Mathematical Logic

Part One

An Important Question

How do we formalize the logic we've been using in our proofs?

Where We're Going

- **Propositional Logic** (Today)
  - Basic logical connectives.
  - Truth tables.
  - Logical equivalences.
- **First-Order Logic** (Today/Friday)
  - Reasoning about properties of multiple objects.

Propositional Logic

A **proposition** is a statement that is, by itself, either true or false.

Some Sample Propositions

- Puppies are cuter than kittens.
- Kittens are cuter than puppies.
- Usain Bolt can outrun everyone in this room.
- CS103 is useful for cocktail parties.
- This is the last entry on this list.

More Propositions

- I came in like a wrecking ball.
- I am a champion.
- You're going to hear me roar.
- We all just entertainers.

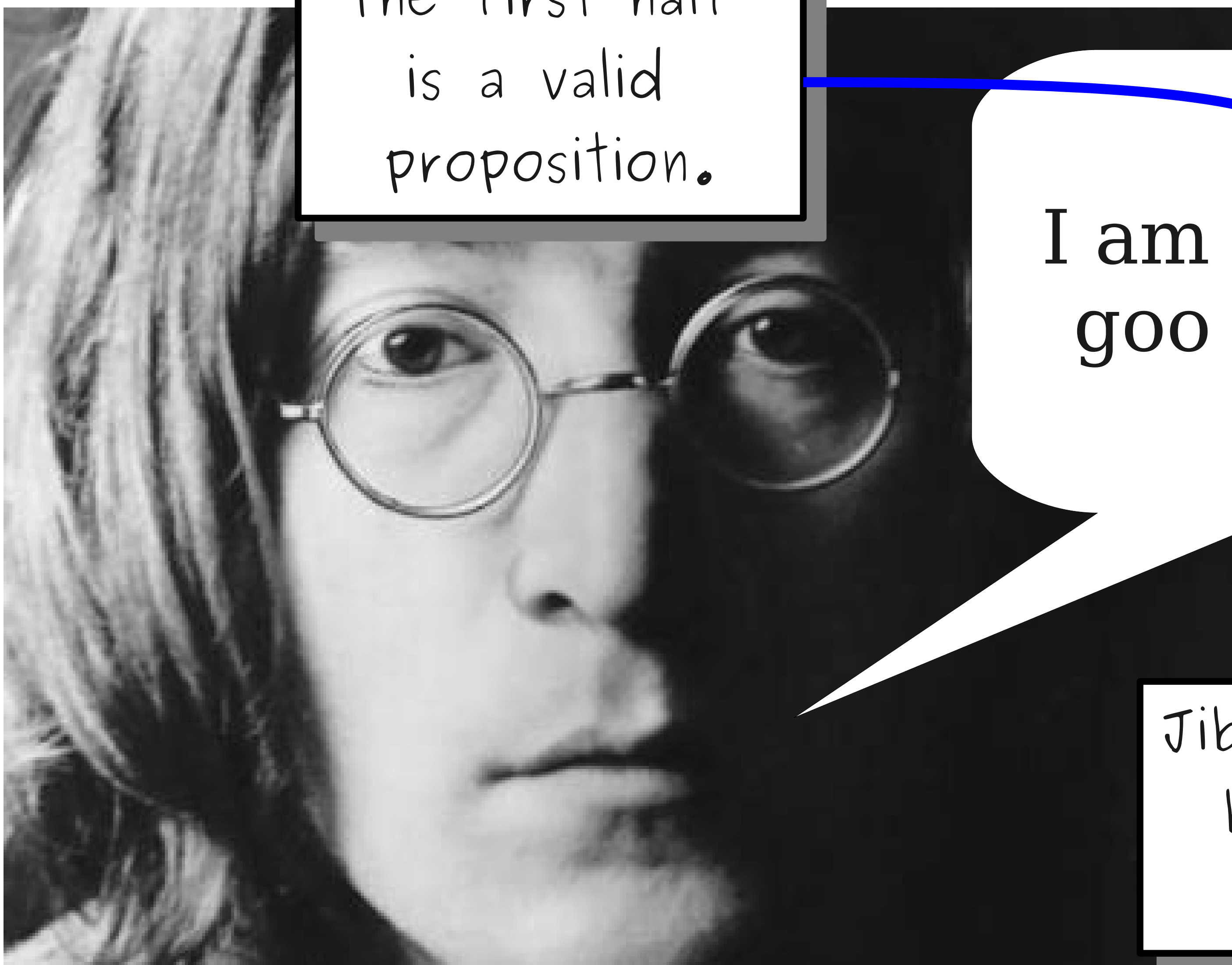
Things That Aren't Propositions

Commands cannot be true or false.

Things That Aren't Propositions

Questions cannot be true or false.

Things That Aren't Propositions



The first half is a valid proposition.

I am the walrus, goo goo g'joob

Jibberish cannot be true or false.

