

Mathematical Induction

Everybody – do the wave!

The Wave

- If done properly, everyone will eventually end up joining in.
- Why is that?
 - Someone (me!) started everyone off.
 - Once the person before you did the wave, you did the wave.

The **principle of mathematical induction** states that if for some $P(n)$ the following hold:

$P(0)$ is true

If it starts
true...

and

...and it stays
true...

For any $n \in \mathbb{N}$, we have $P(n) \rightarrow P(n + 1)$

then

...then it's
always true.

For any $n \in \mathbb{N}$, $P(n)$ is true.

Induction, Intuitively

- It's true for 0.
- Since it's true for 0, it's true for 1.
- Since it's true for 1, it's true for 2.
- Since it's true for 2, it's true for 3.
- Since it's true for 3, it's true for 4.
- Since it's true for 4, it's true for 5.
- Since it's true for 5, it's true for 6.
- ...

Proof by Induction

- Suppose that you want to prove that some property $P(n)$ holds of all natural numbers. To do so:
 - Prove that $P(0)$ is true.
 - This is called the **basis** or the **base case**.
 - Prove that for all $n \in \mathbb{N}$, that if $P(n)$ is true, then $P(n + 1)$ is true as well.
 - This is called the **inductive step**.
 - $P(n)$ is called the **inductive hypothesis**.
 - Conclude by induction that $P(n)$ holds for all n .

Some Summations

$$2^0 = 1 = 2^1 - 1$$

$$2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7 = 2^3 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15 = 2^4 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$$

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: By induction.

Just as in a proof by contradiction or contrapositive, we should mention this proof is by induction.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: By induction. Let $P(n)$ be “the sum of the first n powers of two is $2^n - 1$.”

Now, we state what property $P(n)$ we are going to prove holds for all $n \in \mathbb{N}$.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: By induction. Let $P(n)$ be “the sum of the first n powers of two is $2^n - 1$.” We will show $P(n)$ is true for all $n \in \mathbb{N}$.

For our base case, we need to show $P(0)$ is true, meaning the sum of the first zero powers of two is $2^0 - 1$.

The first step of an inductive proof is to show $P(0)$. We explicitly state what $P(0)$ is, then try to prove it. We can prove $P(0)$ using any proof technique we'd like.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: By induction. Let $P(n)$ be “the sum of the first n powers of two is $2^n - 1$.” We will show $P(n)$ is true for all $n \in \mathbb{N}$.

For our base case, we need to show $P(0)$ is true, meaning the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is $0 = 2^0 - 1$, we see $P(0)$ is true.

For the inductive step, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, meaning that $2^0 + 2^1 + \dots + 2^{n-1} = 2^n - 1$. We need to show that $P(n + 1)$ holds, meaning that the sum of the first $n + 1$ powers of two is $2^{n+1} - 1$.

The goal of this step is to prove

“For any $n \in \mathbb{N}$, if $P(n)$, then $P(n + 1)$ ”

To do this, we'll choose an arbitrary n , assume that $P(n)$ holds, then try to prove $P(n + 1)$.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: By induction. Let $P(n)$ be “the sum of the first n powers of two is $2^n - 1$.” We will show $P(n)$ is true for all $n \in \mathbb{N}$.

For our base case, we need to show $P(0)$ is true, meaning the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is $0 = 2^0 - 1$, we see $P(0)$ is true.

For the inductive step, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, meaning that $2^0 + 2^1 + \dots + 2^{n-1} = 2^n - 1$. We need to show that $P(n + 1)$ holds, meaning that the sum of the first $n + 1$ powers of two is $2^{n+1} - 1$.

Here, we're explicitly stating $P(n + 1)$, which is what we want to prove. Now, we can use any proof technique we want to try to prove it.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: By induction. Let $P(n)$ be “the sum of the first n powers of two is $2^n - 1$.” We will show $P(n)$ is true for all $n \in \mathbb{N}$.

For our base case, we need to show $P(0)$ is true, meaning the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is $0 = 2^0 - 1$, we see $P(0)$ is true.

For the inductive step, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, meaning that $2^0 + 2^1 + \dots + 2^{n-1} = 2^n - 1$. We need to show that $P(n + 1)$ holds, meaning that the sum of the first $n + 1$ powers of two is $2^{n+1} - 1$.

We're assuming that $P(n)$ is true, so we can replace this sum with the value $2^n - 1$.

