Mathematical Induction

Everybody – do the wave!

The Wave

- If done properly, everyone will eventually end up joining in.
- Why is that?
  - Someone (me!) started everyone off.
  - Once the person before you did the wave, you did the wave.

The **principle of mathematical induction** states that if for some $P(n)$ the following hold:

$$\boldsymbol{P(0)} \textbf{ is true}$$

If it starts true…

and

…and it stays true…

For any $n \in \mathbb{N}$, we have $P(n) \to P(n + 1)$

then

…then it's always true.

For any $n \in \mathbb{N}$, $P(n)$ is true.

Induction, Intuitively

- It's true for 0.
- Since it's true for 0, it's true for 1.
- Since it's true for 1, it's true for 2.
- Since it's true for 2, it's true for 3.
- Since it's true for 3, it's true for 4.
- Since it's true for 4, it's true for 5.
- Since it's true for 5, it's true for 6.
- …

Proof by Induction

- Suppose that you want to prove that some property $P(n)$ holds of all natural numbers. To do so:
  - Prove that $P(0)$ is true.
    - This is called the **basis** or the **base case**.
  - Prove that for all $n \in \mathbb{N}$, that if $P(n)$ is true, then $P(n + 1)$ is true as well.
    - This is called the **inductive step**.
    - $P(n)$ is called the **inductive hypothesis**.
  - Conclude by induction that $P(n)$ holds for all n.

Some Summations

$$\mathbf{2^0} = 1$$

$$\mathbf{2^0 + 2^1} = 1 + 2 = 3$$

$$\mathbf{2^0 + 2^1 + 2^2} = 1 + 2 + 4 = 7$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3} = 1 + 2 + 4 + 8 = 15$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3 + 2^4} = 1 + 2 + 4 + 8 + 16 = 31$$

$$2^0 = 1 \quad = 2^1 - 1$$

$$2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7 = 2^3 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15 = 2^4 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$$

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Just as in a proof by contradiction or contrapositive, we should mention this proof is by induction.

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Now, we state what property $P(n)$ we are going to prove holds for all $n \in \mathbb{N}$.

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The first step of an inductive proof is to show $P(0)$. We explicitly state what $P(0)$ is, then try to prove it. We can prove $P(0)$ using any proof technique we'd like.

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The goal of this step is to prove

"For any $n \in \mathbb{N}$, if $P(n)$, then $P(n + 1)$"

To do this, we'll choose an arbitrary n, assume that $P(n)$ holds, then try to prove $P(n + 1)$.

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Here, we're explicitly stating $P(n + 1)$, which is what we want to prove. Now, we can use any proof technique we want to try to prove it.

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We're assuming that $P(n)$ is true, so we can replace this sum with the value $\mathbf{2^n - 1}$.

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1 powers of two. This is

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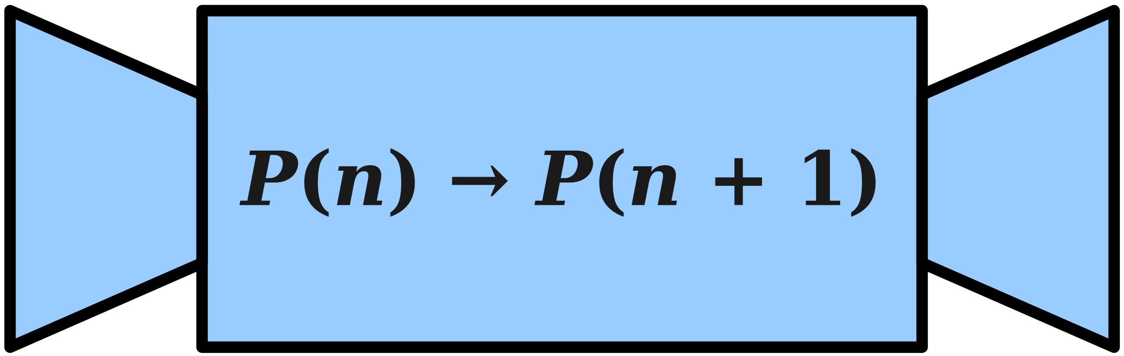
Structuring a Proof by Induction

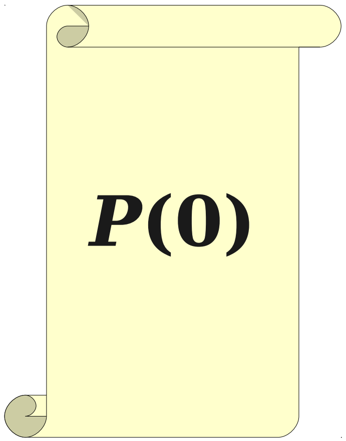
- State that your proof works by induction.
- State your choice of $P(n)$.
- Prove the base case:
  - State what $P(0)$ is, then prove it using any technique you'd like.
- Prove the inductive step:
  - State that for some arbitrary $n \in \mathbb{N}$ that you're assuming $P(n)$ and mention what $P(n)$ is.
  - State that you are trying to prove $P(n + 1)$ and what $P(n + 1)$ means.
  - Prove $P(n + 1)$ using any technique you'd like.
- This is very rigorous, so as we gain more familiarity with induction we will start being less formal in our proofs.