Propositional Logic

- **Propositional logic** is a mathematical system for reasoning about propositions and how they relate to one another.
- Every statement in propositional logic consists of **propositional variables** combined via **logical connectives**.
  - Each variable represents some proposition, such as “You liked it” or “You should have put a ring on it.”
  - Connectives encode how propositions are related, such as “If you liked it, then you should have put a ring on it.”

Propositional Variables

- Each proposition will be represented by a **propositional variable**.
- Propositional variables are usually represented as lower-case letters, such as *p*, *q*, *r*, *s*, etc.
- Each variable can take one one of two values: true or false.

Logical Connectives

- **Logical NOT: $\neg p$**
  - Read "**not** *p*"
  - $\neg p$ is true if and only if p is false.
  - Also called **logical negation**.
- **Logical AND: $p \wedge q$**
  - Read "*p* **and** *q*."
  - $p \wedge q$ is true if both p and q are true.
  - Also called **logical conjunction**.
- **Logical OR: $p \vee q$**
  - Read "*p* **or** *q*."
  - $p \vee q$ is true if at least one of p or q are true (inclusive OR)
  - Also called **logical disjunction**.

Truth Tables

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Truth Tables

p	q	$p \land q$
F	F	F
F	T	F
T	F	F
T	T	T

If p is false and q is false, then "both p and q" is false.

Truth Tables

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Truth Tables

p	q	$p \land q$
F	F	F
F	T	F
T	F	F
T	T	T

Truth Tables

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Truth Tables

p	q	$p \land q$
F	F	F
F	T	F
T	F	F
T	T	T

"Both p and q" is true only when both p and q are true.

Truth Tables

Truth Tables

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Truth Tables

p	q	$p \lor q$
F	F	F
F	T	T
T	F	T
T	T	T

This "or" is an inclusive or.

Truth Tables

p	$\neg p$
F	T
T	F

Truth Table for Implication

p	q	$p \to q$
F	F	
F	T	
T	F	
T	T	

Truth Table for Implication

p	q	$p \to q$
F	F	
F	T	
T	F	
T	T	

Truth Table for Implication

p	q	$p \to q$
F	F	
F	T	
T	F	
T	T	

In both of these cases, p is false, so the statement "if p, then q" is vacuously true.

Truth Table for Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	
T	T	

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T	T	

Truth Table for Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	
T	T	

$p \to q$ should mean when p is true, q is true as well. But here p is true and q is false!

Truth Table for Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	

$p \to q$ should mean when p is true, q is true as well. But here p is true and q is false!

Truth Table for Implication

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F	F	T
F	T	T
T	F	F
T	T	

Truth Table for Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	

$p \to q$ means that if we ever find that p is true, we'll find that q is true as well.

Truth Table for Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

$p \to q$ means that if we ever find that p is true, we'll find that q is true as well.

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F	T	T
T	F	F
T	T	T

Truth Table for Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

The only way for $p \to q$ to be false is for p to be true and q to be false.

The Biconditional

- The **biconditional** connective $p \leftrightarrow q$ is read "p if and only if q."
- Intuitively, either both p and q are true, or neither of them are.

p	q	$p \leftrightarrow q$
F	F	
F	T	
T	F	
T	T	

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F	T	
T	F	
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One of p or q is true without the other.

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F	T	F
T	F	F
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p	q	$p \leftrightarrow q$
F	F	
F	T	F
T	F	F
T	T	T

Both p and q are false here, so the statement "p if and only if q" is true.

The Biconditional

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F	T	F
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- Intuitively, either both p and q are true, or neither of them are.

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

One interpretation of $\leftrightarrow$ is to think of it as equality: the two propositions must have equal truth values.

True and False

- There are two more “connectives” to speak of: true and false.
  - The symbol $\top$ is a value that is always true.
  - The symbol $\bot$ is value that is always false.
- These are often called connectives, though they don't connect anything.
  - (Or rather, they connect zero things.)

Operator Precedence

- How do we parse this statement?

$$\neg x \to y \vee z \to x \vee y \wedge z$$

- Operator precedence for propositional logic:

$$\neg$$

$$\wedge$$

$$\vee$$

$$\to$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$\neg x \to y \lor z \to x \lor y \land z$$

- Operator precedence for propositional logic:

$$\neg$$
$$\land$$
$$\lor$$
$$\to$$
$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to y \lor z \to x \lor y \land z$$

- Operator precedence for propositional logic:

$$\neg$$

$$\land$$

$$\lor$$

$$\to$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to y \vee z \to x \vee y \wedge z$$

- Operator precedence for propositional logic:

$$\neg$$

$$\boldsymbol{\wedge}$$

$$\vee$$

$$\to$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to y \lor z \to x \lor (y \land z)$$

- Operator precedence for propositional logic:

$$\neg$$
$$\land$$
$$\lor$$
$$\to$$
$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \rightarrow y \vee z \rightarrow x \vee (y \wedge z)$$

- Operator precedence for propositional logic:

$$\neg$$

$$\wedge$$

$$\vee$$

$$\rightarrow$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \rightarrow (y \vee z) \rightarrow (x \vee (y \wedge z))$$

- Operator precedence for propositional logic:

$$\neg$$

$$\wedge$$

$$\vee$$

$$\rightarrow$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to (y \lor z) \to (x \lor (y \land z))$$