1 powers of two. This is two, plus 2^n . Using the

$$2^0 + 2^1 + \dots + 2^{n-1} + 2^n = (2^0 + 2^1 + \dots + 2^{n-1}) + 2^n$$

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: By induction. Let $P(n)$ be “the sum of the first n powers of two is $2^n - 1$.” We will show $P(n)$ is true for all $n \in \mathbb{N}$.

For our base case, we need to show $P(0)$ is true, meaning the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is $0 = 2^0 - 1$, we see $P(0)$ is true.

For the inductive step, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, meaning that $2^0 + 2^1 + \dots + 2^{n-1} = 2^n - 1$. We need to show that $P(n + 1)$ holds, meaning that the sum of the first $n + 1$ powers of two is $2^{n+1} - 1$.

Consider the sum of the first $n + 1$ powers of two. This is the sum of the first n powers of two, plus 2^n . Using the inductive hypothesis, we see that

$$\begin{aligned} 2^0 + 2^1 + \dots + 2^{n-1} + 2^n &= (2^0 + 2^1 + \dots + 2^{n-1}) + 2^n \\ &= 2^n - 1 + 2^n \\ &= 2(2^n) - 1 \\ &= 2^{n+1} - 1 \end{aligned}$$

Thus $P(n + 1)$ is true, completing the induction. ■

Structuring a Proof by Induction

- State that your proof works by induction.
- State your choice of $P(n)$.
- Prove the base case:
 - State what $P(0)$ is, then prove it using any technique you'd like.
- Prove the inductive step:
 - State that for some arbitrary $n \in \mathbb{N}$ that you're assuming $P(n)$ and mention what $P(n)$ is.
 - State that you are trying to prove $P(n + 1)$ and what $P(n + 1)$ means.
 - Prove $P(n + 1)$ using any technique you'd like.
- This is very rigorous, so as we gain more familiarity with induction we will start being less formal in our proofs.

Induction, Intuitively

- You can imagine an “machine” that turns proofs of $P(n)$ into proofs of $P(n + 1)$.
- Starting with a proof of $P(0)$, we can run the machine as many times as we'd like to get proofs of $P(1)$, $P(2)$, $P(3)$,
- The principle of mathematical induction says that this style of reasoning is a rigorous argument.

A Quick Aside

- This result helps explain the range of numbers that can be stored in an **int**.
- If you have an unsigned 32-bit integer, the largest value you can store is given by $1 + 2 + 4 + 8 + \dots + 2^{31} = 2^{32} - 1$.
- This formula for sums of powers of two has many other uses as well. We'll see one next week.