Induction, Intuitively

- You can imagine an “machine” that turns proofs of $P(n)$ into proofs of $P(n+1)$.
- Starting with a proof of $P(0)$, we can run the machine as many times as we'd like to get proofs of $P(1)$, $P(2)$, $P(3)$, … .
- The principle of mathematical induction says that this style of reasoning is a rigorous argument.

Why This Proof Works





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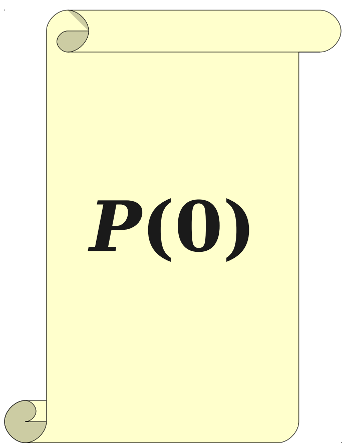

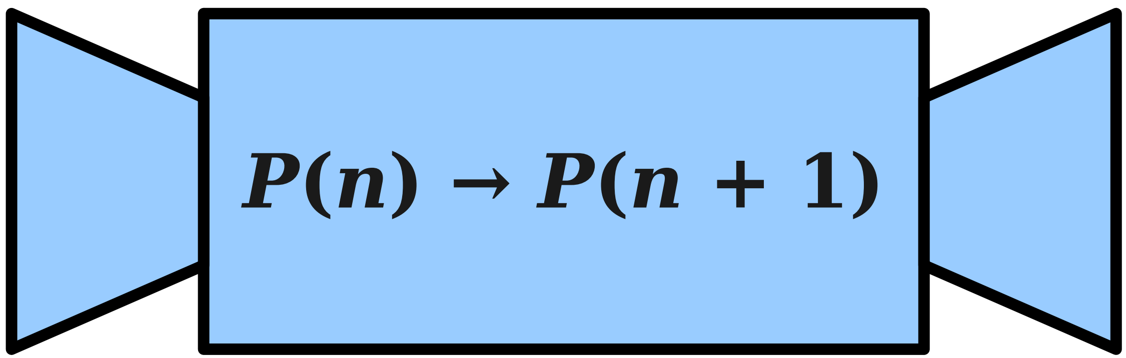

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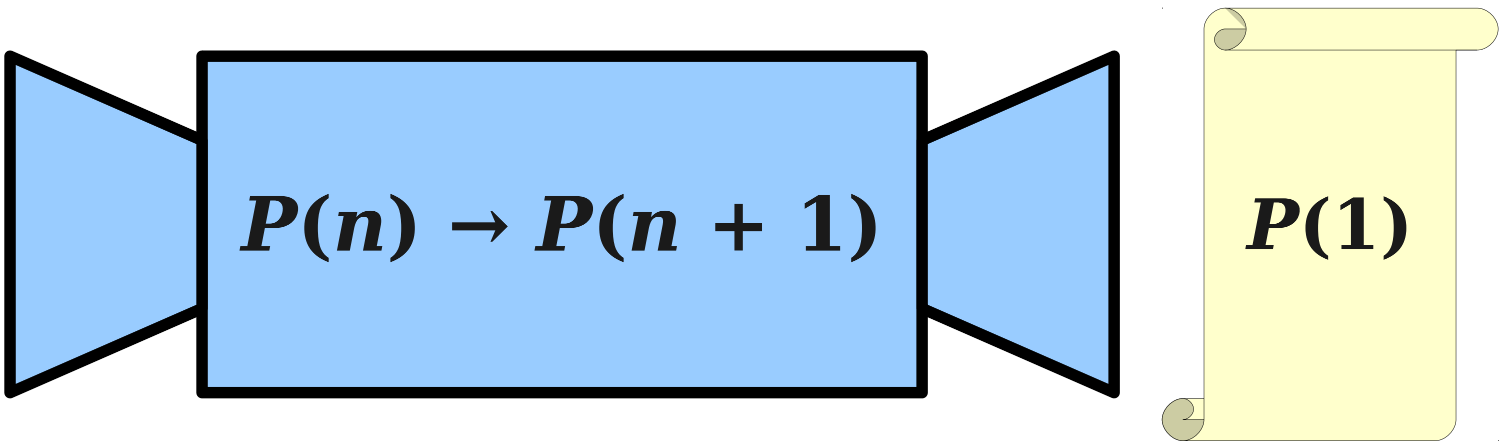

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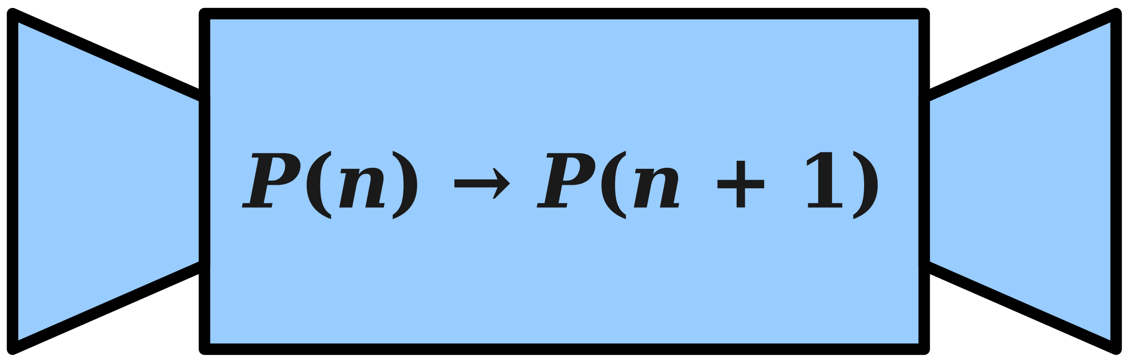

