$



# Why This Matters

- Many software tools work by matching regular expressions against text.
- One possible algorithm for doing so:
  - Convert the regular expression to an NFA.
  - (Optionally) Convert the NFA to a DFA using the subset construction.
  - Run the text through the finite automaton and look for matches.
- Runs extremely quickly!

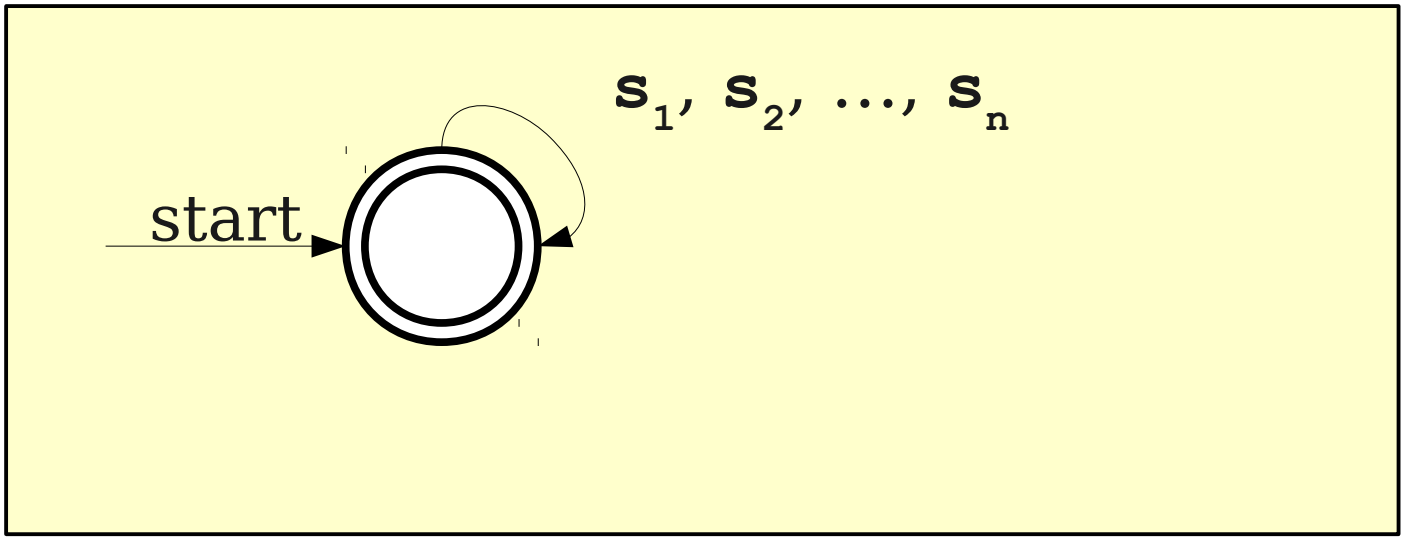
# The Power of Regular Expressions

***Theorem:*** If $L$ is a regular language, then there is a regular expression for $L$.

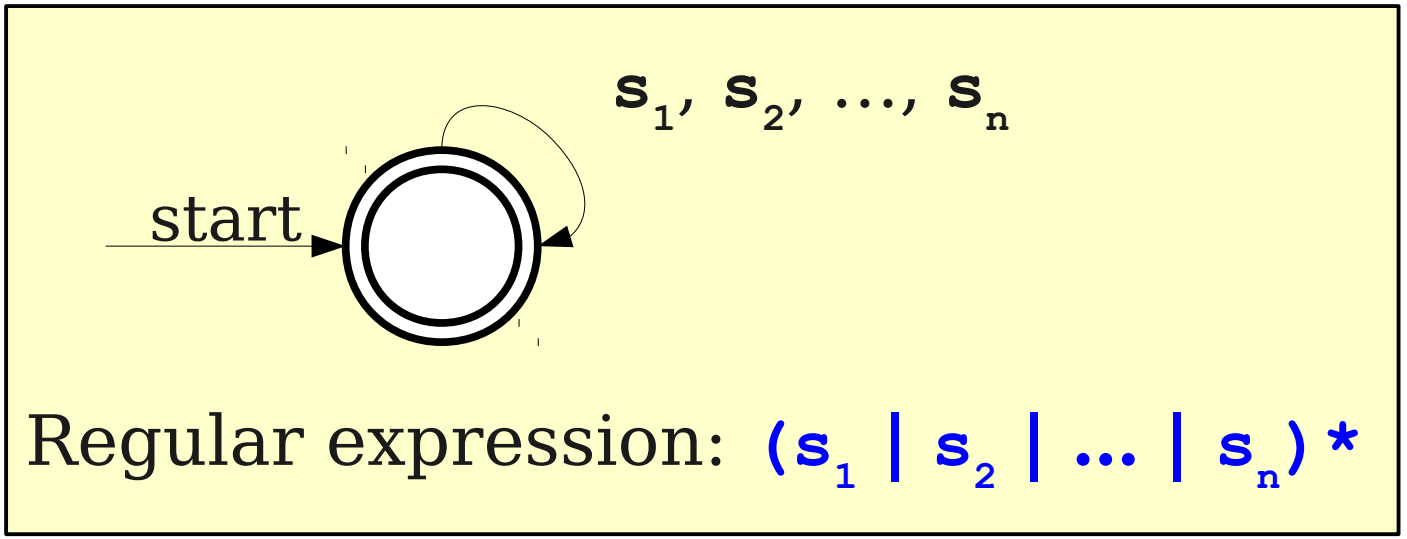
***This is not obvious!***

***Proof idea:*** Show how to convert an arbitrary NFA into a regular expression.

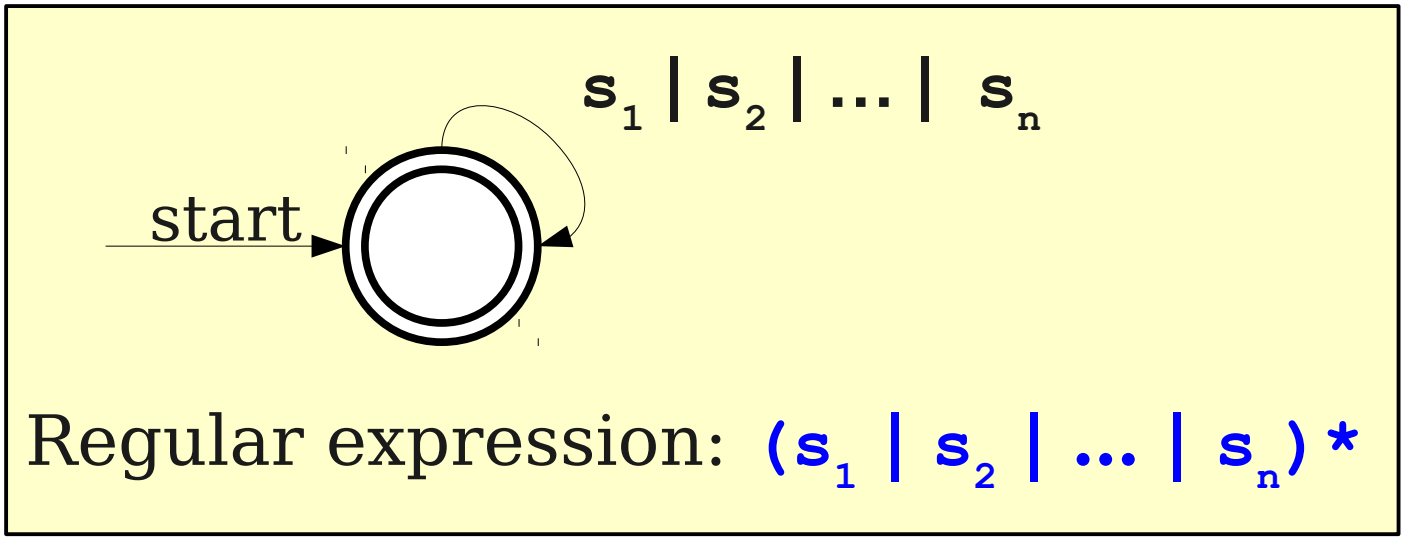
# From NFAs to Regular Expressions



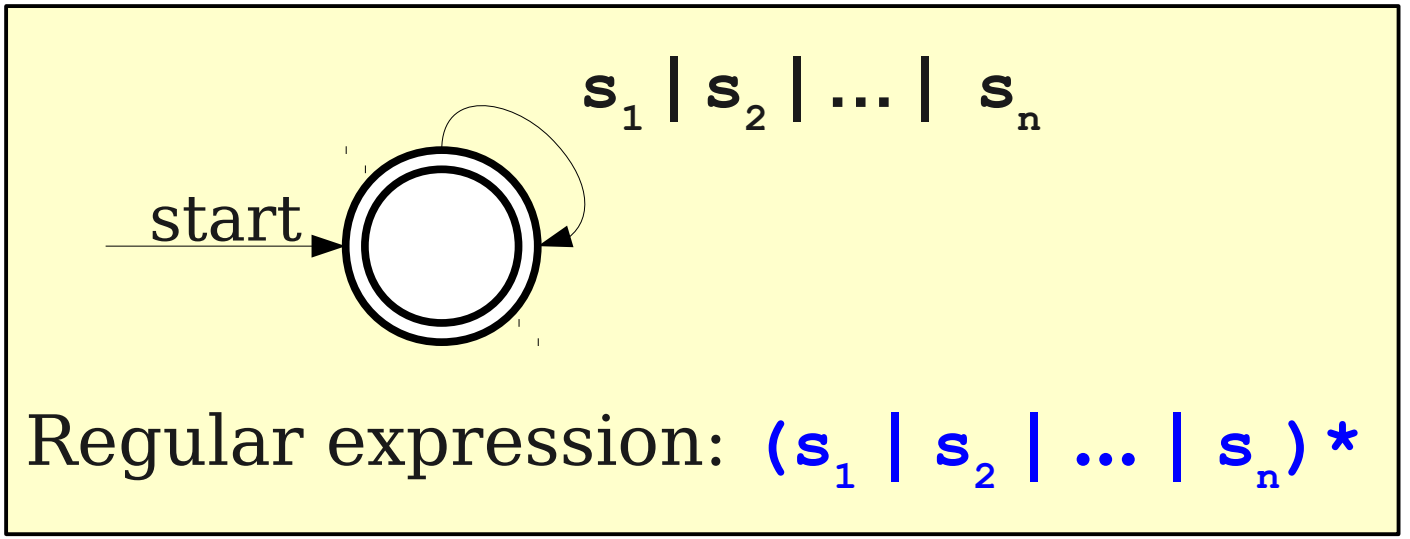
# From NFAs to Regular Expressions



# From NFAs to Regular Expressions



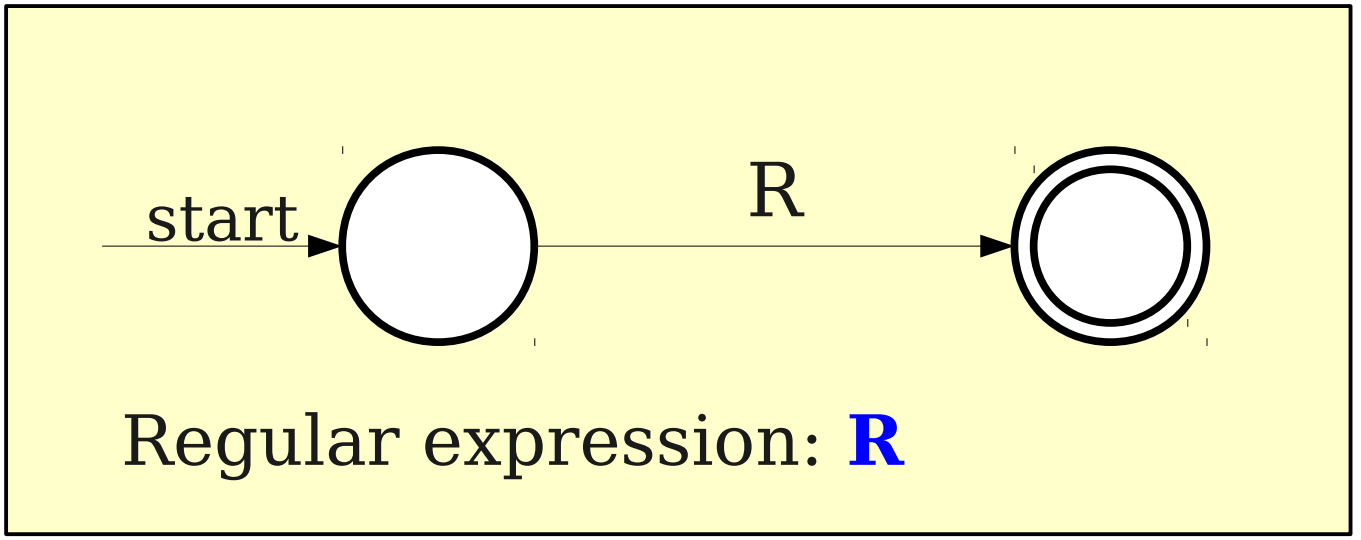
# From NFAs to Regular Expressions



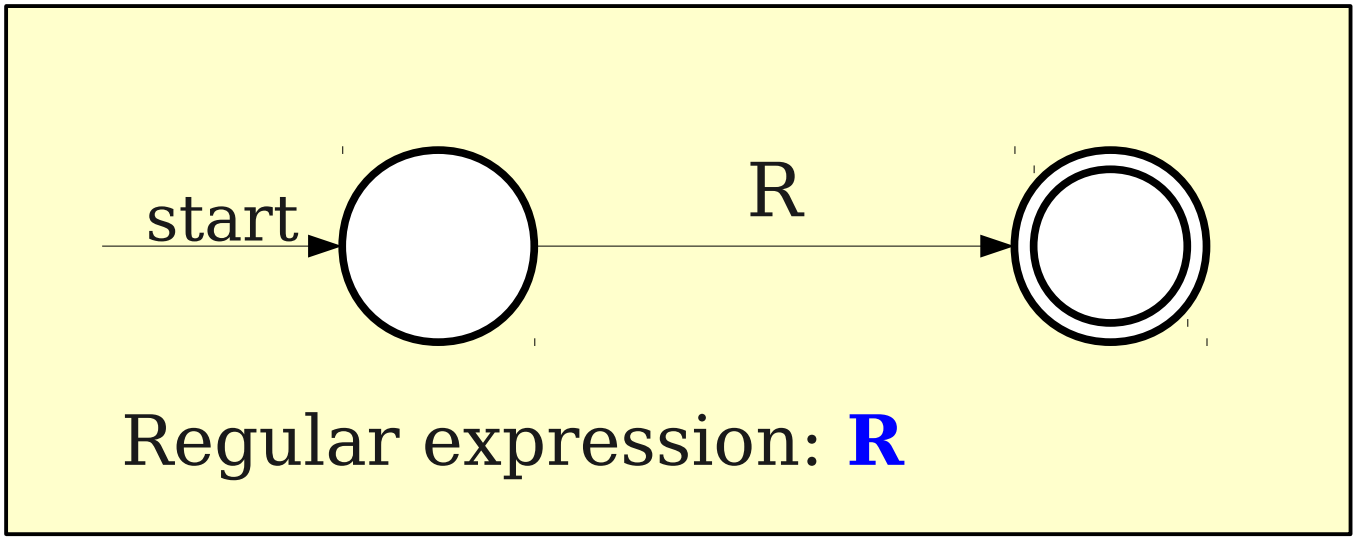

Key idea: Label transitions with arbitrary regular expressions.