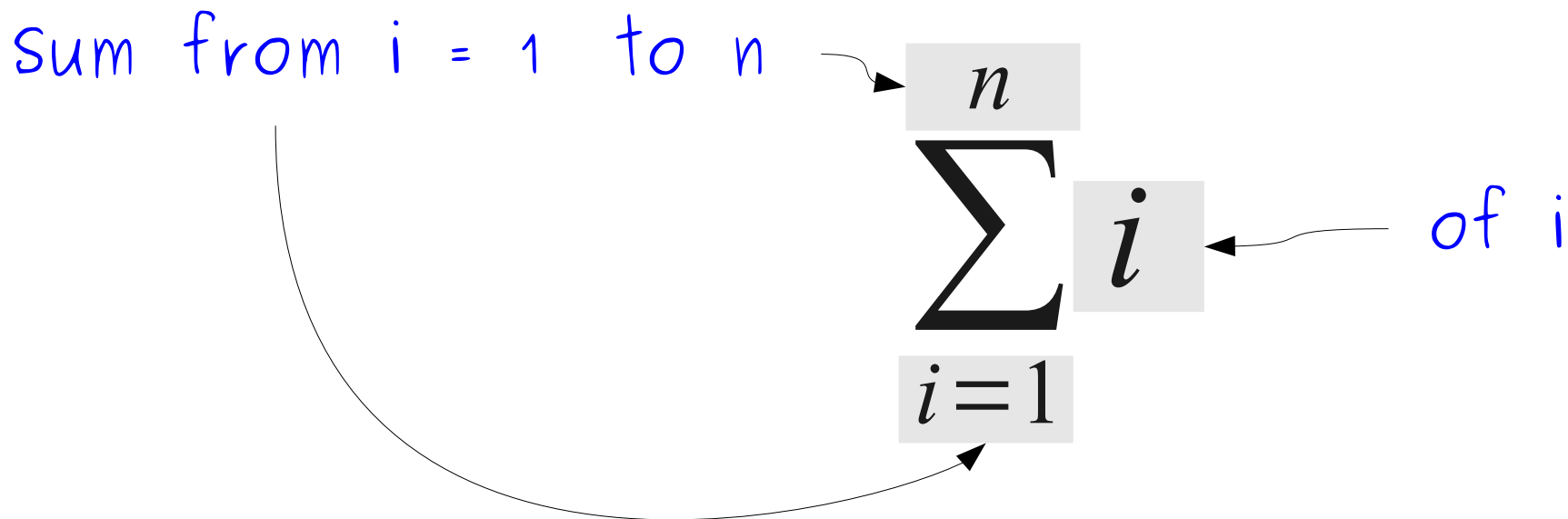
Notation: Summations

- **Summation notation** gives a compact way for discussing sums of multiple terms.
- For example, instead of writing the sum $1 + 2 + 3 + \dots + n$, we can write

sum from $i = 1$ to n

$$\sum_{i=1}^n i$$

of i



The diagram illustrates the components of the summation notation $\sum_{i=1}^n i$. The text 'sum from $i = 1$ to n ' is written in blue and has two arrows: one pointing to the upper limit n and another pointing to the lower limit $i=1$. The variable i is enclosed in a gray box, and the text 'of i ' is written in blue with an arrow pointing to this box.

Summation Examples

$$\sum_{i=1}^5 i = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{i=0}^3 2^i = 2^0 + 2^1 + 2^2 + 2^3 = 15$$

$$\sum_{i=0}^2 (i^2 - i) = (0^2 - 0) + (1^2 - 1) + (2^2 - 2) = 2$$

The Empty Sum

- A sum of no numbers is called the **empty sum** and is defined to be zero.
- Examples:

$$\sum_{i=1}^0 2^i = 0$$

$$\sum_{i=137}^{42} i^i = 0$$

$$\sum_{i=0}^{-1} i = 0$$

- Why do you think it's defined to be zero as opposed to some other number?

Theorem: For any natural number n , $\sum_{i=0}^{n-1} 2^i = 2^n - 1$

Proof: By induction. Let $P(n)$ be

$$P(n) \equiv \sum_{i=0}^{n-1} 2^i = 2^n - 1$$

For our base case, we need to show $P(0)$ is true, meaning that

$$\sum_{i=0}^{-1} 2^i = 2^0 - 1$$

Since $2^0 - 1 = 0$ and the left-hand side is the empty sum, $P(0)$ holds.

For the inductive step, assume that for some $n \in \mathbb{N}$, that $P(n)$ holds, so

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

We need to show that $P(n + 1)$ holds, meaning that

$$\sum_{i=0}^n 2^i = 2^{n+1} - 1$$

To see this, note that

$$\sum_{i=0}^n 2^i = \left(\sum_{i=0}^{n-1} 2^i \right) + 2^n = 2^n - 1 + 2^n = 2(2^n) - 1 = 2^{n+1} - 1$$

Thus $P(n + 1)$ holds, completing the induction. ■

A Brief Interlude for Announcements

Recitation Sections

- Handout #06 contains several discussion questions for this week.
- We will set up several recitation sections where you can work through these problems with one of the TAs.
 - Dates/times announced later today.
 - All sections cover the same material.
- Solutions distributed at recitation sections and online later this week.

Problem Set Clarification

- All problem sets are designed to use only the material up to and include the lecture in which they are released.
- We'll explicitly mark any problems for which we won't have covered the requisite material.

Ask Us A Question:

“What are the criteria used to grade proofs in our problem sets?”

Back to our regularly
scheduled ~~programming~~...
math

How Not To Induct

An Incorrect Proof

Theorem: For any $n \in \mathbb{N}$, we have $\sum_{i=0}^{n-1} 2^i = 2^n$.

Proof: By induction. Let $P(n)$ be defined as $P(n) \equiv \sum_{i=0}^{n-1} 2^i = 2^n$.

Assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, so

$$\sum_{i=0}^{n-1} 2^i = 2^n$$

We want to show that $P(n + 1)$ is true, which means that we want to show

$$\sum_{i=0}^n 2^i = 2^{n+1}$$

Where did we
prove the base
case?