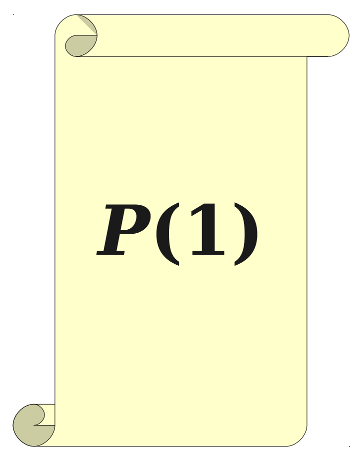

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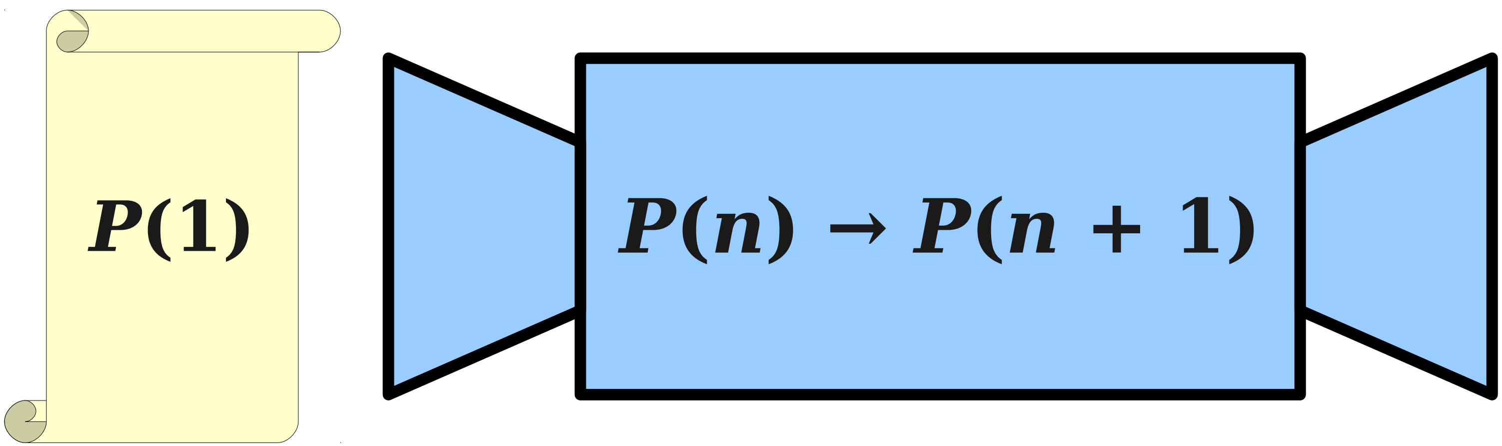

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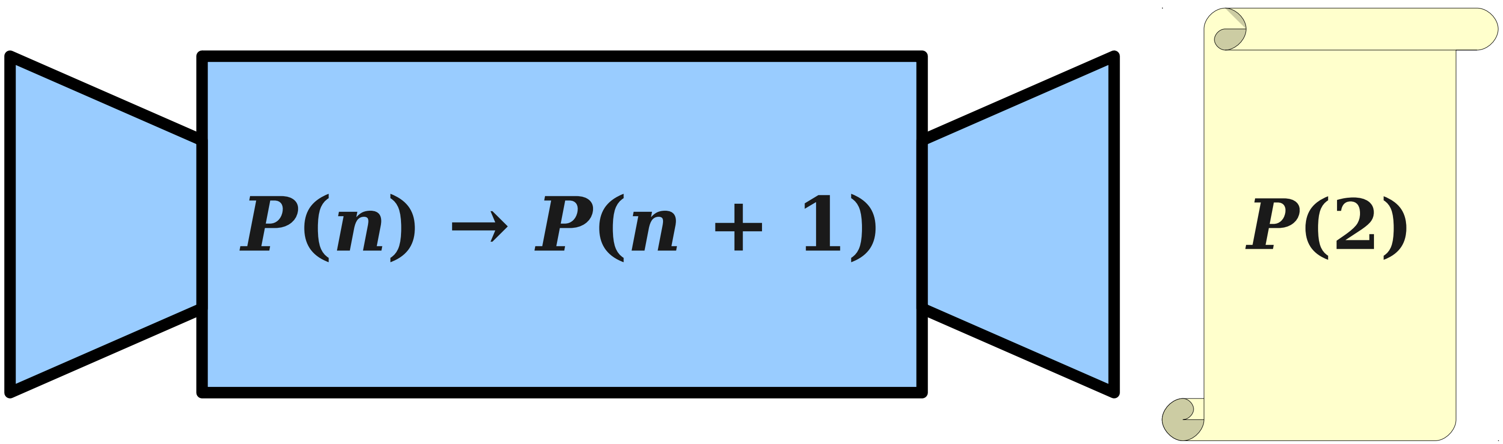