# From NFAs to Regular Expressions

# From NFAs to Regular Expressions

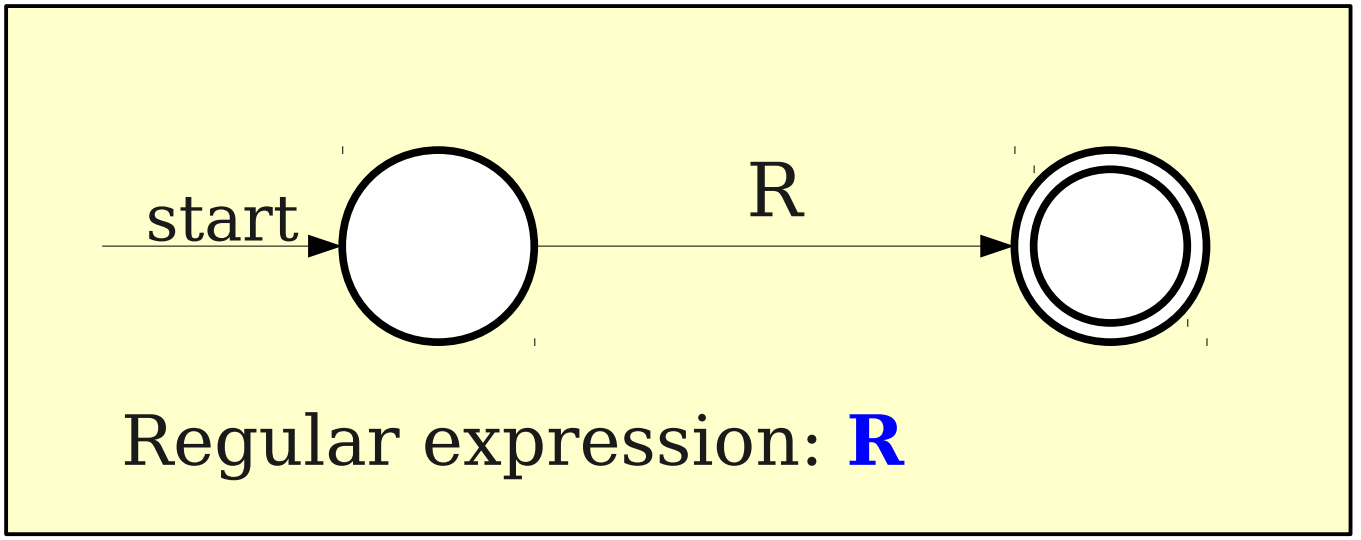


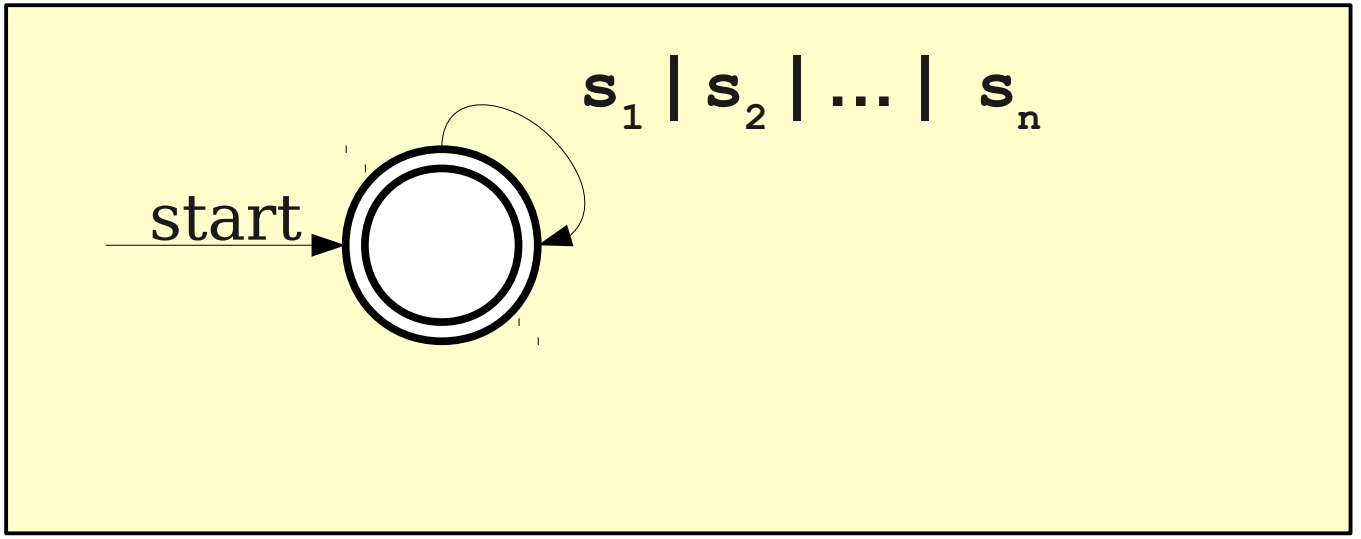
# From NFAs to Regular Expressions



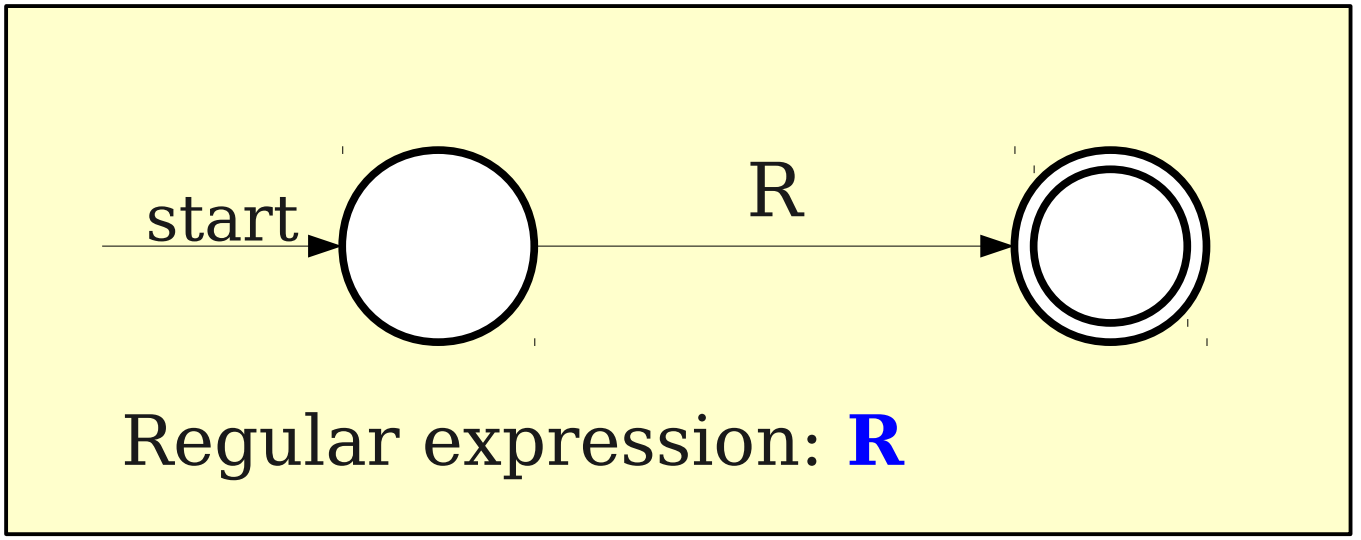

Key idea: If we can convert any NFA into something that looks like this, we can easily read off the regular expression.