To see this, note that

$$\sum_{i=0}^n 2^i = \sum_{i=0}^{n-1} 2^i + 2^n = 2^n + 2^n = 2^{n+1}$$

So $P(n + 1)$ holds, completing the induction. ■

When proving $P(n)$ is true
for all $n \in \mathbb{N}$ by induction,

make sure to show the base case!

Otherwise, your argument is invalid!

Why This Worked

- The math internally checked out because we made an incorrect assumption!
- Induction requires both the base case and the inductive step.
 - The base case shows that the property initially holds true.
 - The inductive step shows how each step influences the next.

The Counterfeit Coin Problem

Problem Statement

- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

A Harder Problem

- You are given a set of **nine** seemingly identical coins, eight of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only **two** weighings on the balance, find the counterfeit coin.

If we have n weighings on the scale, what is the largest number of coins out of which we can find the counterfeit?

A Pattern

- Assume out of the coins that are given, exactly one is counterfeit and weighs more than the other coins.
- If we have no weighings, how many coins can we have while still being able to find the counterfeit?
 - **One coin**, since that coin has to be the counterfeit!
- If we have one weighing, we can find the counterfeit out of **three** coins.
- If we have two weighings, we can find the counterfeit out of **nine** coins.

So far, we have

$$\mathbf{1, 3, 9 = 3^0, 3^1, 3^2}$$

Does this pattern continue?

Theorem: Given n weighings, we can detect which of 3^n coins is counterfeit.

Proof: By induction. Let $P(n)$ be “Given n weighings, we can detect which of 3^n coins is counterfeit.” We prove that $P(n)$ is true for all $n \in \mathbb{N}$.

For the base case, we show $P(0)$, that we can detect which of $3^0 = 1$ coins is counterfeit in no weighings. Exactly one coin is counterfeit, so the sole coin must be counterfeit and we can find it with no weighings.

For the inductive step, suppose $P(n)$ holds for some $n \in \mathbb{N}$, so we can detect which of 3^n coins is counterfeit using n weighings. We will show $P(n + 1)$ holds, meaning we can detect a counterfeit in 3^{n+1} coins using $n + 1$ weighings.

Given 3^{n+1} coins, split them into three groups of size 3^n ; call them A , B , and C . Put the coins in A on one side of the scale and the coins in B on the other. We consider three cases based on how the scale tips:

Case 1: Side A is heavier. Then the counterfeit must be in group A .

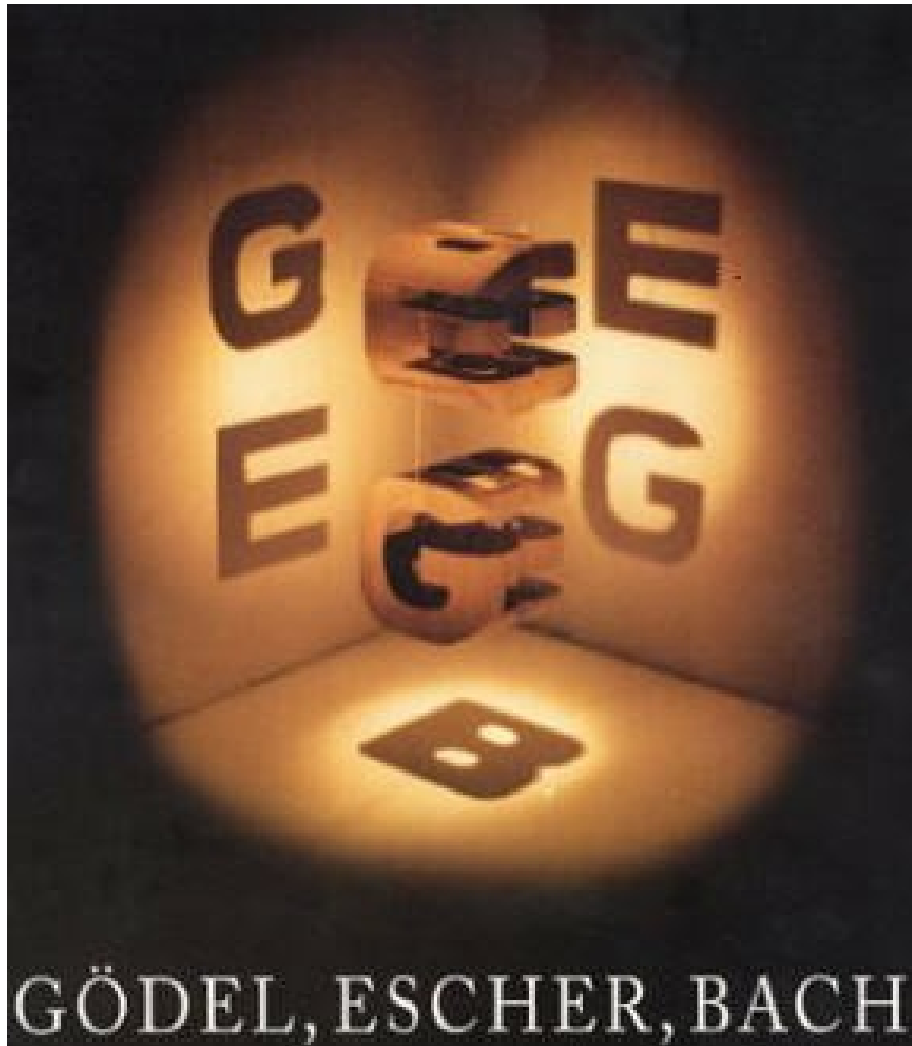
Case 2: Side B is heavier. Then the counterfeit must be in group B .