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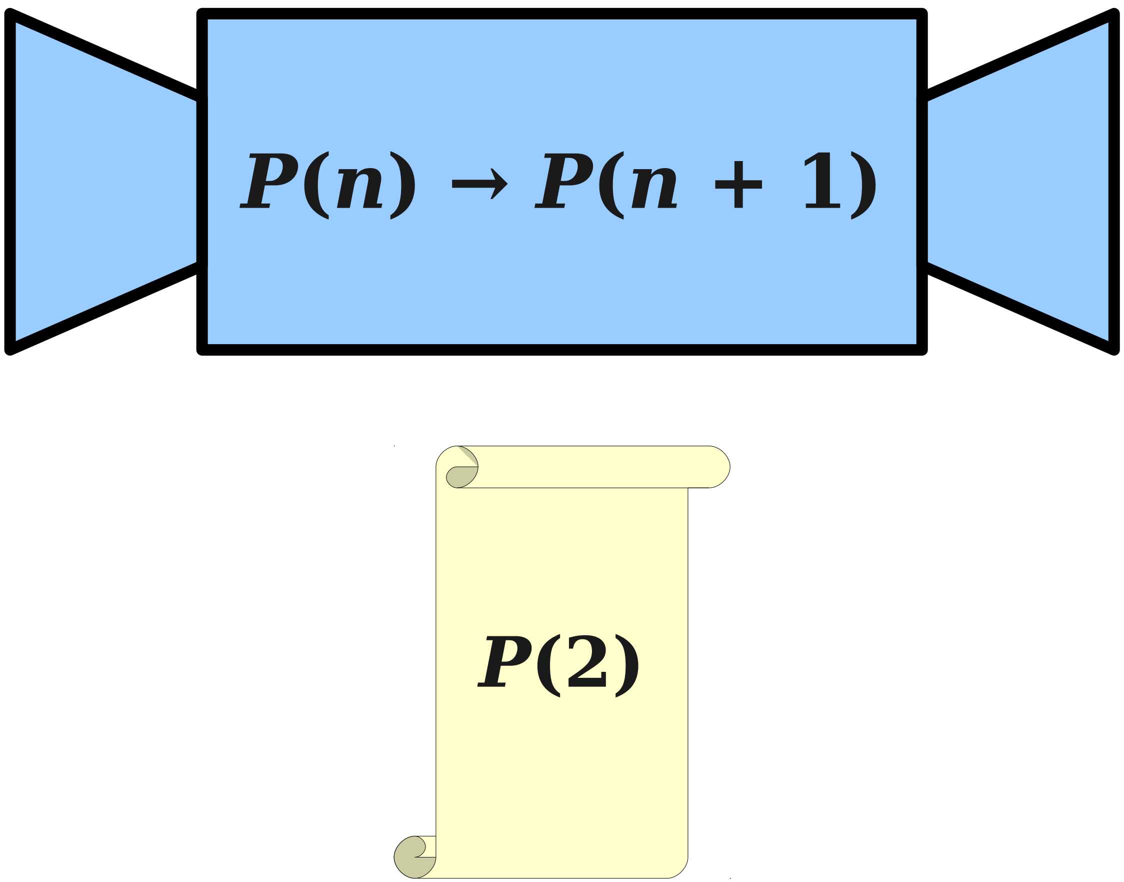

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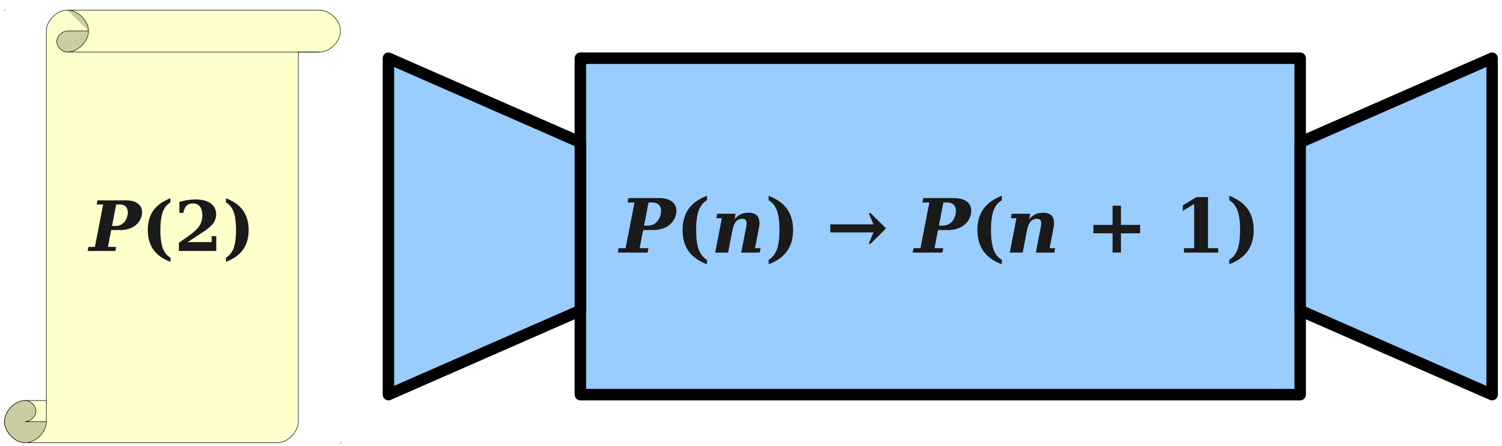