- Operator precedence for propositional logic:

$$\neg$$
$$\land$$
$$\lor$$
$$\to$$
$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to ((y \lor z) \to (x \lor (y \land z)))$$

- Operator precedence for propositional logic:

$$\neg$$
$$\land$$
$$\lor$$
$$\to$$
$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to ((y \vee z) \to (x \vee (y \wedge z)))$$

- Operator precedence for propositional logic:

$$\neg$$
$$\wedge$$
$$\vee$$
$$\to$$
$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Recap So Far

- A **propositional variable** is a variable that is either true or false.
- The **logical connectives** are
  - Negation: $\neg p$
  - Conjunction: $p \land q$
  - Disjunction: $p \lor q$
  - Implication: $p \to q$
  - Biconditional: $p \leftrightarrow q$
  - True: $\top$
  - False: $\bot$

Translating into Propositional Logic

Some Sample Propositions

a: There is a velociraptor outside my apartment.

b: Velociraptors can open windows.

c: I am in my apartment right now.

d: My apartment has windows.

e: I am going to be eaten by a velociraptor

"I won't be eaten by a velociraptor if there isn't a velociraptor outside my apartment."

$$\neg a \to \neg e$$

“p if q”

translates to

$$q \to p$$

It does *not* translate to

$$p \to q$$

Some Sample Propositions

a: There is a velociraptor outside my apartment.

b: Velociraptors can open windows.

c: I am in my apartment right now.

d: My apartment has windows.

e: I am going to be eaten by a velociraptor

"If there is a velociraptor outside my apartment, but velociraptors can't open windows, I am not going to be eaten by a velociraptor."

$$a \land \neg b \to \neg e$$

“p, but q”

translates to

$$p \wedge q$$

The Takeaway Point

- When translating into or out of propositional logic, be very careful not to get tripped up by nuances of the English language.
  - In fact, this is one of the reasons we have a symbolic notation in the first place!
- Many prepositional phrases lead to counterintuitive translations; make sure to double-check yourself!

More Elaborate Truth Tables

p	q	$p \land (p \to q)$
F	F	
F	T	
T	F	
T	T	

More Elaborate Truth Tables

We can't evaluate this until we have a value for p → q.

p	q	$p \land (p \to q)$
F	F	
F	T	
T	F	
T	T	

More Elaborate Truth Tables

So let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	
F	T	
T	F	
T	T	

More Elaborate Truth Tables

so let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	
F	T	
T	F	
T	T	

More Elaborate Truth Tables

so let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	T
F	T	
T	F	
T	T	

More Elaborate Truth Tables

so let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	T
F	T	
T	F	
T	T	

More Elaborate Truth Tables

so let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	
T	T	

More Elaborate Truth Tables

so let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	
T	T	

More Elaborate Truth Tables

so let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	F
T	T	

More Elaborate Truth Tables

so let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	F
T	T	

More Elaborate Truth Tables

so let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	F
T	T	T

More Elaborate Truth Tables

p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	F
T	T	T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	F
T	T	T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	F
T	T	T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	$p \land (p \to q)$
F	F	F T
F	T	T
T	F	F
T	T	T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	$p \land (p \to q)$
F	F	F T
F	T	T
T	F	F
T	T	T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	p	$\land$	$(p$	$\to$	$q)$
F	F		F		T	
F	T		F		T	
T	F				F	
T	T				T	

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	$p \land (p \to q)$	
F	F	F	T
F	T	F	T
T	F		F
T	T		T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	$p \land (p \to q)$	
F	F	F	T
F	T	F	T
T	F	F	F
T	T		T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	$p \land (p \to q)$	
F	F	F	T
F	T	F	T
T	F	F	F
T	T		T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	$p \land (p \to q)$	
F	F	F	T
F	T	F	T
T	F	F	F
T	T	T	T

More Elaborate Truth Tables

p	q	$p \land (p \to q)$	
F	F	F	T
F	T	F	T
T	F	F	F
T	T	T	T

More Elaborate Truth Tables

This gives the final truth value for the expression.

p	q	$p \land (p \to q)$	
F	F	F	T
F	T	F	T
T	F	F	F
T	T	T	T

Logical Equivalence

Negations

- $p \land q$ is false if and only if $\neg(p \land q)$ is true.
- Intuitively, this is only possible if either p is false or q is false (or both!)
- In propositional logic, we can write this as $\neg p \lor \neg q$.
- How would we prove that $\neg(p \land q)$ and $\neg p \lor \neg q$ are equivalent?
- **Idea**: Build truth tables for both expressions and confirm that they always agree.