Case 3: The scale is balanced. Then the counterfeit must be in group C , since it isn't in groups A or B .

In all cases, we use one weighing to find a set of 3^n coins containing the counterfeit coin. By the inductive hypothesis, with n more weighings, we can find which of these 3^n coins is counterfeit. This means that we can find the counterfeit of 3^{n+1} coins in $n + 1$ weighings. Thus $P(n + 1)$ holds, completing the induction. ■

The MU Puzzle

Gödel, Escher Bach: An Eternal Golden Braid

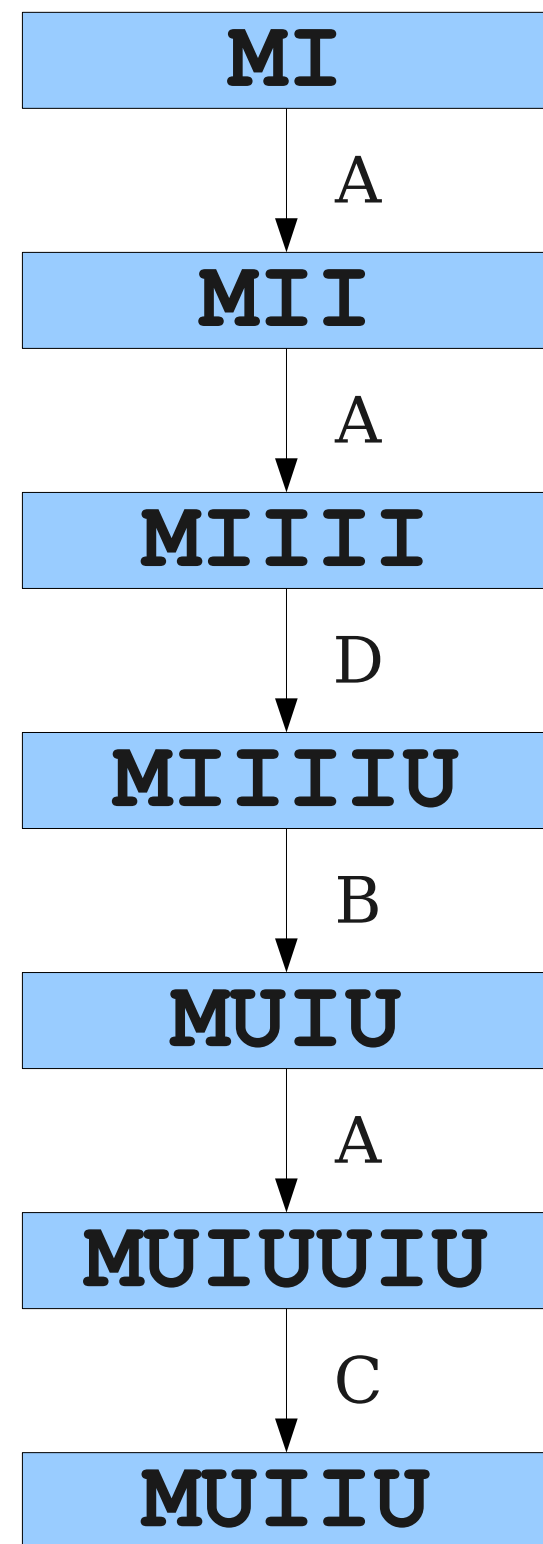


- Pulitzer-Prize winning book exploring recursion, computability, and consciousness.
- Written by Douglas Hofstadter, computer scientist at Indiana University.
- A great (but dense!) read.