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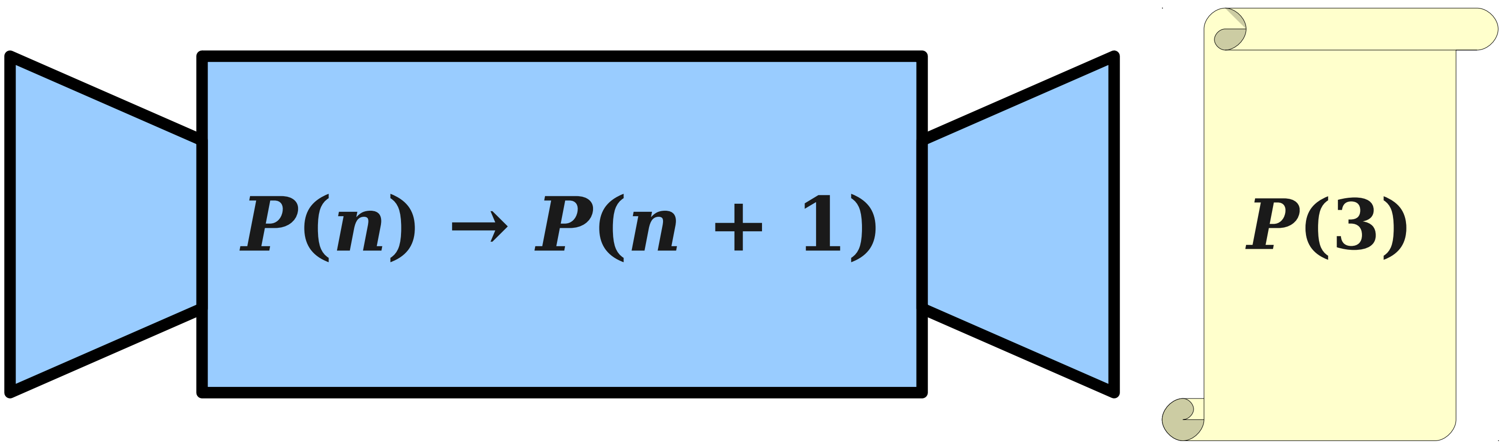

A Quick Aside

- This result helps explain the range of numbers that can be stored in an **`int`**.
- If you have an unsigned 32-bit integer, the largest value you can store is given by $1 + 2 + 4 + 8 + \ldots + 2^{31} = 2^{32} - 1$.
- This formula for sums of powers of two has many other uses as well. We'll see one next week.

Notation: Summations

- **Summation notation** gives a compact way for discussing sums of multiple terms.
- For example, instead of writing the sum $1 + 2 + 3 + \ldots + n$, we can write

sum from i = 1 to n

$$\sum_{i=1}^{n} i$$

of i

Summation Examples

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{i=0}^{3} 2^i = 2^0 + 2^1 + 2^2 + 2^3 = 15$$

$$\sum_{i=0}^{2} \left(i^2 - i\right) = \left(0^2 - 0\right) + \left(1^2 - 1\right) + \left(2^2 - 2\right) = 2$$

The Empty Sum

- A sum of no numbers is called the **empty sum** and is defined to be zero.
- Examples:

$$\sum_{i=1}^{0} 2^i = 0 \qquad \sum_{i=137}^{42} i^i = 0 \qquad \sum_{i=0}^{-1} i = 0$$

- Why do you think it's defined to be zero as opposed to some other number?

Theorem: For any natural number *n*, $\sum_{i=0}^{n-1} 2^i = 2^n - 1$

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In this context, ≡ means "is defined as." $P(n)$ is defined to be the assertion that the sum of the first n powers of two is $2^n - 1$.

Theorem: For any natural number n, $\sum_{i=0}^{n-1} 2^i = 2^n - 1$

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Here, we're "peeling off" the last term of the sum. Many inductive proofs on sums will use this trick.

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A Brief Interlude for Announcements

Recitation Sections

- Handout #06 contains several discussion questions for this week.
- We will set up several recitation sections where you can work through these problems with one of the TAs.
  - Dates/times announced later today.
  - All sections cover the same material.
- Solutions distributed at recitation sections and online later this week.

Problem Set Clarification

- All problem sets are designed to use only the material up to and include the lecture in which they are released.
- We'll explicitly mark any problems for which we won't have covered the requisite material.

Ask Us A Question:

"What are the criteria used to grade proofs in our problem sets?"

Back to our regularly scheduled programming…

Back to our regularly scheduled ~~programming~~...

math

How Not To Induct

An Incorrect Proof

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Where did we prove the base case?