Negating AND

p	q	$\neg(p \wedge q)$
F	F	
F	T	
T	F	
T	T	

Negating AND

p	q	$\neg(p \wedge q)$
F	F	F
F	T	F
T	F	F
T	T	T

Negating AND

p	q	$\neg(p$	$\land$	$q)$
F	F	T	F	
F	T	T	F	
T	F	T	F	
T	T	F	T	

Negating AND

p	q	$\neg(p$	$\wedge$	$q)$
F	F	T	F	
F	T	T	F	
T	F	T	F	
T	T	F	T	

Negating AND

p	q	$\neg(p \land q)$	
F	F	T	F
F	T	T	F
T	F	T	F
T	T	F	T

p	q	$\neg p \lor \neg q$
F	F	
F	T	
T	F	
T	T	

Negating AND

p	q	$\neg(p \wedge q)$	
F	F	T	F
F	T	T	F
T	F	T	F
T	T	F	T

p	q	$\neg p \vee \neg q$
F	F	T
F	T	T
T	F	F
T	T	F

Negating AND

p	q	$\neg(p$	$\wedge$	$q)$
F	F	T	F	
F	T	T	F	
T	F	T	F	
T	T	F	T	

p	q	$\neg p$	$\vee$	$\neg q$
F	F	T		T
F	T	T		F
T	F	F		T
T	T	F		F

Negating AND

p	q	$\neg$	$(p \wedge q)$
F	F	T	F
F	T	T	F
T	F	T	F
T	T	F	T

p	q	$\neg p$	$\vee$	$\neg q$
F	F	T	T	T
F	T	T	T	F
T	F	F	T	T
T	T	F	F	F

Negating AND

p	q	$\neg(p$	$\wedge$	$q)$
F	F	T	F	
F	T	T	F	
T	F	T	F	
T	T	F	T	

p	q	$\neg p$	$\vee$	$\neg q$
F	F	T	T	T
F	T	T	T	F
T	F	F	T	T
T	T	F	F	F

Negating AND

p	q	$\neg(p \land q)$	
F	F	T	F
F	T	T	F
T	F	T	F
T	T	F	T

p	q	$\neg p$	$\lor$	$\neg q$
F	F	T	T	T
F	T	T	T	F
T	F	F	T	T
T	T	F	F	F

These two statements are always the same!

Logical Equivalence

- If two propositional logic statements φ and ψ always have the same truth values as one another, they are called **logically equivalent**.
- We denote this by **φ ≡ ψ**.
- ≡ is not a connective. It is a statement used to describe propositional formulas.
  - **φ ↔ ψ** is a propositional statement that can take on different truth values based on how φ and ψ evaluate. Think of it as a function of φ and ψ.
  - **φ ≡ ψ** is an assertion that the formulas always take on the same values. It is either true or it isn't.

De Morgan's Laws

- Using truth tables, we concluded that

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

- We can also use truth tables to show that

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

- These two equivalences are called **De Morgan's Laws**.

Another Important Equivalence

- When is $p \to q$ false?
- **Answer**: p must be true and q must be false.
- In propositional logic:

$$\boldsymbol{p \land \neg q}$$

- Is the following true?

$$\boldsymbol{\neg(p \to q) \equiv p \land \neg q}$$

Negating Implications

Negating Implications

p	q	$\neg(p \to q)$
F	F	
F	T	
T	F	
T	T	

Negating Implications

p	q	$\neg(p \to q)$
F	F	T
F	T	T
T	F	F
T	T	T

Negating Implications

p	q	$\neg(p \to q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

Negating Implications

p	q	$\neg(p \to q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

Negating Implications

p	q	$\neg(p \to q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	$p \land \neg q$
F	F	
F	T	
T	F	
T	T	

Negating Implications

p	q	$\neg(p \to q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	$p \land \neg q$
F	F	F
F	T	F
T	F	T
T	T	T

Negating Implications

p	q	$\neg(p \to q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	$p \land \neg q$	
F	F	F	T
F	T	F	F
T	F	T	T
T	T	T	F

Negating Implications

p	q	$\neg(p \to q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	$p \land \neg q$		
F	F	F	F	T
F	T	F	F	F
T	F	T	T	T
T	T	T	F	F

Negating Implications

p	q	$\neg$	$(p \to q)$
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	p	$\wedge$	$\neg q$
F	F	F	F	T
F	T	F	F	F
T	F	T	T	T
T	T	T	F	F

Negating Implications

p	q	$\neg(p \to q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	p	$\wedge$	$\neg q$
F	F	F	F	T
F	T	F	F	F
T	F	T	T	T
T	T	T	F	F

$$\neg(p \to q) \equiv p \wedge \neg q$$

An Important Observation

- We have just proven that

$$\boldsymbol{\neg(p \to q) \equiv p \land \neg q}$$

- If we negate both sides, we get that

$$\boldsymbol{p \to q \equiv \neg(p \land \neg q)}$$

- By De Morgan's laws:

$$p \to q \equiv \neg(p \land \neg q)$$

$$p \to q \equiv \neg p \lor \neg\neg q$$

$$p \to q \equiv \neg p \lor q$$

- Thus $\boldsymbol{p \to q \equiv \neg p \lor q}$

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If p is false, the whole thing is true and we gain no information. If p is true, then q has to be true for the whole expression to be true.

Why This Matters

- Understanding these equivalences helps justify how proofs work and what to prove.
- Unsure what to prove? Try translating it into logic first and see what happens.

Announcements!