The **MU** Puzzle

- Begin with the string **MI**.
- Repeatedly apply one of the following operations:
 - Double the contents of the string after the **M**: for example, **MIIU** becomes **MIUIIU** or **MI** becomes **MII**.
 - Replace **III** with **U**: **MIIII** becomes **MUI** or **MIU**
 - Append **U** to the string if it ends in **I**: **MI** becomes **MIU**
 - Remove any **UU**: **MUUU** becomes **MU**
- **Question:** How do you transform **MI** to **MU**?

- A) Double the contents of the string after **M**.
- B) Replace **III** with **U**.
- C) **Remove UU**
- D) Append **U** if the string ends in **I**.



Try It!

Starting with **MI**, apply these operations to make **MU**:

- A) Double the contents of the string after **M**.
- B) Replace **III** with **U**.
- C) Remove **UU**
- D) Append **U** if the string ends in **I**.

Not a single person in this room
was able to solve this puzzle.

Are we even sure that there is a solution?

Counting I's



The Key Insight

- Initially, the number of **I**'s is **not** a multiple of three.
- To make **MU**, the number of **I**'s must end up as a multiple of three.
- Can we *ever* make the number of **I**'s a multiple of three?

Lemma: Beginning with **MI** and applying any legal sequence of moves, the number of **I**s never becomes a multiple of three.

Proof: By induction. Let $P(n)$ be “Starting with **MI** and making n moves, the number of **I**s is not a multiple of 3.” We prove $P(n)$ holds for all $n \in \mathbb{N}$. As a base case, we prove $P(0)$, that after making no moves the number of **I**s is not a multiple of 3. **MI** has one **I** in it, which is not a multiple of 3.

For the inductive step, assume for some $n \in \mathbb{N}$ that $P(n)$ holds: after any sequence of n moves, the number of **I**s is not a multiple of 3. We prove $P(n+1)$: after $n+1$ moves, the number of **I**s is not a multiple of 3. Any sequence of $n+1$ moves is a sequence of n moves followed by an $(n+1)$ st move. By the inductive hypothesis, after the first n moves, the number of **I**s is not a multiple of 3, so before the $(n+1)$ st move, the number of **I**s equals either $3k+1$ or $3k+2$ for some $k \in \mathbb{N}$. Consider the $(n+1)$ st move:

Case 1: “Double the string after the **M**.” Then we end up with either $2(3k+1) = 6k+2 = 3(2k)+2$ or $2(3k+2) = 6k+4 = 3(2k+1) + 1$ **I**s, neither of which is a multiple of 3.

Case 2: “Delete **UU**” or “append **U**.” The number of **I**s is unchanged.

Case 3: “Delete **IIII**.” The number of **I**s either changes from $3k + 1$ to $3k+1 - 3 = 3(k-1)+1$ or from $3k+2$ to $3k+2 - 3 = 3(k-1) + 2$, neither of which is a multiple of 3.

Thus after the $(n+1)$ st move, the number of **I**s is not a multiple of three, so $P(n+1)$ holds, completing the induction. ■

Theorem: The **MU** puzzle has no solution.

Proof: By contradiction; assume it has a solution. By our lemma, the number of **I**'s in the final string must not be a multiple of three. However, for the solution to be valid, the number of **I**'s must be 0, which is a multiple of three. We have reached a contradiction, so our assumption was wrong and the **MU** puzzle has no solution. ■

Algorithms and Loop Invariants

- The proof we just made had the form
 - “If P is true before we perform an action, it is true after we perform an action.”
- We could therefore conclude that after any series of actions of any length, if P was true beforehand, it is true now.
- In algorithmic analysis, this is called a **loop invariant**.
- Proofs on algorithms often use loop invariants to reason about the behavior of algorithms.
 - Take CS161 for more details!

Next Time

- **Variations on Induction**
 - Starting induction later.
 - Taking larger steps.
 - Complete induction.