To see this, note that

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$n-1$

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$P(n)$ holds, so

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Where did we prove the base case?

$^n + 2^n \;=\; 2^{n+1}$

induction. ■

When proving $P(n)$ is true
for all $n \in \mathbb{N}$ by induction,

make sure to show the base case!

Otherwise, your argument is invalid!

Why This Worked

- The math internally checked out because we made an incorrect assumption!
- Induction requires both the base case and the inductive step.
  - The base case shows that the property initially holds true.
  - The inductive step shows how each step influences the next.

The Counterfeit Coin Problem

Problem Statement

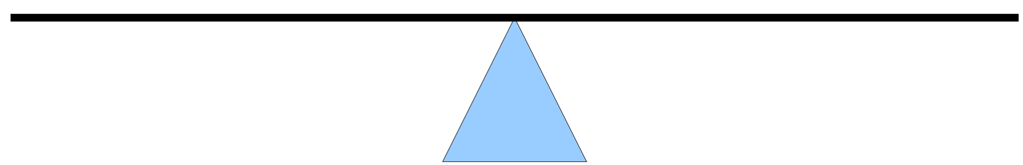
- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

Finding the Counterfeit Coin

1

2

3



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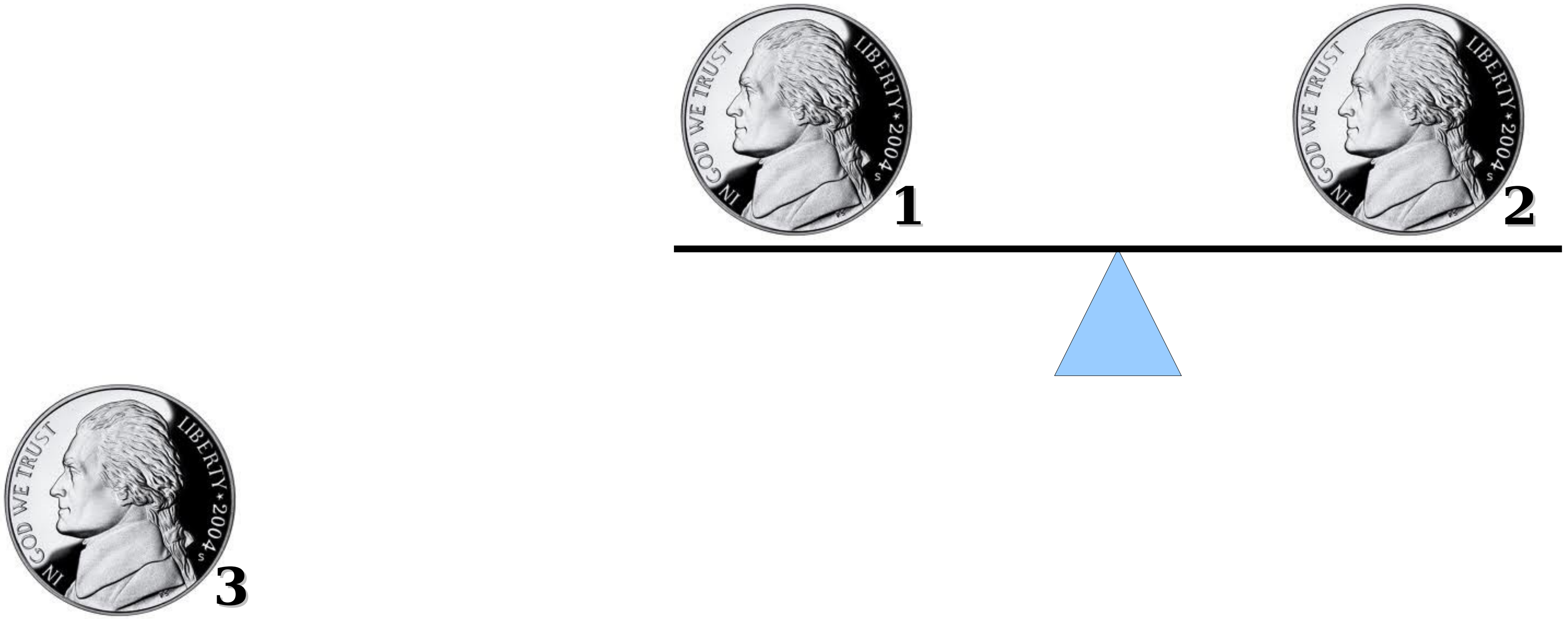