Problem Set Three Checkpoint

- Problem Set Three checkpoints graded and solutions are released.
- ***Please review the feedback and solution set***. Parts (ii) and (iv) are trickier than they might seem.
- On-time Problem Set Two's should be graded and returned by tomorrow at noon in the homework return bin.
  - Please keep everything sorted!
  - Please don't leave papers sitting out!

A Note on Induction

- In an inductive proof, $P(n)$ must be a statement that is either true or false for a particular choice of n.
- Examples:
  - $P(n)$ = "$a_n = 2^n$."
  - $P(n)$ = "any tournament with n players has a winner."
- Non-examples:
  - $P(n)$ = "a game of Nim with n stones in each pile"
  - $P(n)$ = "for any $n \in \mathbb{N}$, $a_n = 2^n$."

Your Questions

What are some practical applications of cardinality? Why is it useful?

First-Order Logic

What is First-Order Logic?

- **First-order logic** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - **predicates** that describe properties of objects, and
  - **functions** that map objects to one another,
  - **quantifiers** that allow us to reason about multiple objects simultaneously.

The Universe of Propositional Logic

The Universe of Propositional Logic

$$p \wedge q \rightarrow \neg r \vee \neg s$$

The Universe of Propositional Logic

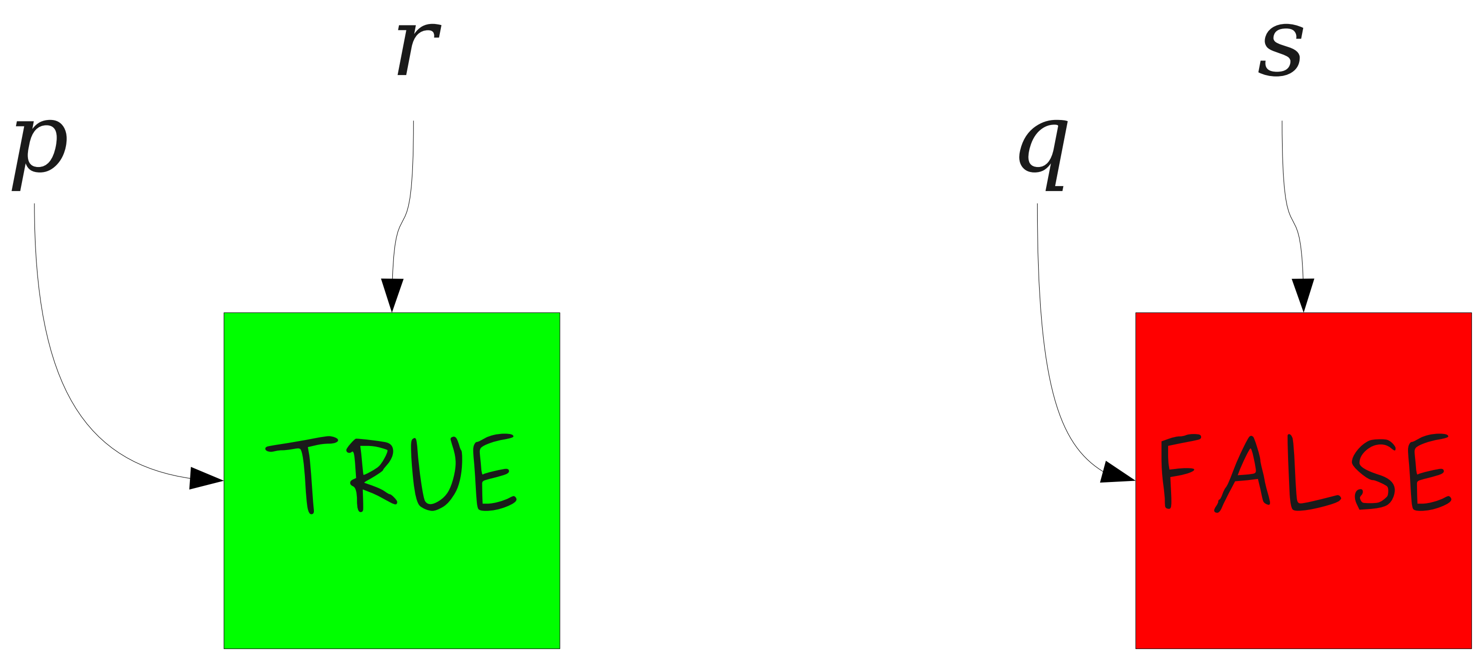
$$p \land q \to \neg r \lor \neg s$$

TRUE

FALSE

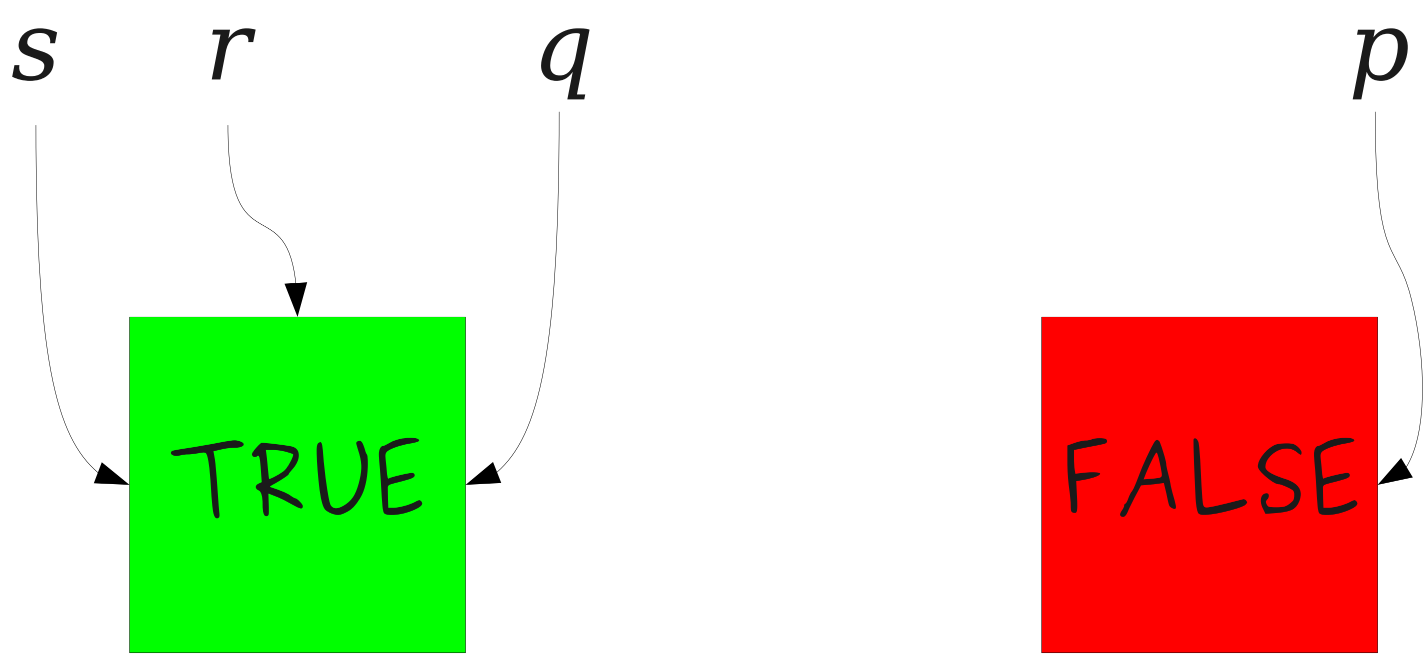
The Universe of Propositional Logic

$$p \land q \to \neg r \lor \neg s$$



The Universe of Propositional Logic

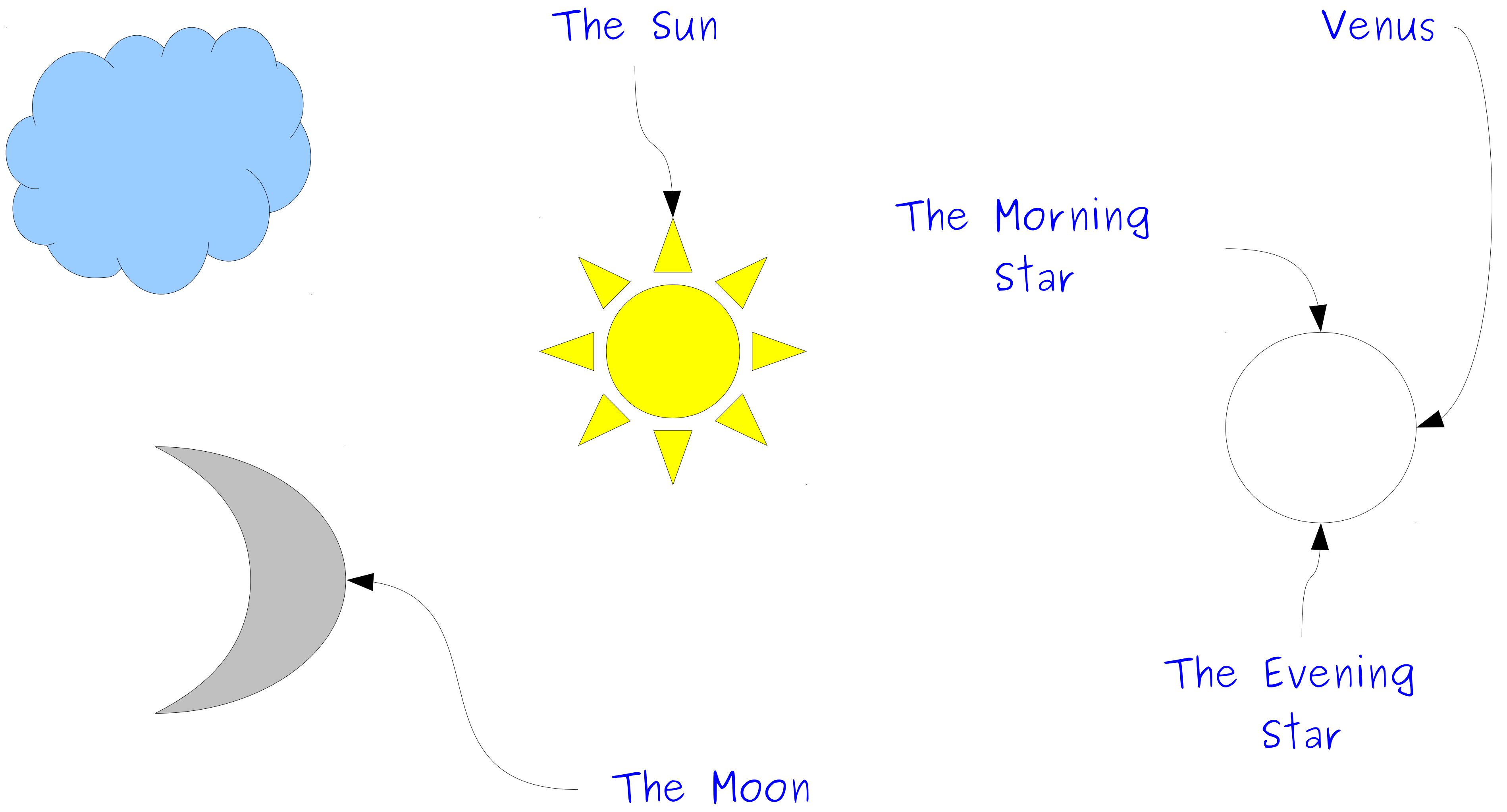