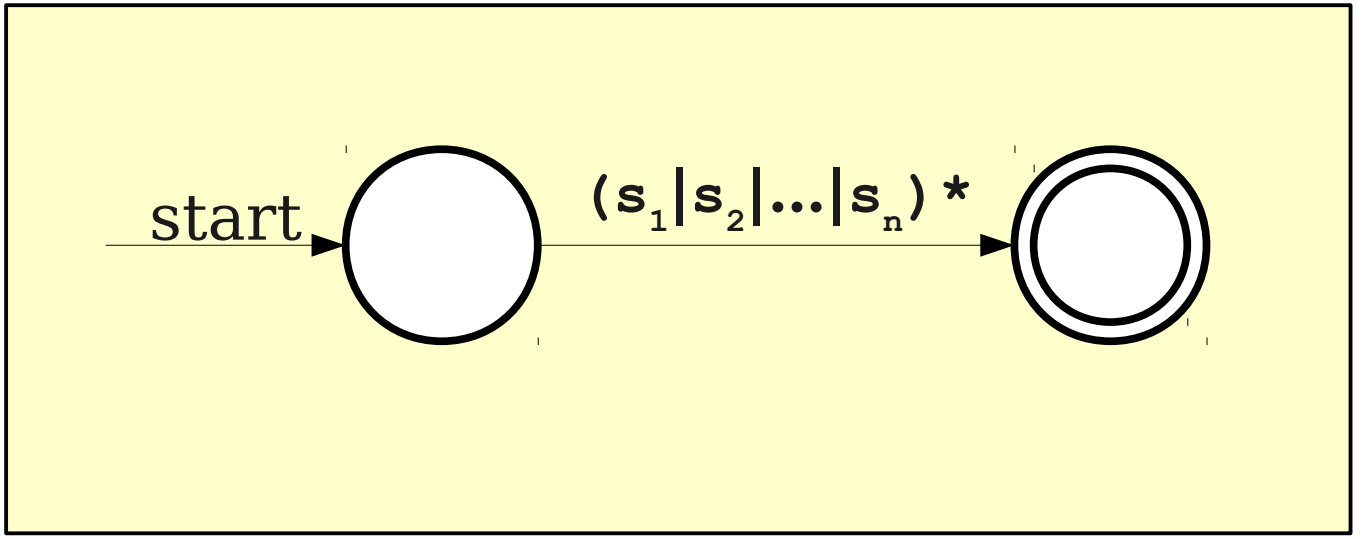
# From NFAs to Regular Expressions



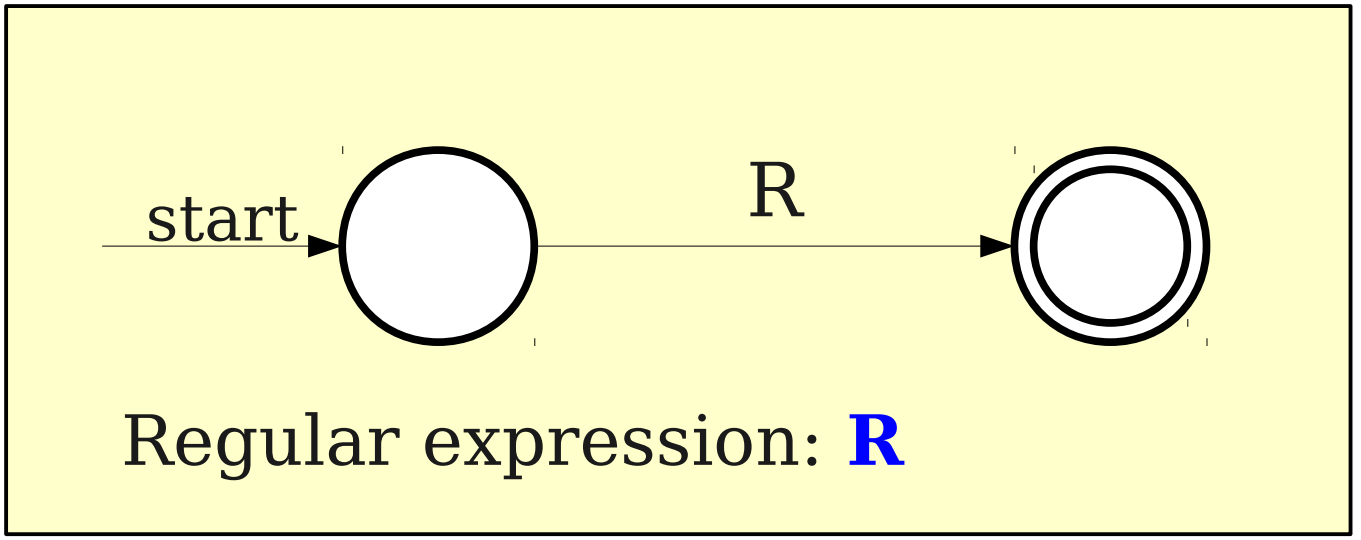



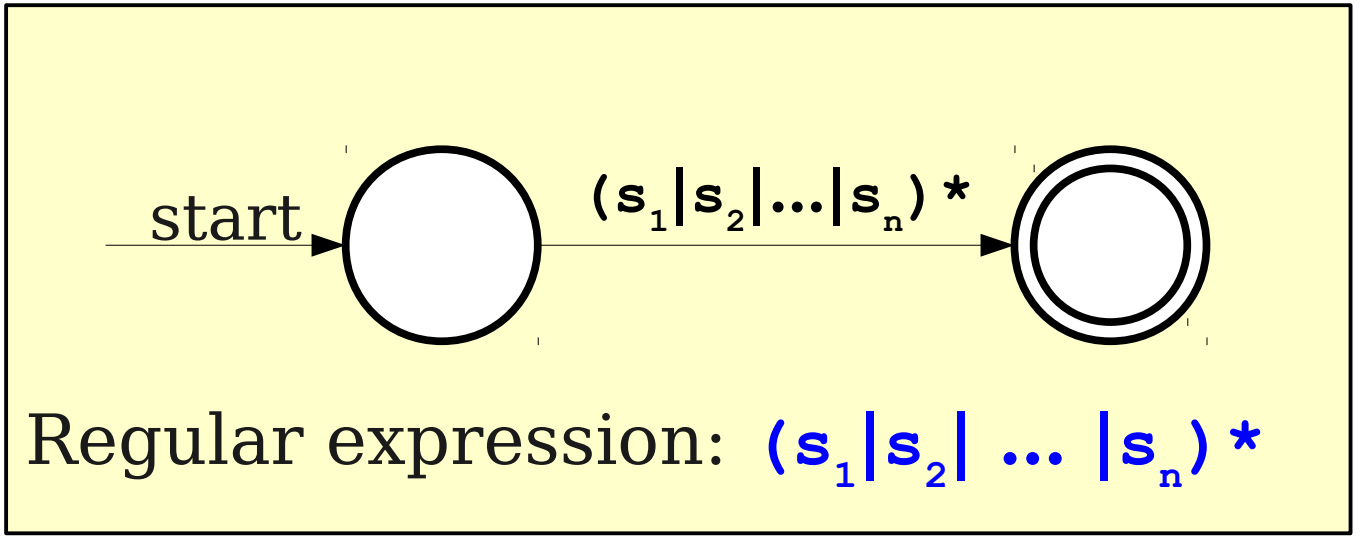
# From NFAs to Regular Expressions



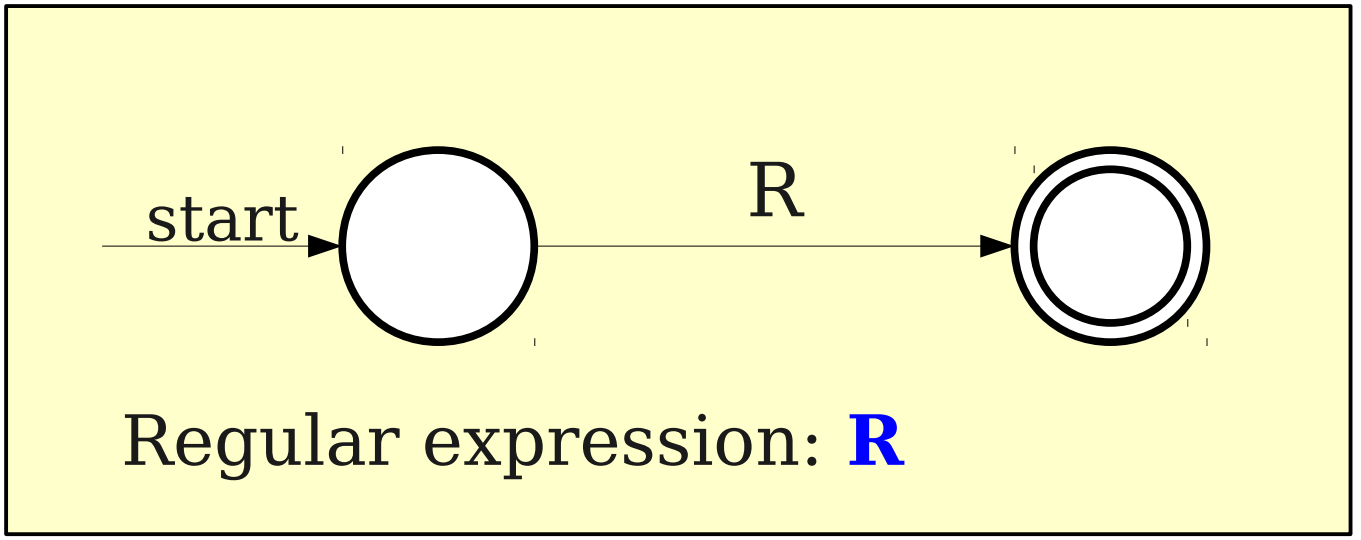



# From NFAs to Regular Expressions

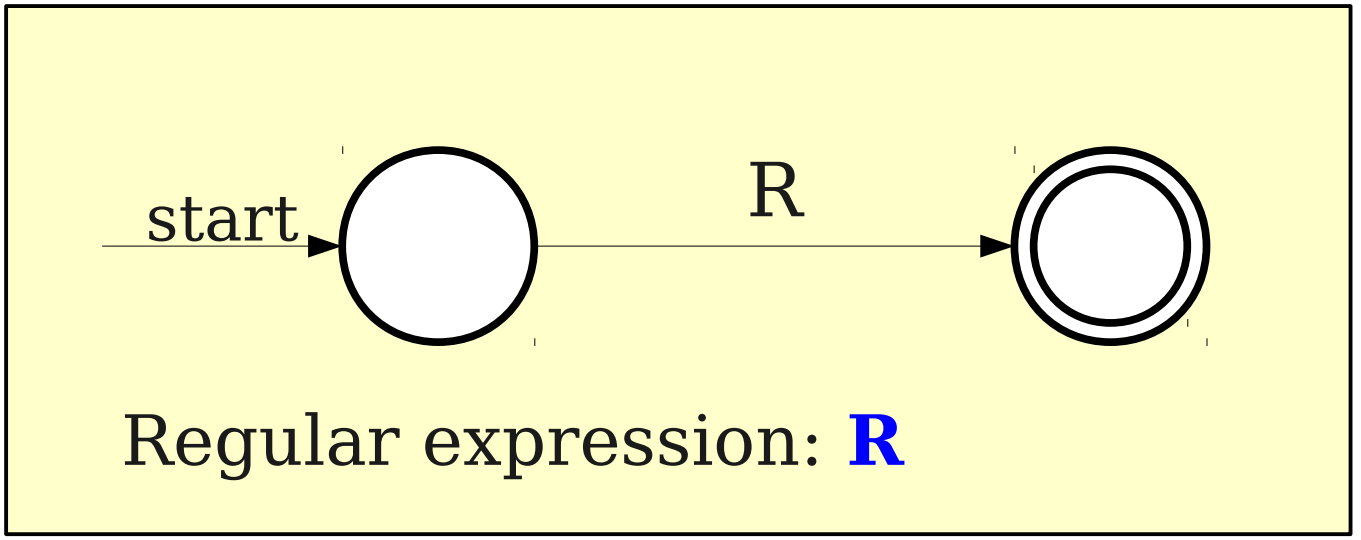


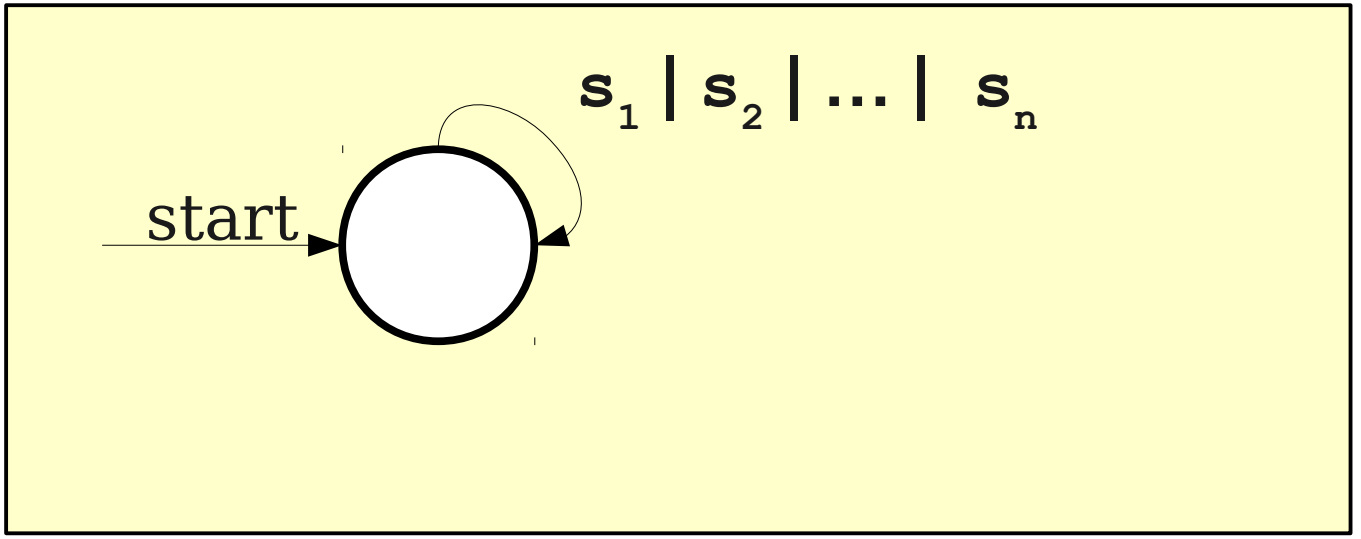



# From NFAs to Regular Expressions

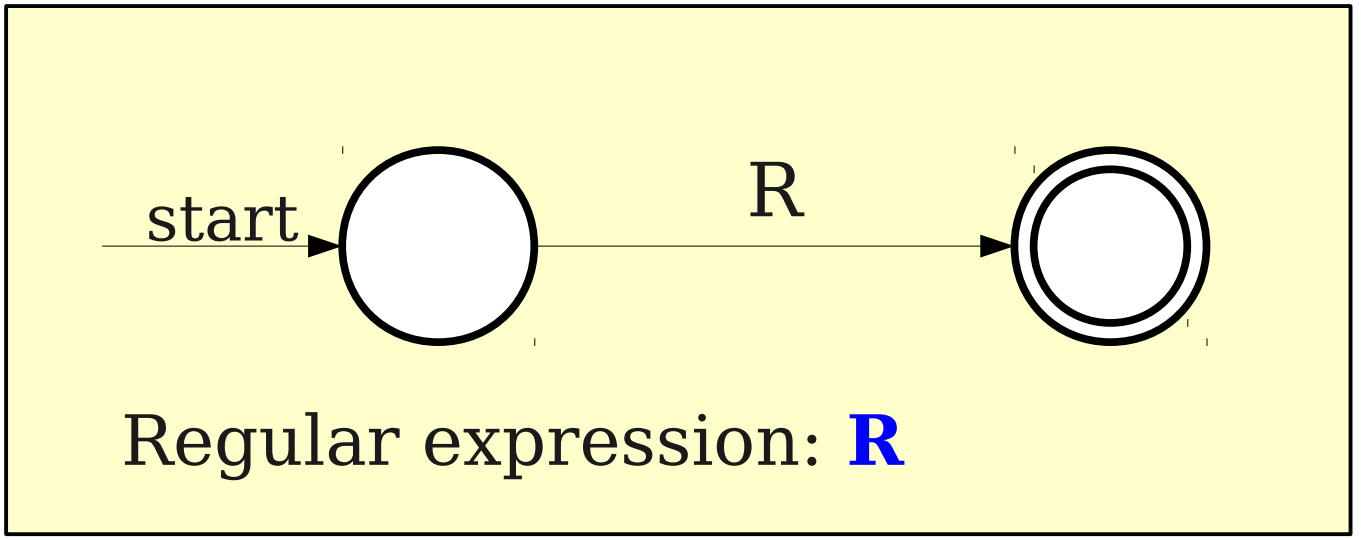


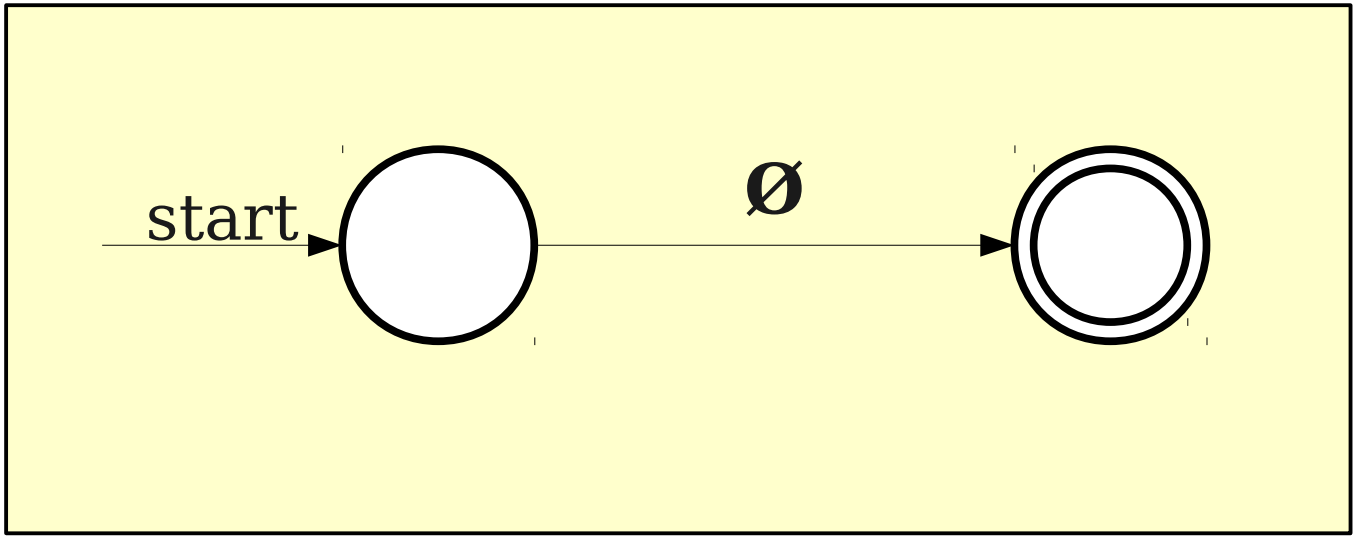
# From NFAs to Regular Expressions



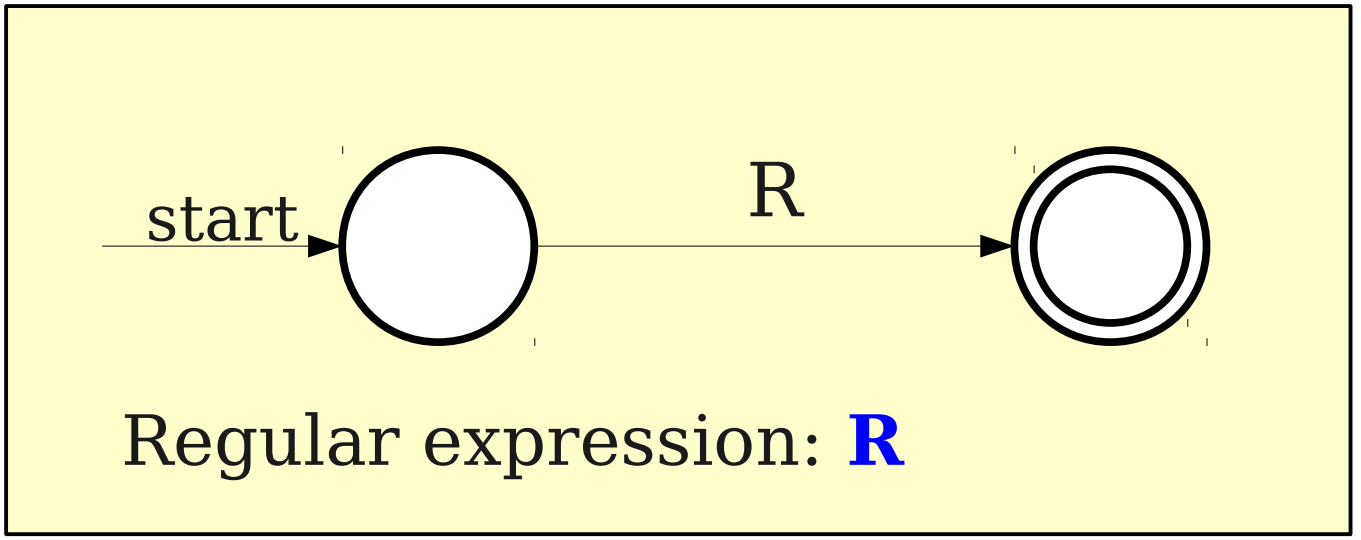