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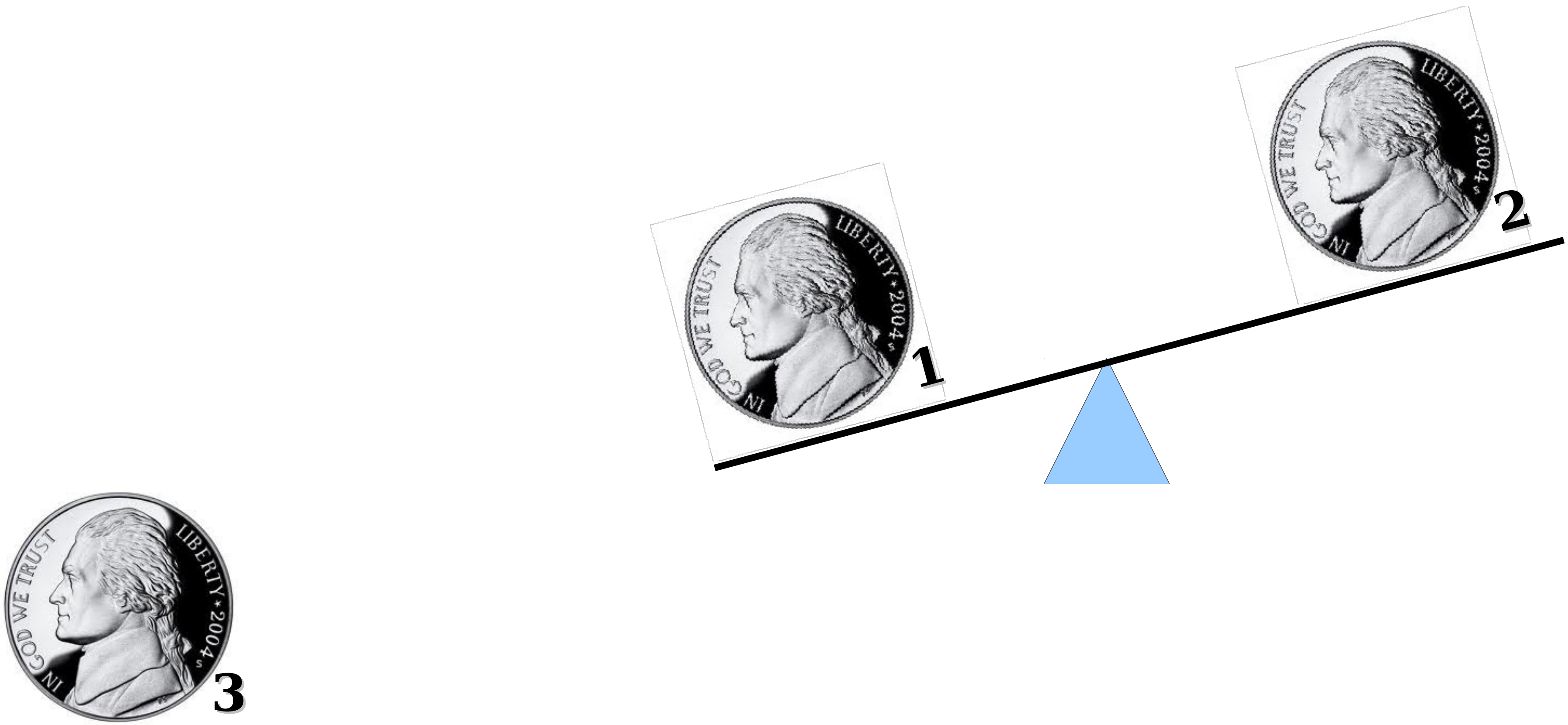

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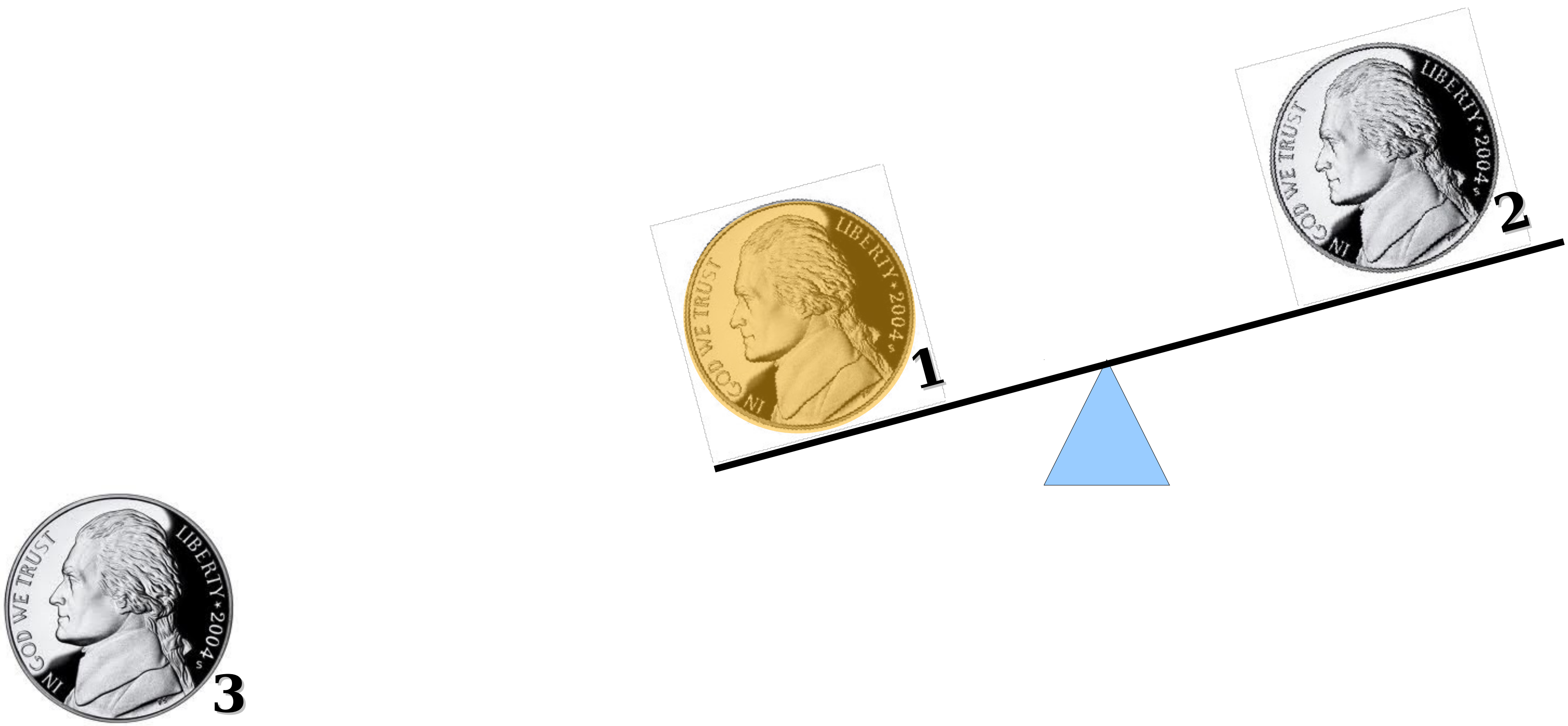