$$p \land q \to \neg r \lor \neg s$$



Propositional Logic

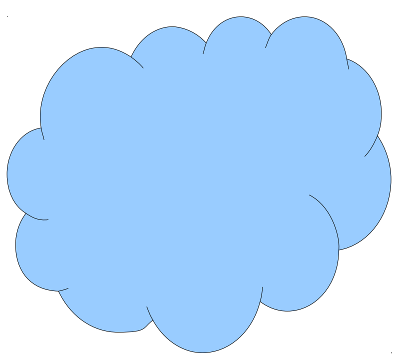
- In propositional logic, each variable represents a **proposition**, which is either true or false.
- We can directly apply connectives to propositions:
  - $p \to q$
  - $\neg p \land q$
- The truth of a statement can be determined by plugging in the truth values for the input propositions and computing the result.
- We can see all possible truth values for a statement by checking all possible truth assignments to its variables.

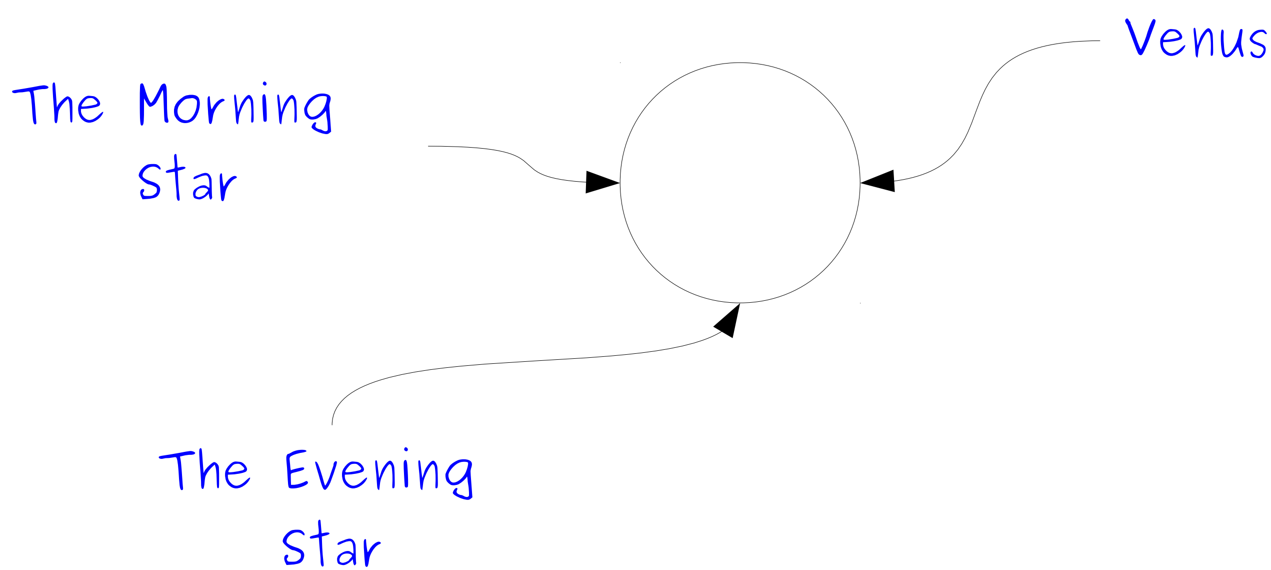
The Universe of First-Order Logic



First-Order Logic

- In first-order logic, each variable refers to some object in a set called the **domain of discourse**.
- Some objects may have multiple names.
- Some objects may have no name at all.