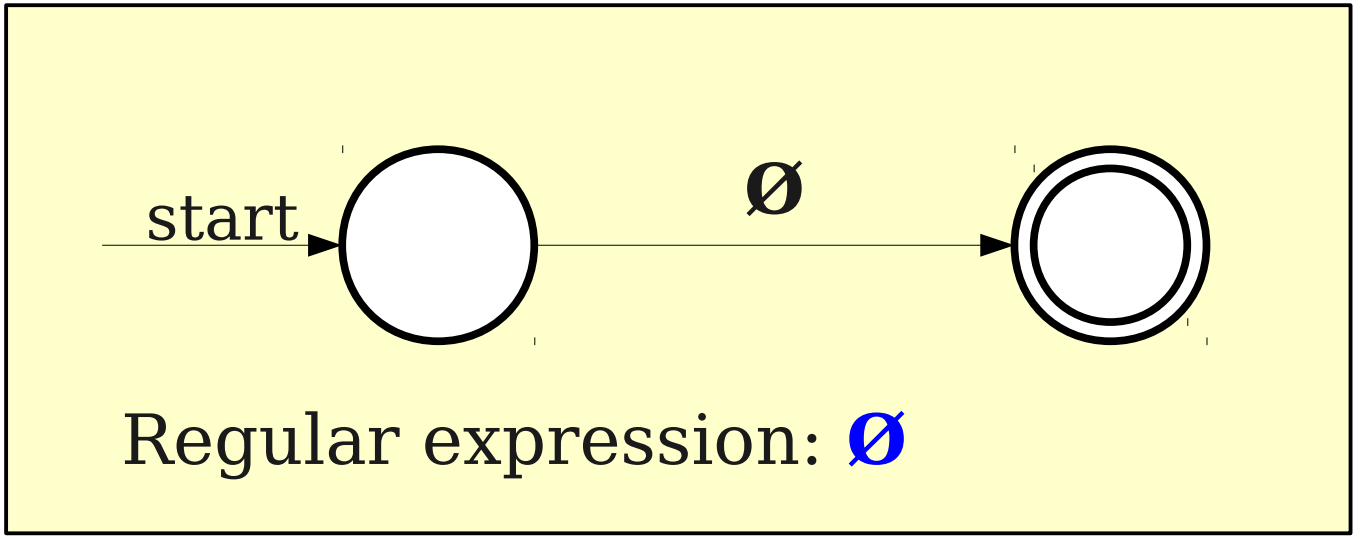
# From NFAs to Regular Expressions



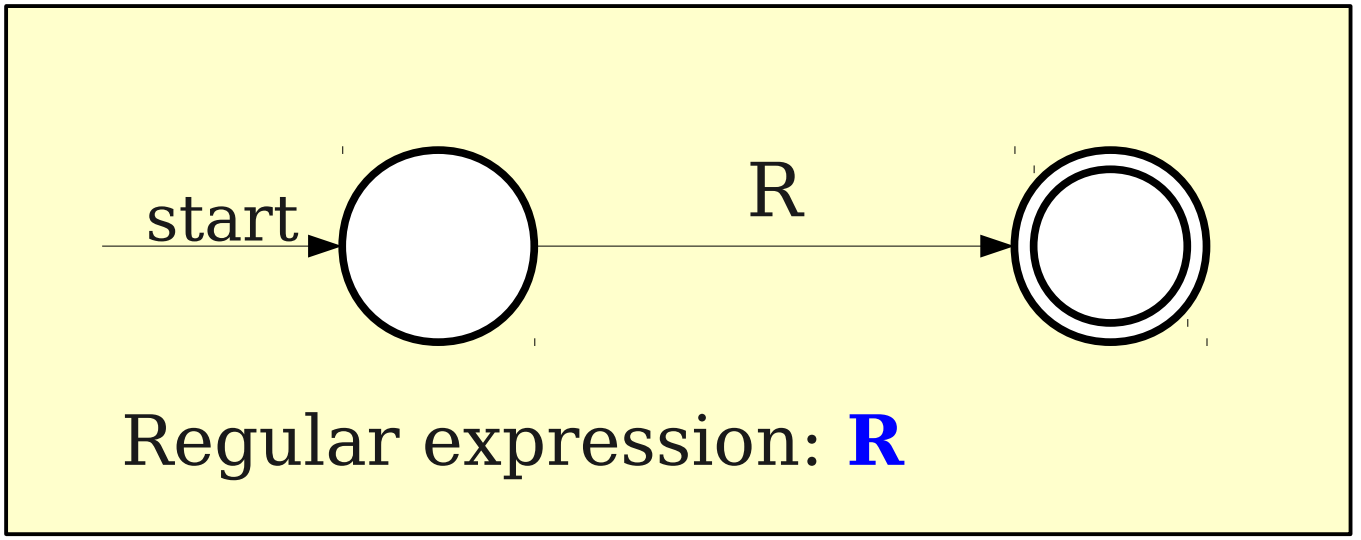



# From NFAs to Regular Expressions

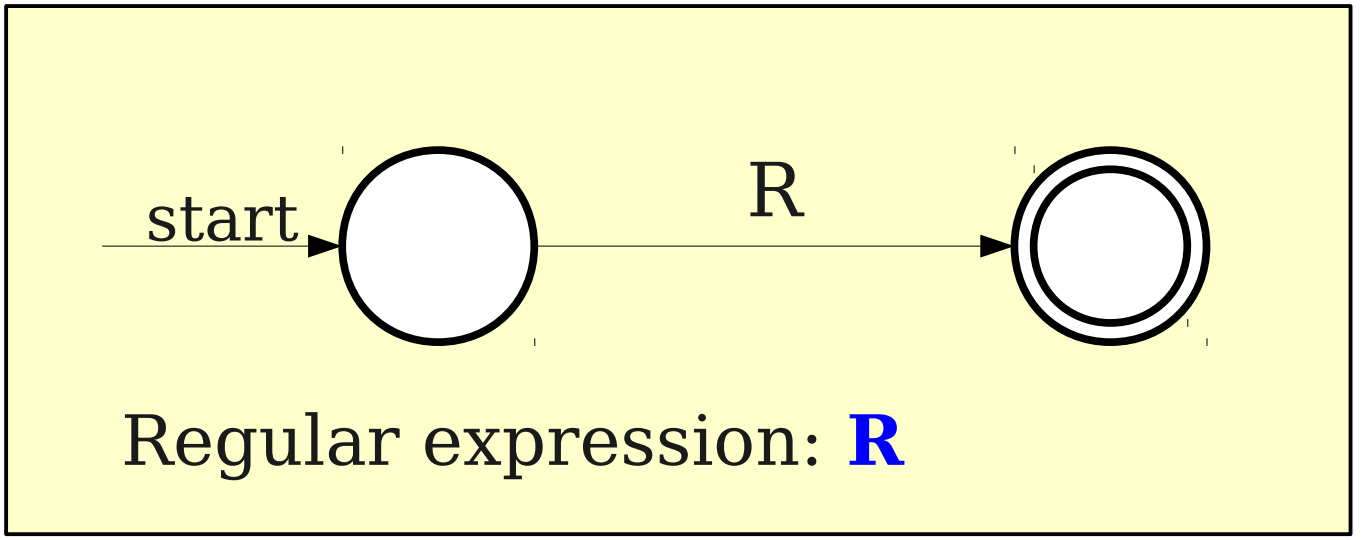


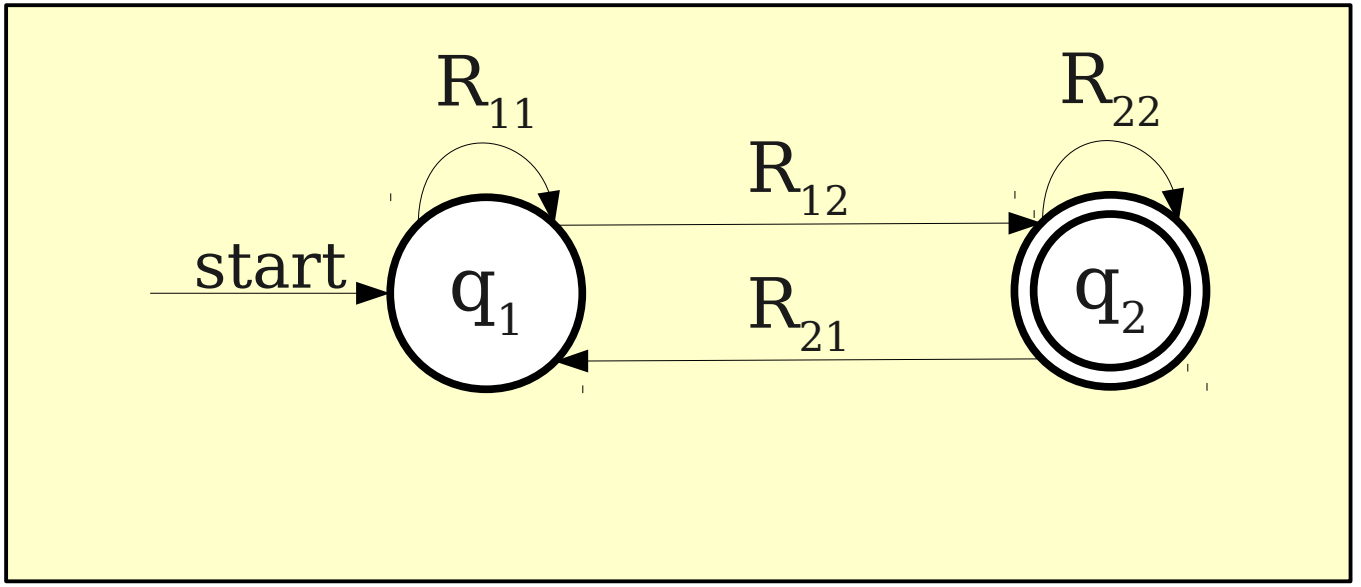



# From NFAs to Regular Expressions

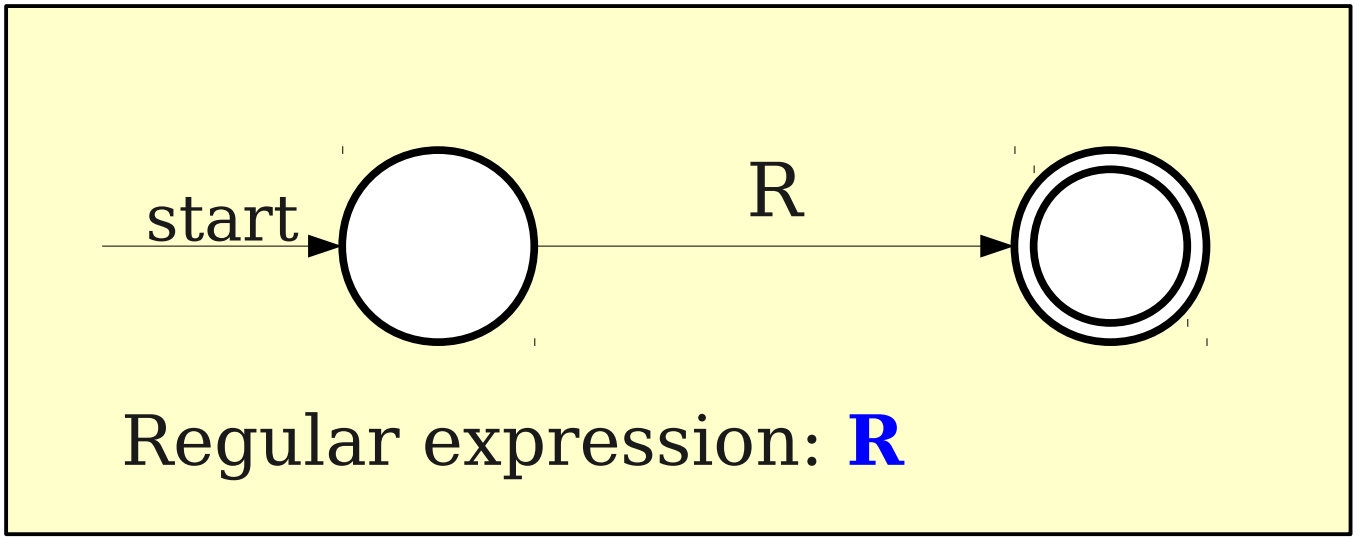


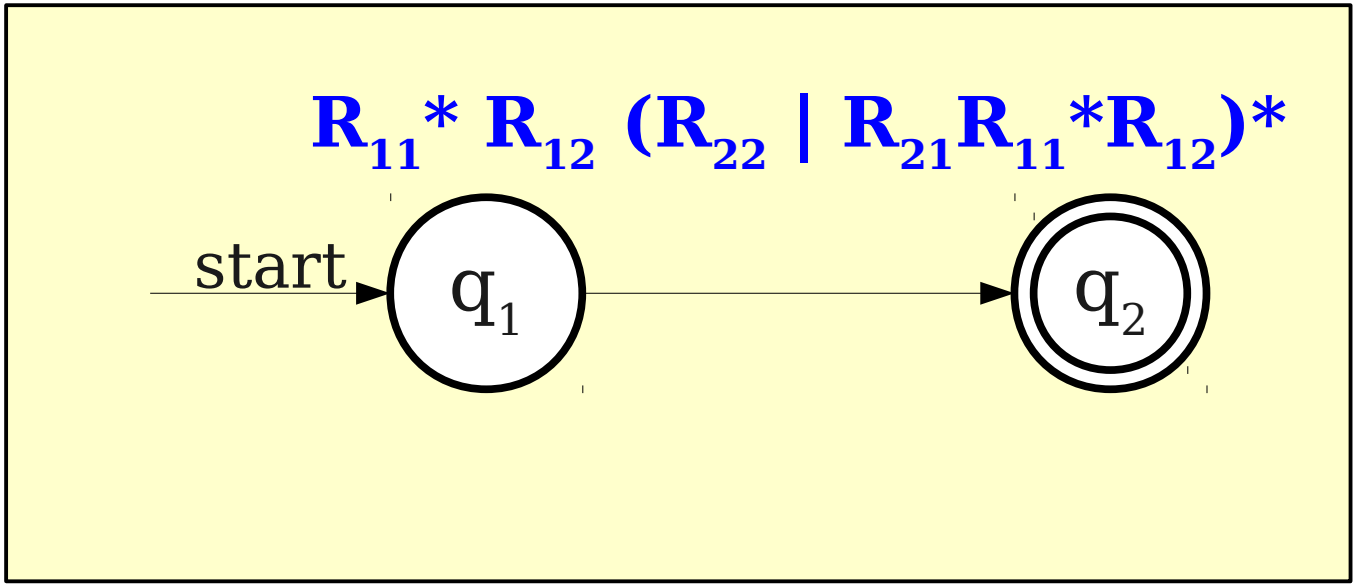
# From NFAs to Regular Expressions



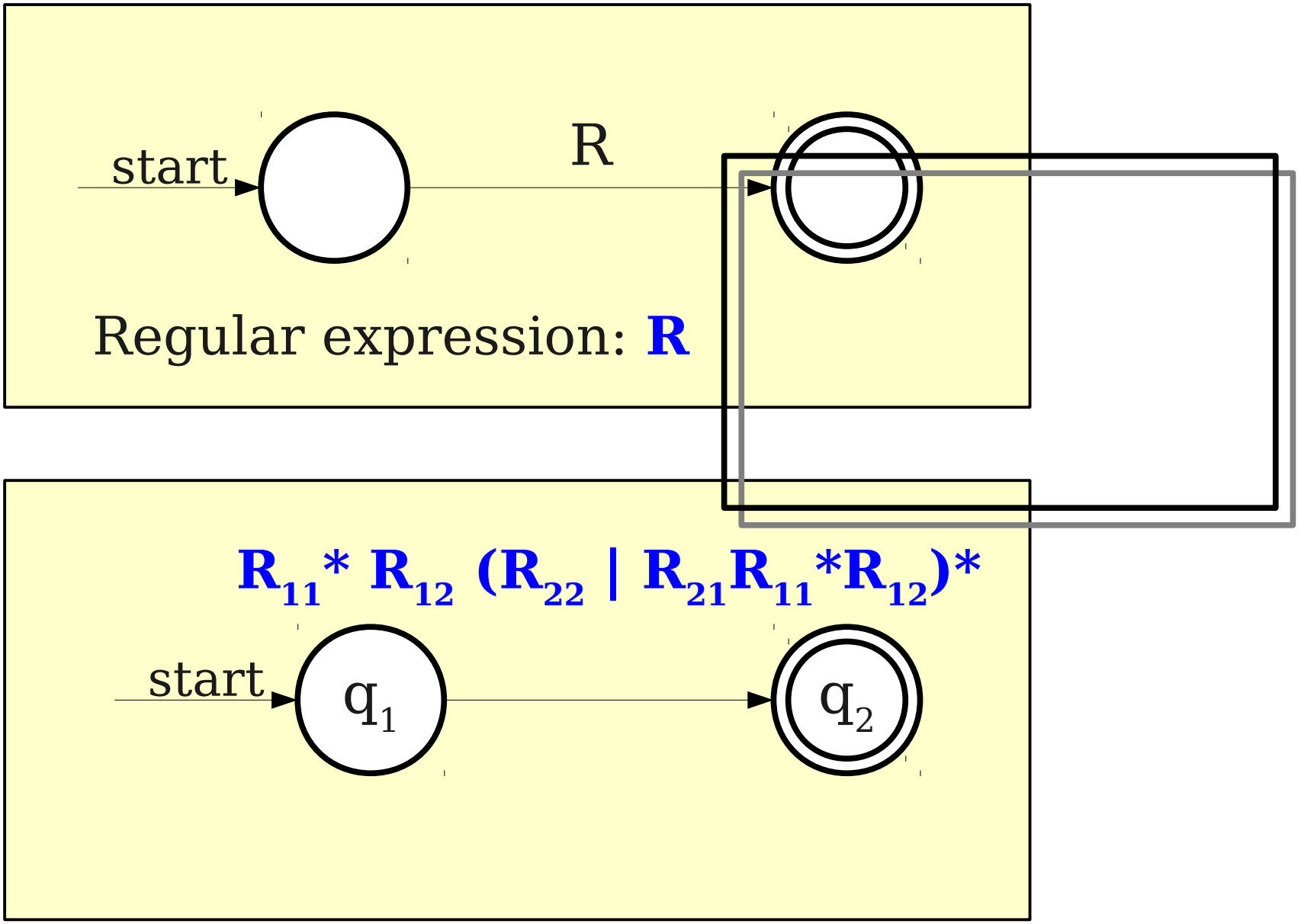



# From NFAs to Regular Expressions

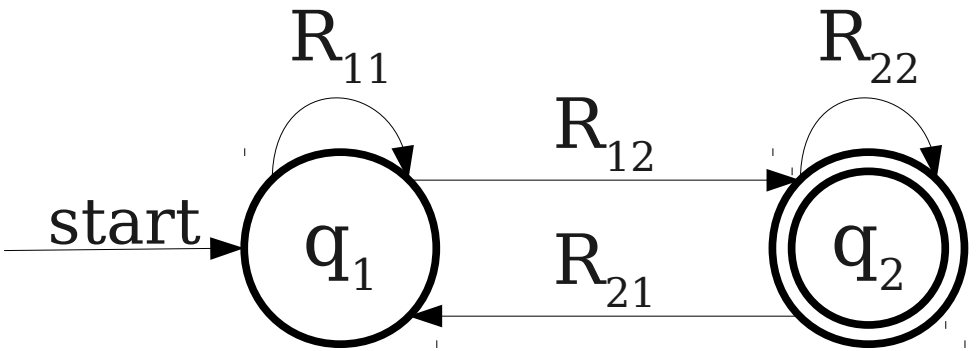


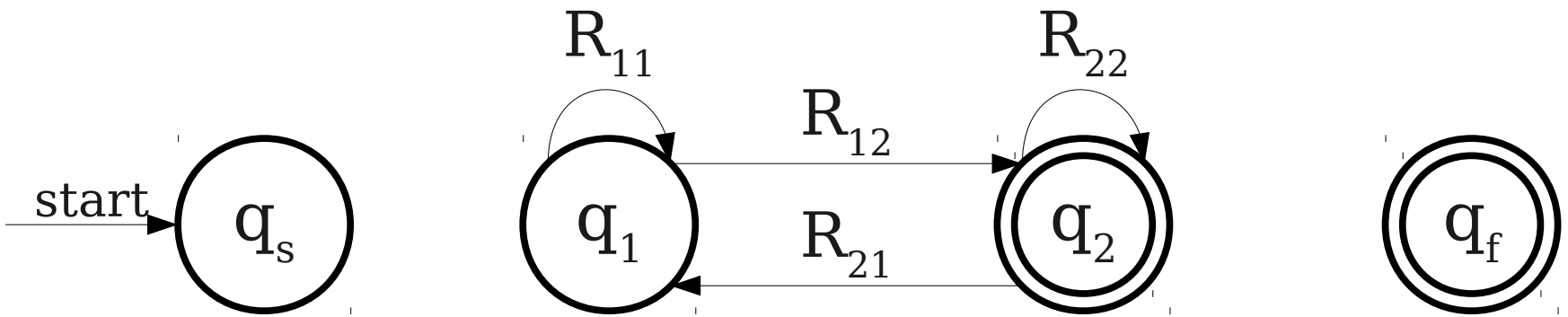