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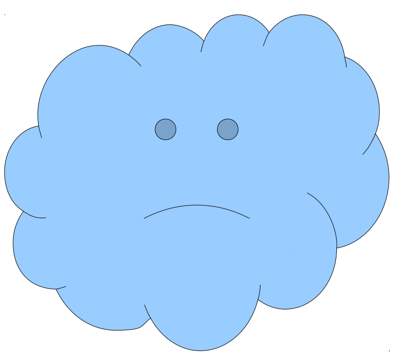

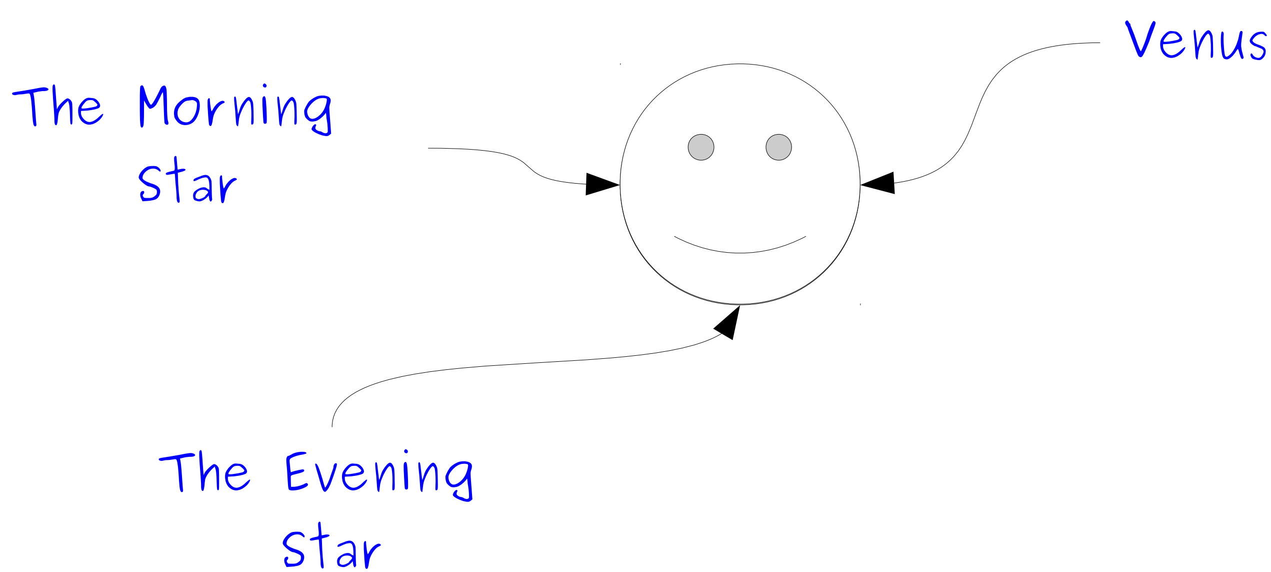

Propositional vs. First-Order Logic

- Because propositional variables are either true or false, we can directly apply connectives to them.

$$\boldsymbol{p \to q} \qquad\qquad \boldsymbol{\neg p \leftrightarrow q \land r}$$

- Because first-order variables refer to arbitrary objects, it does not make sense to apply connectives to them.

$$\textit{\textbf{Venus}} \to \textit{\textbf{Sun}} \qquad\qquad \textit{\textbf{137}} \leftrightarrow \neg \textit{\textbf{42}}$$

- ***This is not C!***

Reasoning about Objects

- To reason about objects, first-order logic uses **predicates**.
- Examples:
  - $\mathit{ExtremelyCute}(\mathit{Quokka})$
  - $\mathit{DeadlockEachOther}(\mathit{House}, \mathit{Senate})$
- Predicates can take any number of arguments, but each predicate has a fixed number of arguments (called its **arity**)
- Applying a predicate to arguments produces a proposition, which is either true or false.

First-Order Sentences

- Sentences in first-order logic can be constructed from predicates applied to objects:

$$LikesToEat(V, M) \land Near(V, M) \to WillEat(V, M)$$

$$Cute(t) \to Dikdik(t) \lor Kitty(t) \lor Puppy(t)$$

$$x < 8 \to x < 137$$

The notation **x < 8** is just a shorthand for something like **LessThan(x, 8)**. Binary predicates in math are often written like this, but symbols like **<** are not a part of first-order logic.

Equality

- First-order logic is equipped with a special predicate **=** that says whether two objects are equal to one another.
- Equality is a part of first-order logic, just as $\to$ and $\neg$ are.
- Examples:

$$\mathit{MorningStar} = \mathit{EveningStar}$$

$$\mathit{Glinda} = \mathit{GoodWitchOfTheNorth}$$

- Equality can only be applied to **objects**; to see if **propositions** are equal, use $\leftrightarrow$.

For notational simplicity, define $\neq$ as

$$x \neq y \quad \equiv \quad \neg(x = y)$$

Next Time

- **First-Order Logic II**
  - Functions and quantifiers.
  - How do we translate statements into first-order logic?
  - Why does any of this matter?