# From NFAs to Regular Expressions



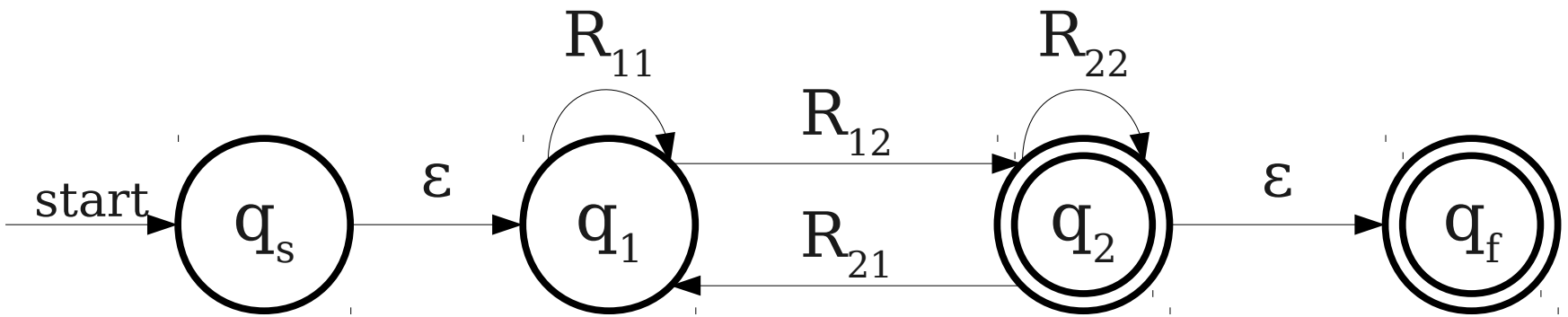
# From NFAs to Regular Expressions



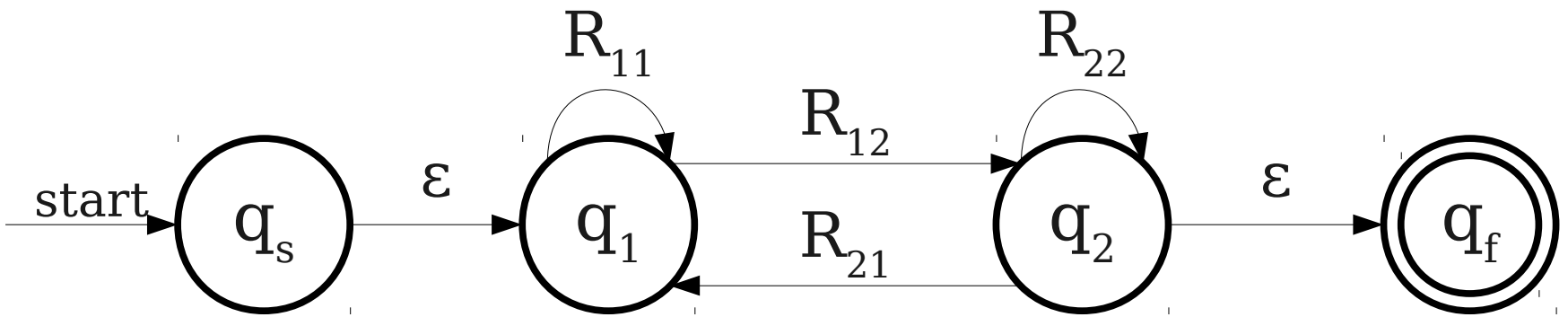
# From NFAs to Regular Expressions



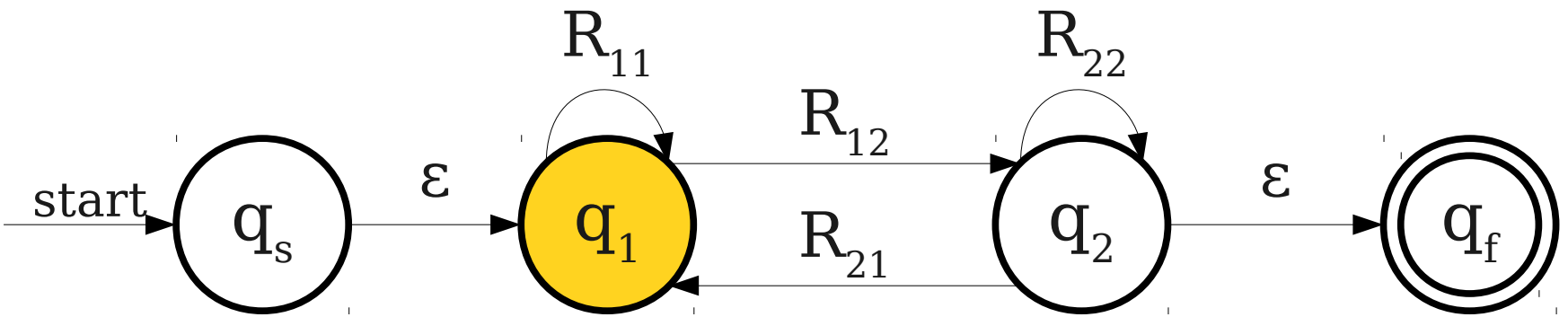
# From NFAs to Regular Expressions



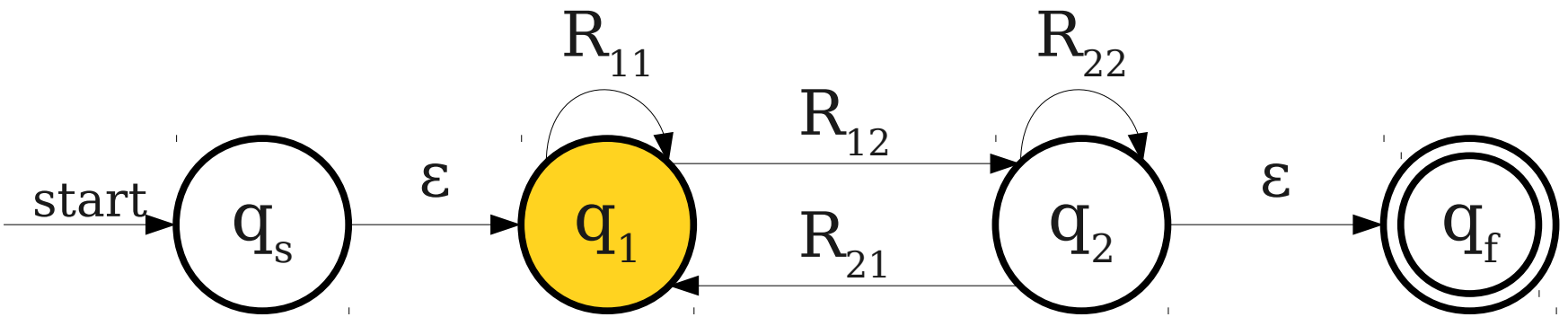
# From NFAs to Regular Expressions



# From NFAs to Regular Expressions

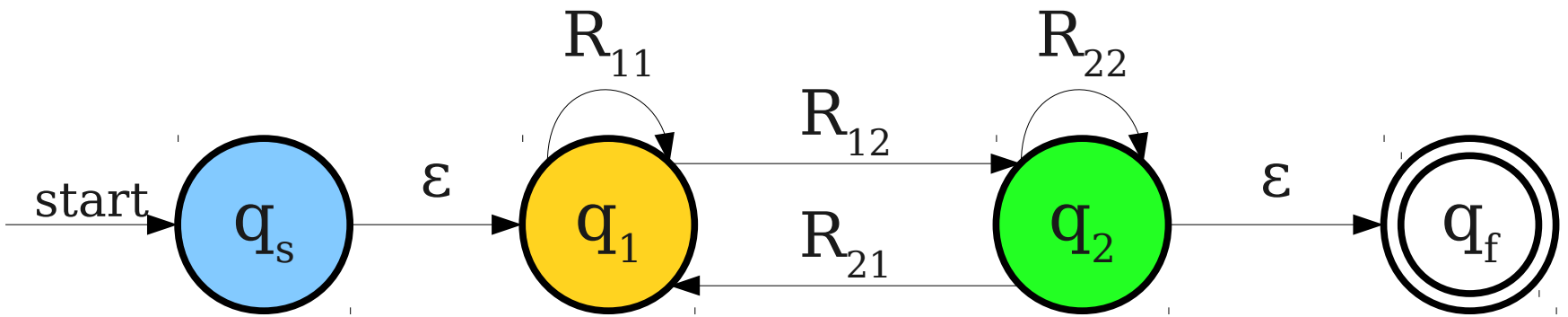


# From NFAs to Regular Expressions

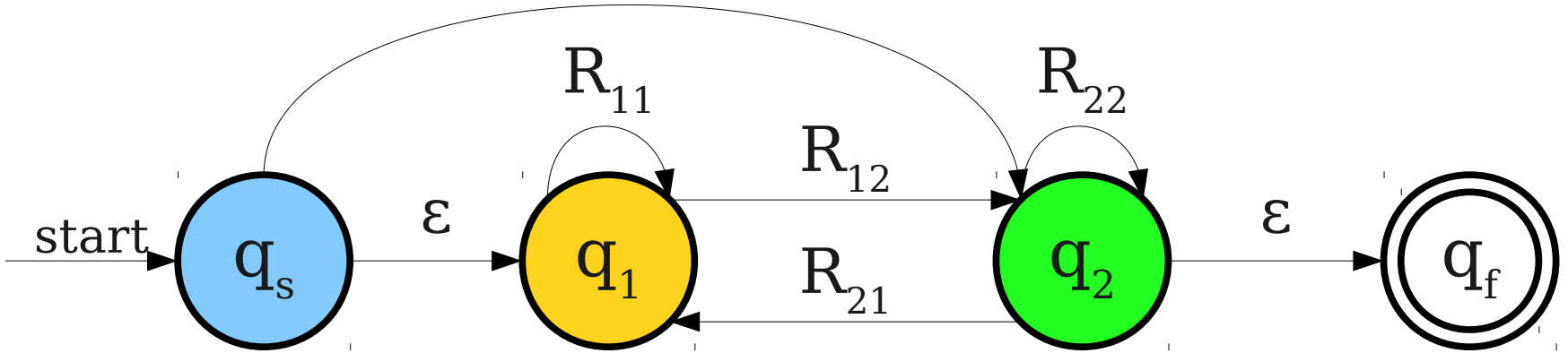



Could we eliminate this state from the NFA?

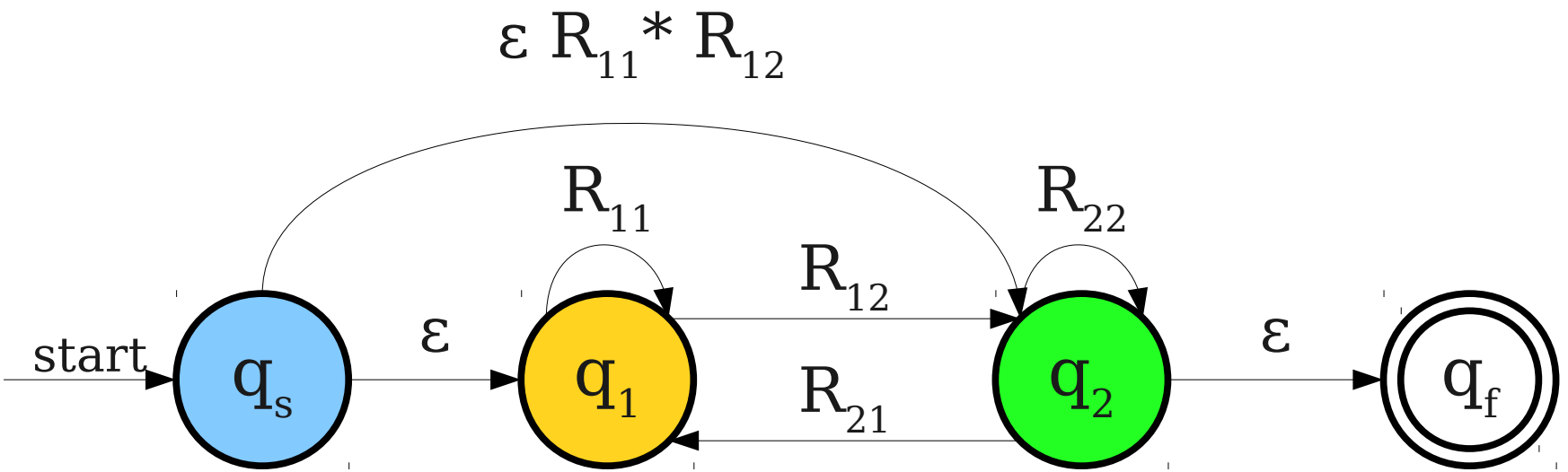
# From NFAs to Regular Expressions



# From NFAs to Regular Expressions

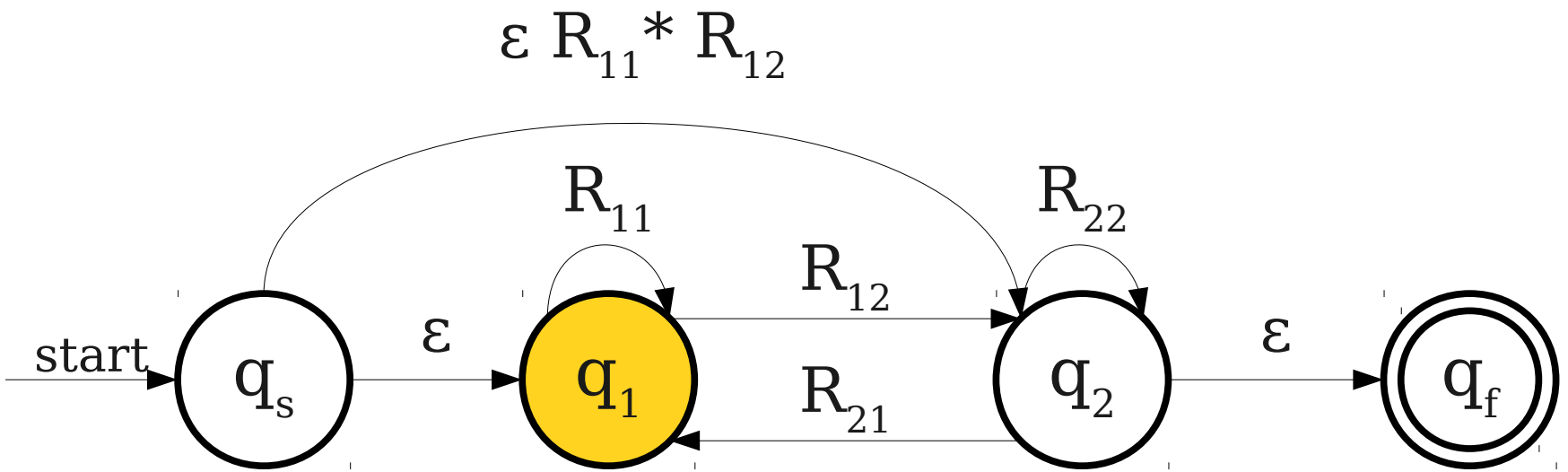


# From NFAs to Regular Expressions

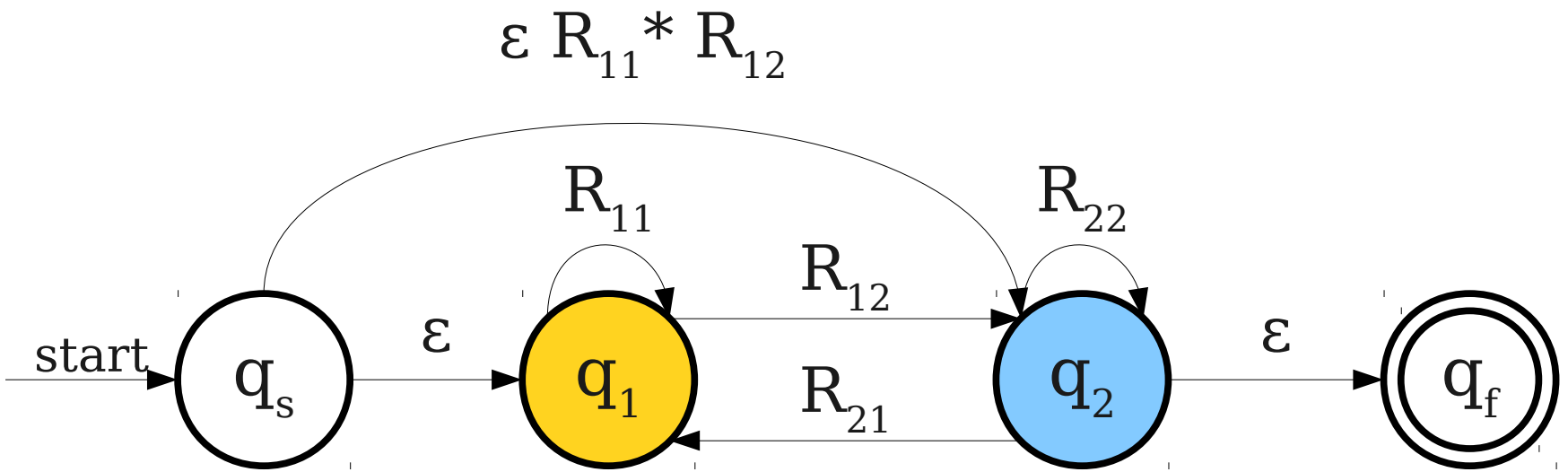



Note: We're using concatenation and Kleene closure in order to skip this state.

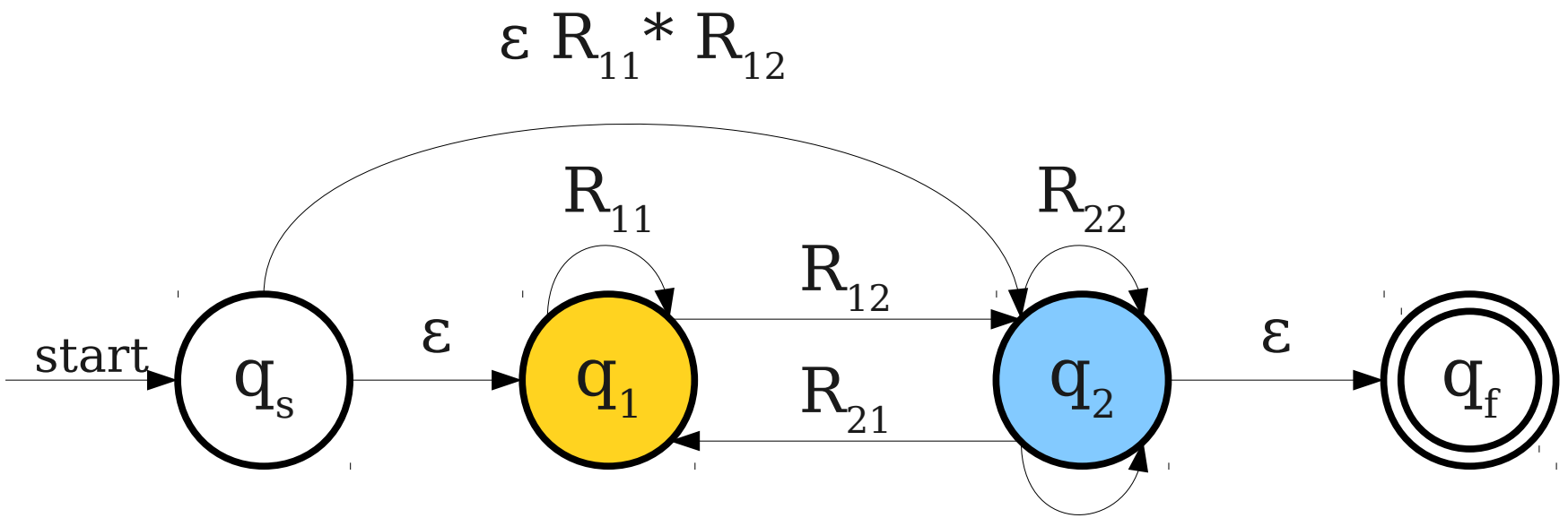
# From NFAs to Regular Expressions



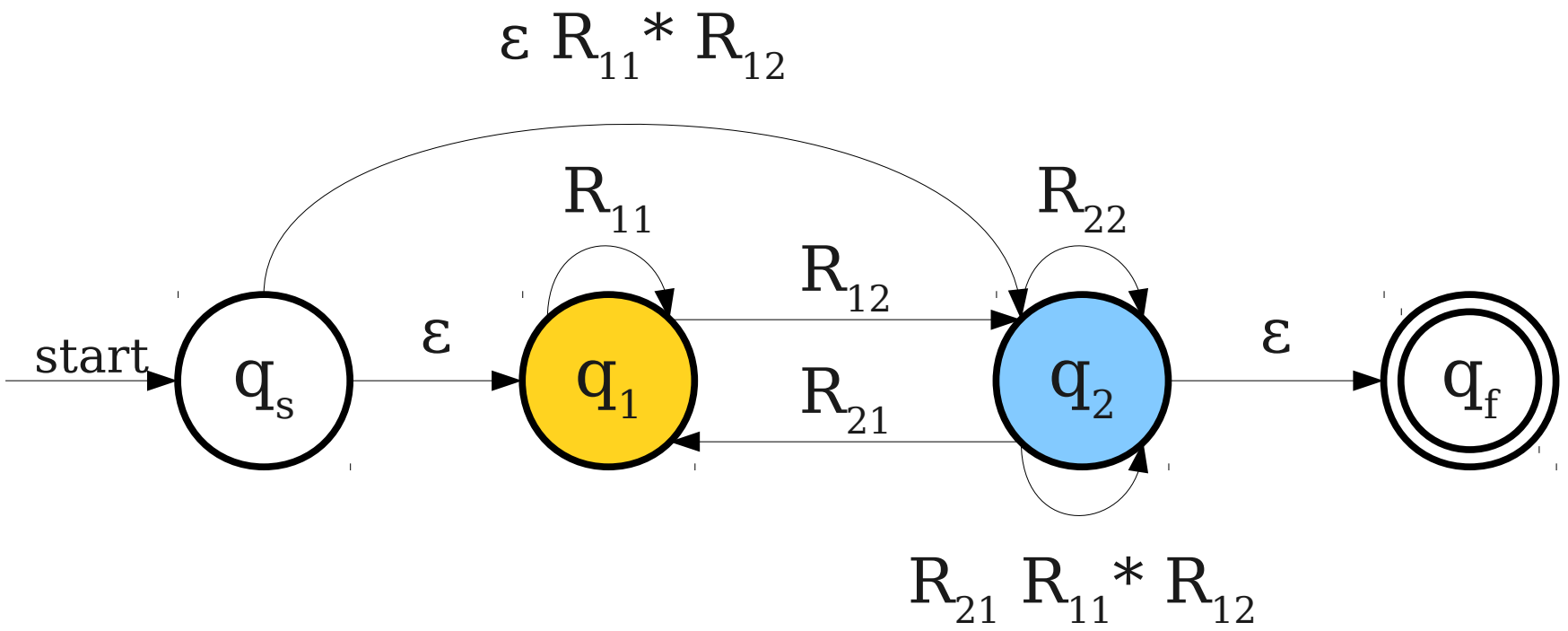
# From NFAs to Regular Expressions



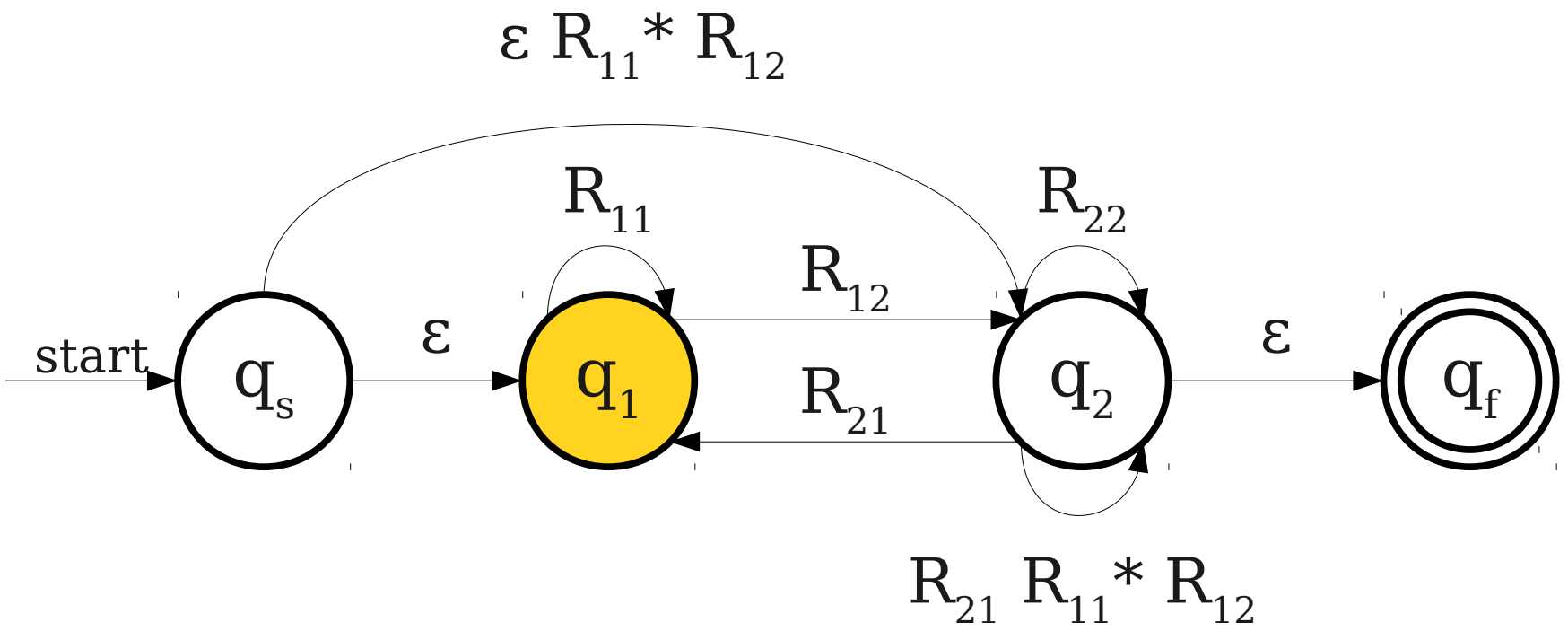
# From NFAs to Regular Expressions



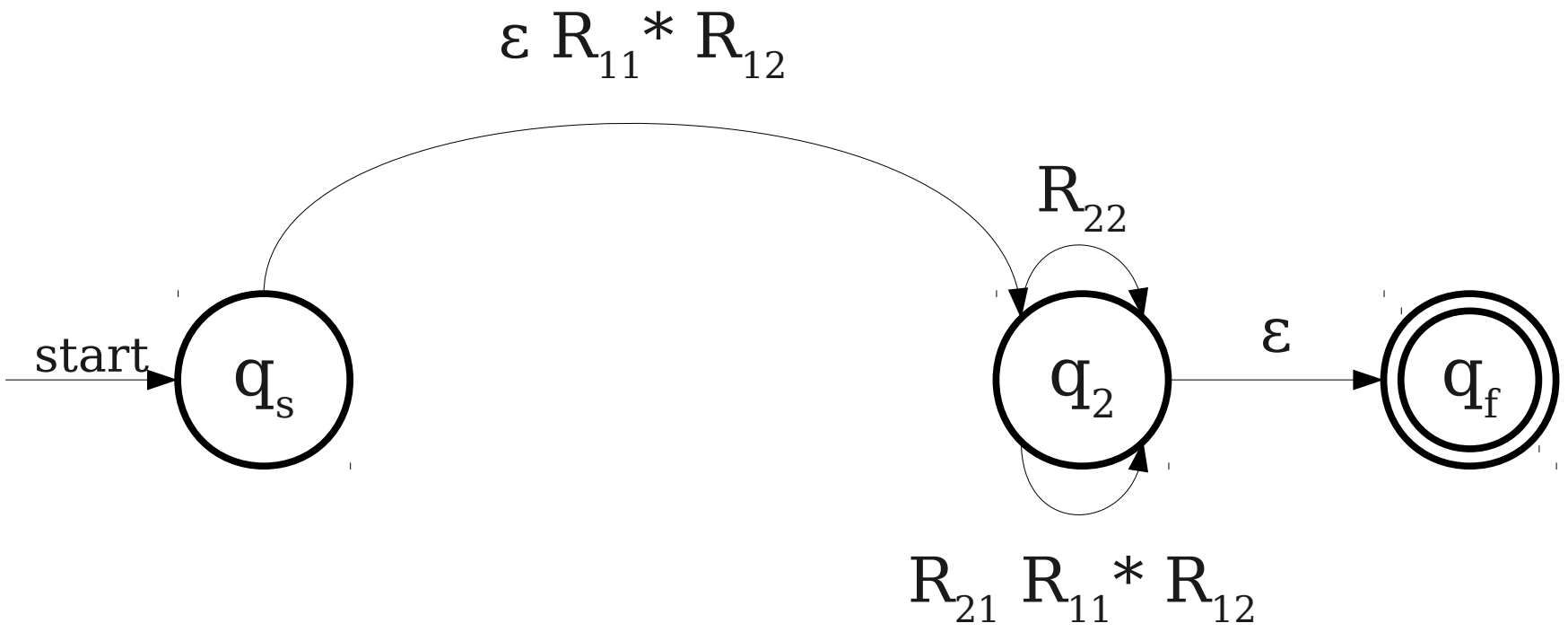
# From NFAs to Regular Expressions



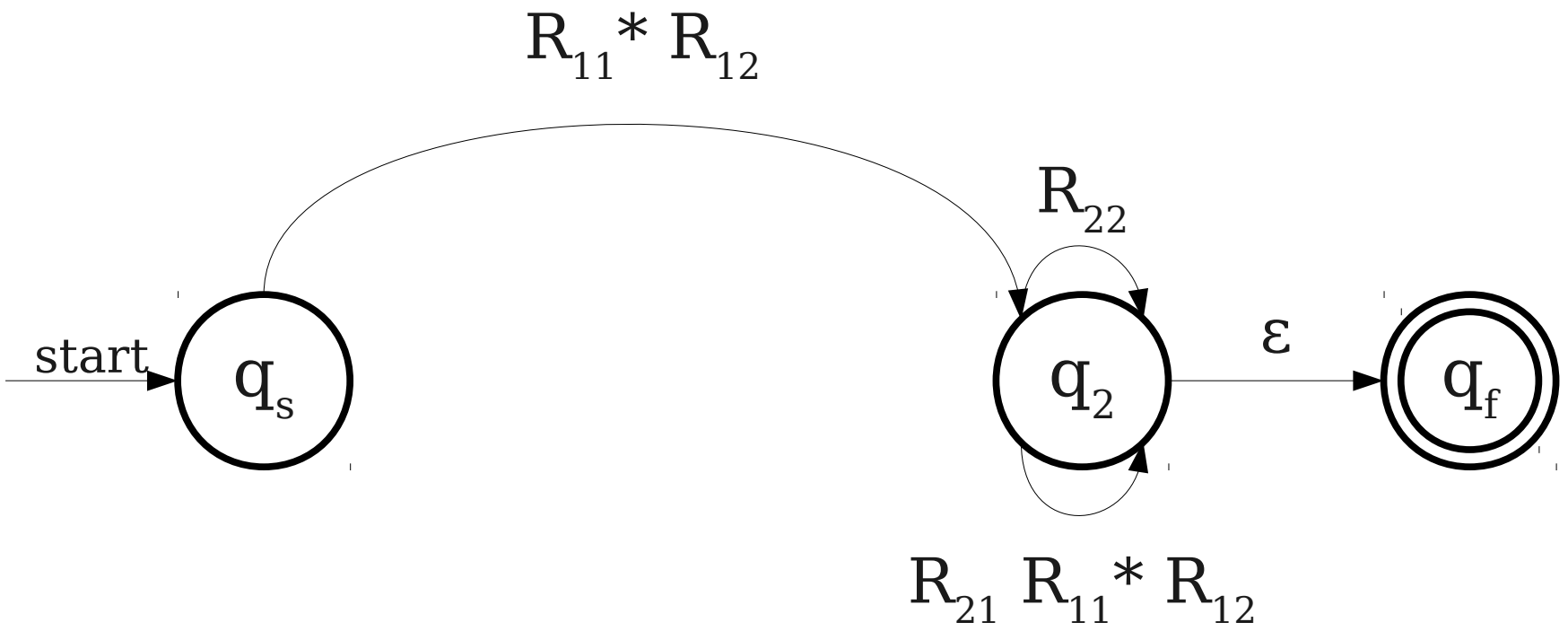
# From NFAs to Regular Expressions



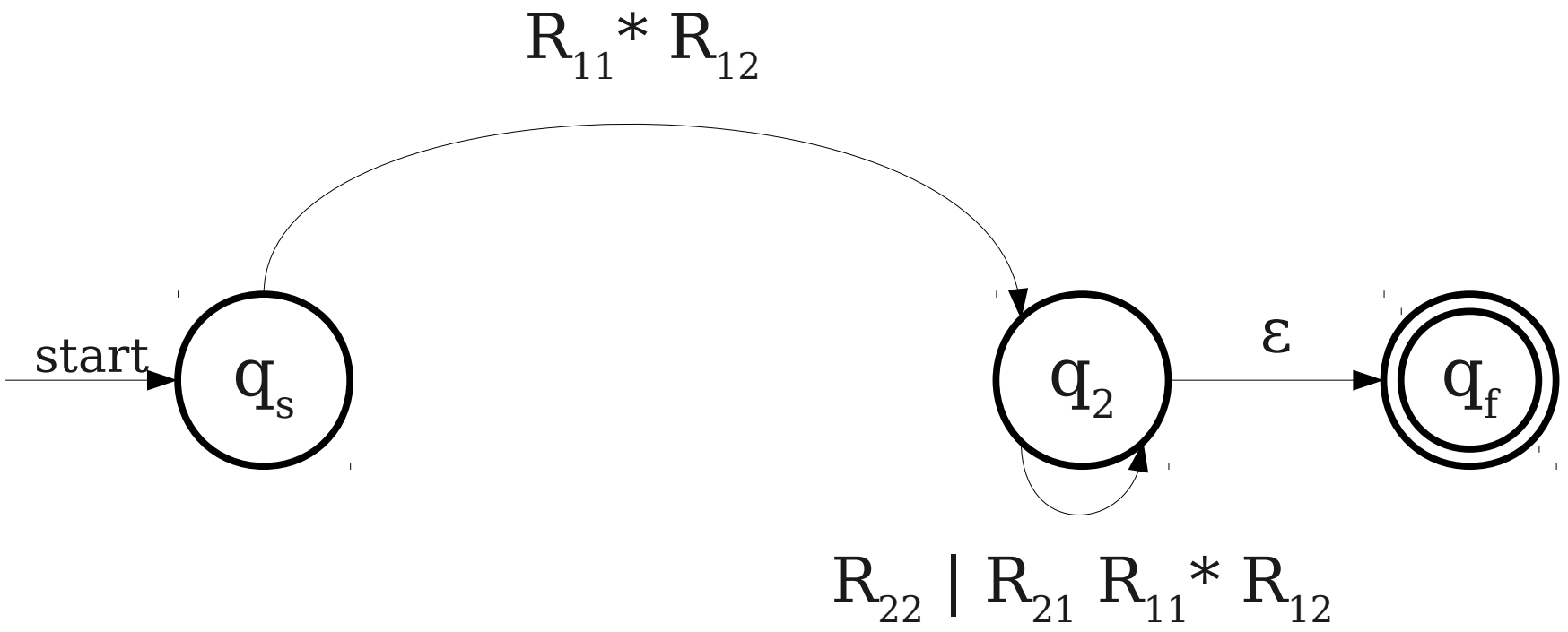
# From NFAs to Regular Expressions



# From NFAs to Regular Expressions

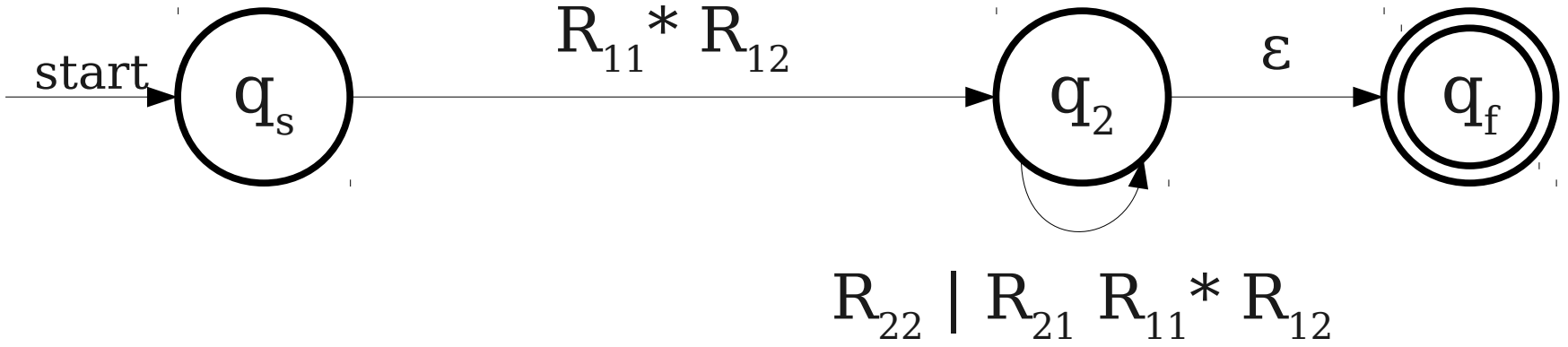


# From NFAs to Regular Expressions

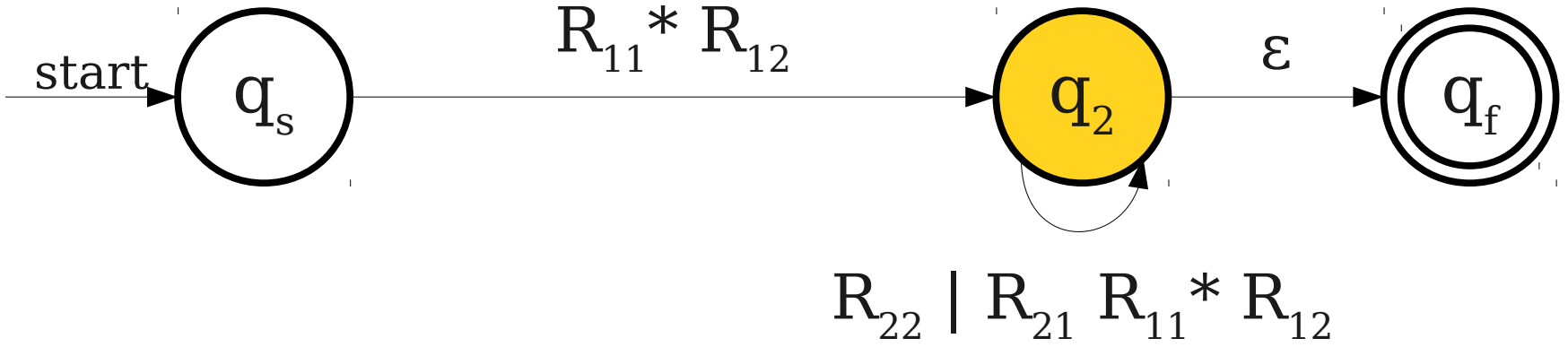



Note: We're using union to combine these transitions together.

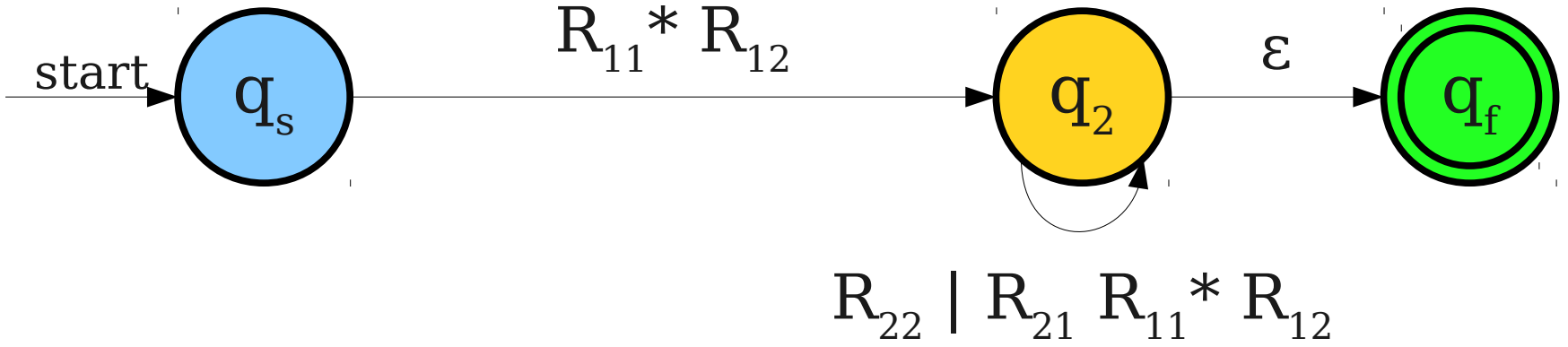
# From NFAs to Regular Expressions



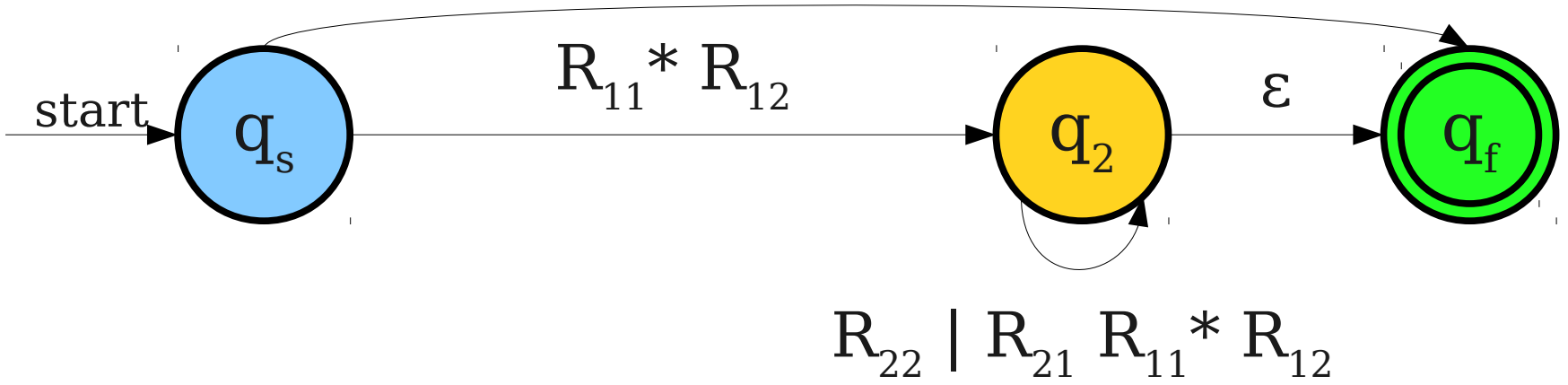
# From NFAs to Regular Expressions



# From NFAs to Regular Expressions

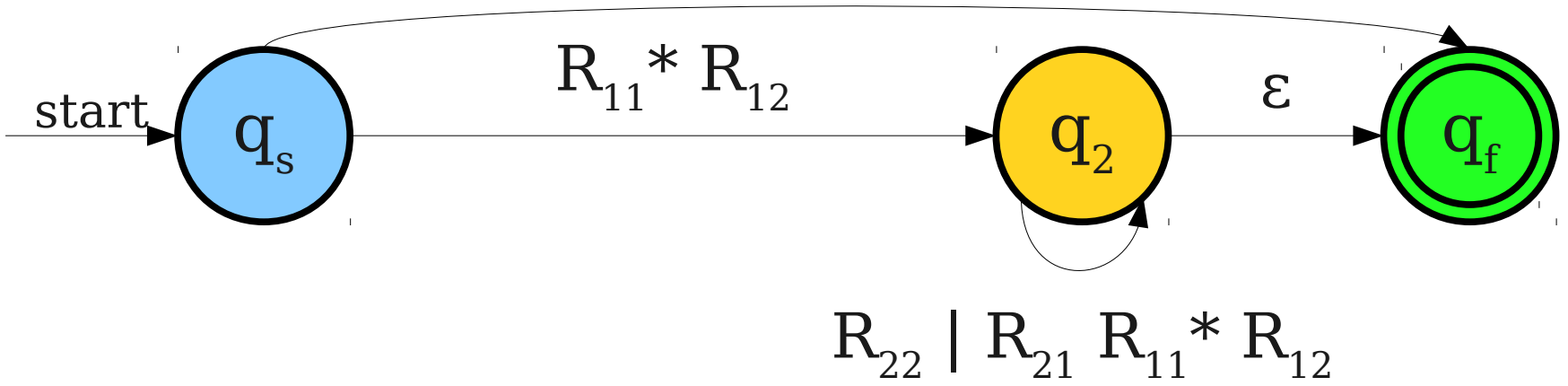


# From NFAs to Regular Expressions



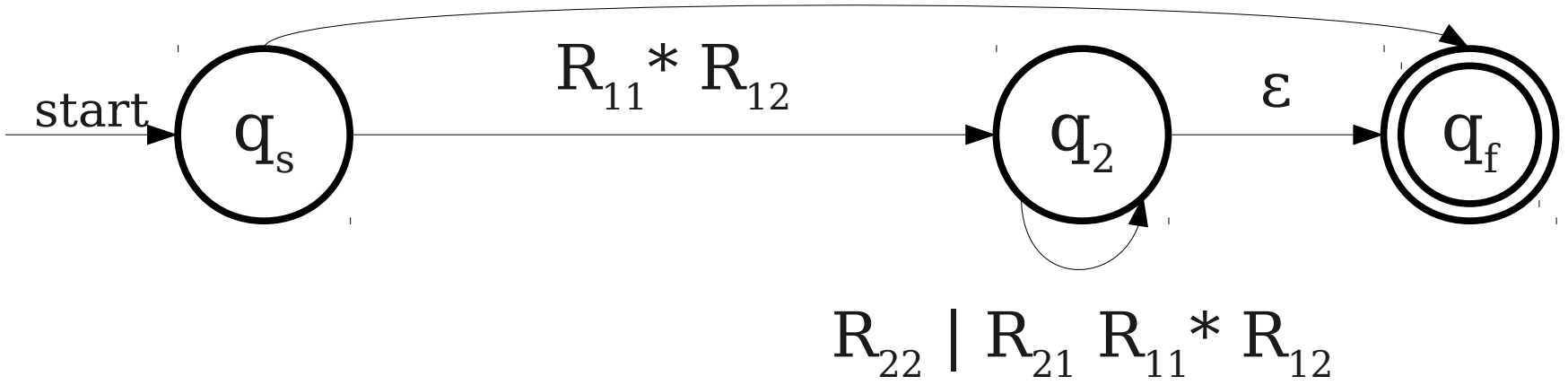
# From NFAs to Regular Expressions

$R_{11}^* R_{12} (R_{22} \mid R_{21} R_{11}^* R_{12})^* \varepsilon$



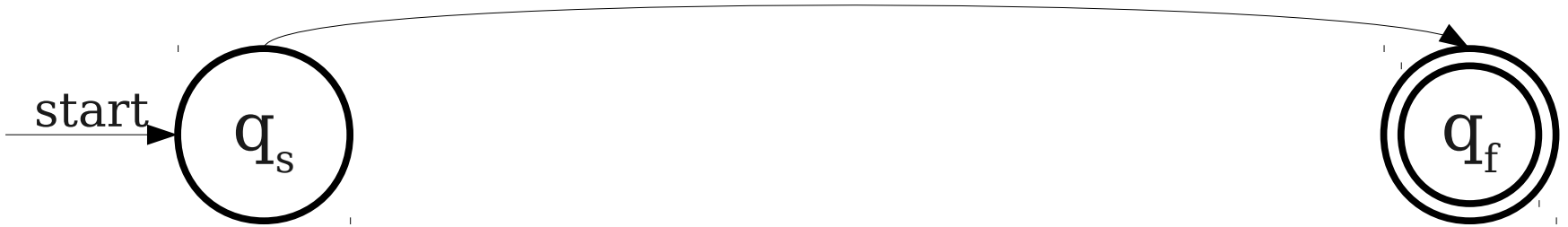
# From NFAs to Regular Expressions

$$R_{11}^* R_{12} (R_{22} \mid R_{21} R_{11}^* R_{12})^* \varepsilon$$



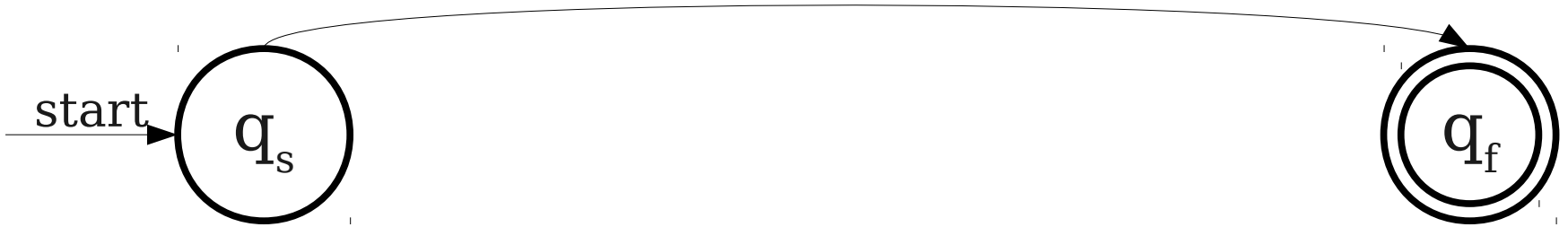
# From NFAs to Regular Expressions

$$R_{11}^* R_{12} (R_{22} \mid R_{21} R_{11}^* R_{12})^* \varepsilon$$

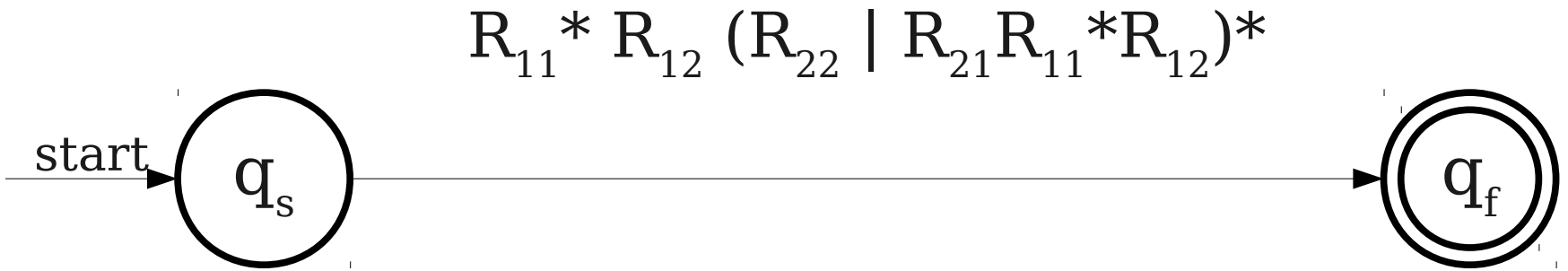


# From NFAs to Regular Expressions

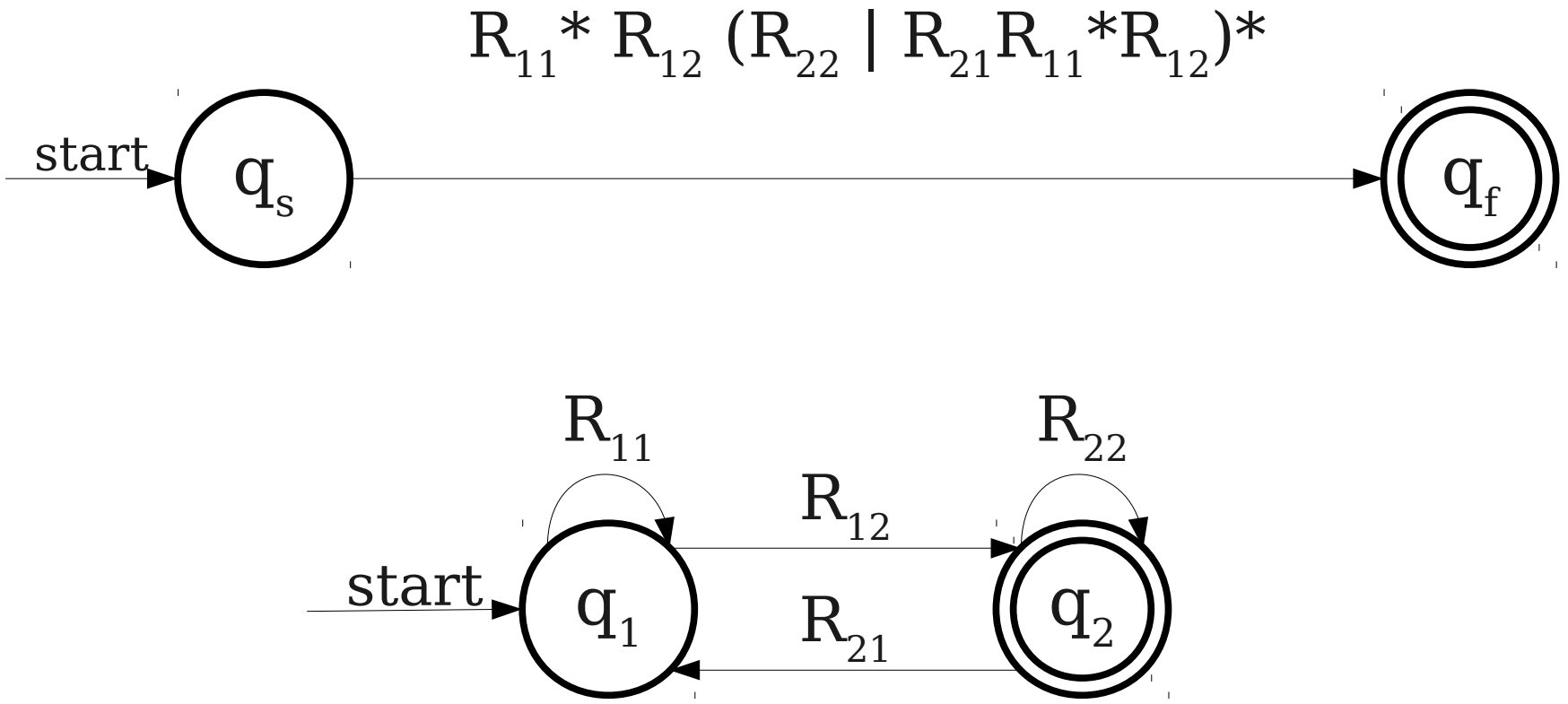
$$R_{11}^* R_{12} (R_{22} \mid R_{21} R_{11}^* R_{12})^*$$



# From NFAs to Regular Expressions



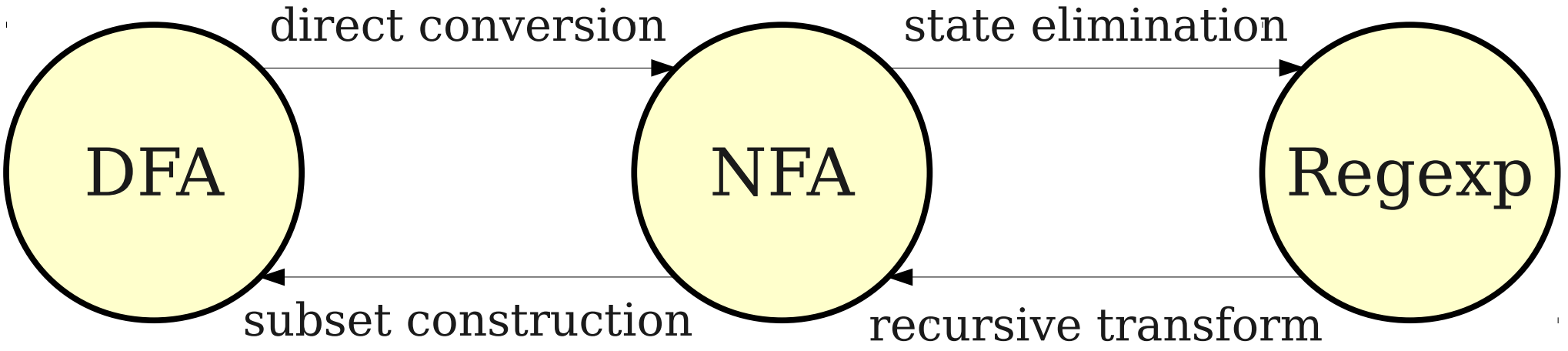
# From NFAs to Regular Expressions



# The Construction at a Glance

- Start with an NFA for the language $L$.
- Add a new start state $q_s$ and accept state $q_f$ to the NFA.
  - Add ε-transitions from each original accepting state to $q_f$, then mark them as not accepting.
- Repeatedly remove states other than $q_s$ and $q_f$ from the NFA by "shortcutting" them until only two states remain: $q_s$ and $q_f$.
- The transition from $q_s$ to $q_f$ is then a regular expression for the NFA.

# Our Transformations



***Theorem:*** The following are all equivalent:

- $L$ is a regular language.
- There is a DFA $D$ such that $\mathscr{L}(D) = L$.
- There is an NFA $N$ such that $\mathscr{L}(N) = L$.
- There is a regular expression $R$ such that $\mathscr{L}(R) = L$.

# Next Time

- **Applications of Regular Languages**
  - Answering “so what?”
- **Intuiting Regular Languages**
  - What makes a language regular?
- **The Pumping Lemma**
  - Proving languages aren't regular.