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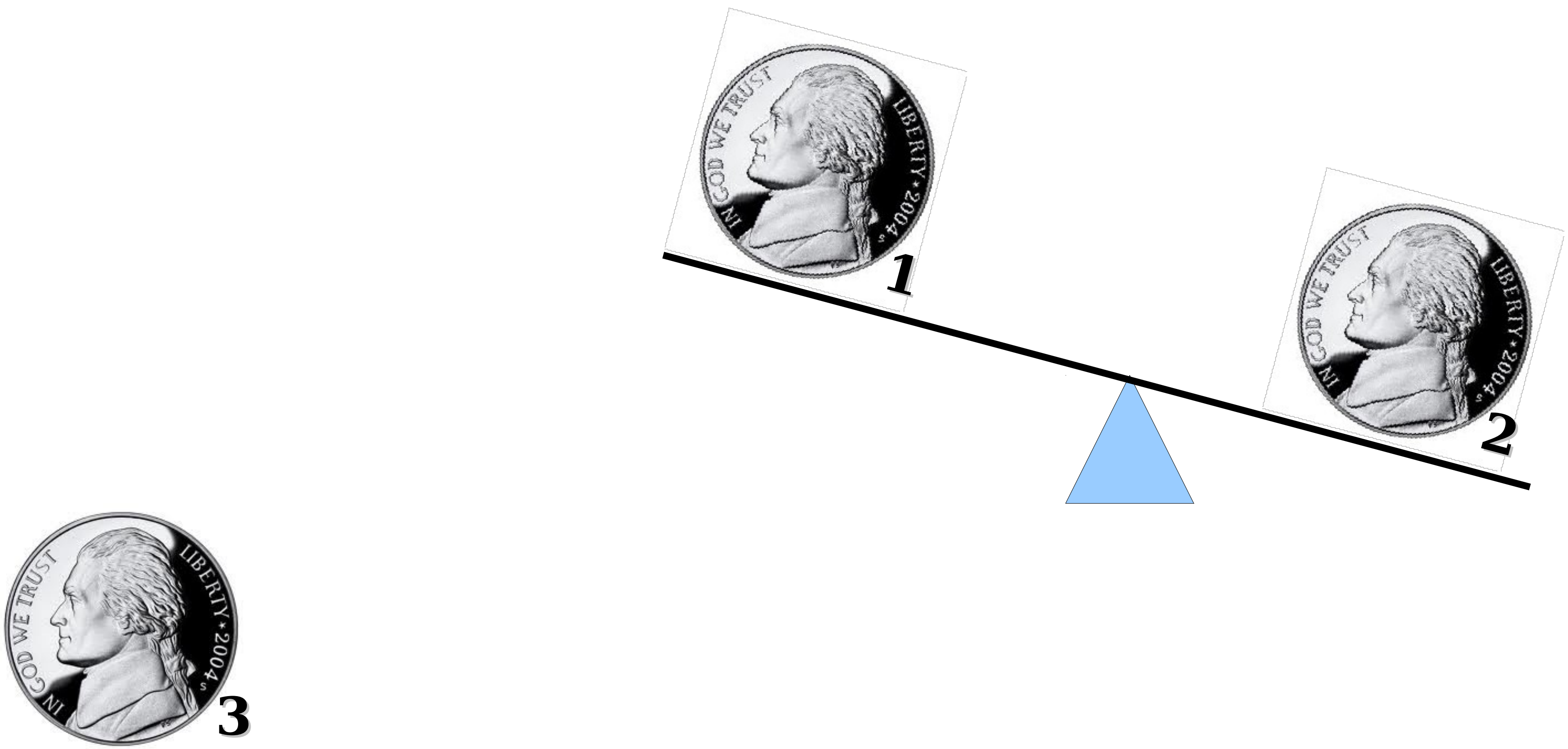

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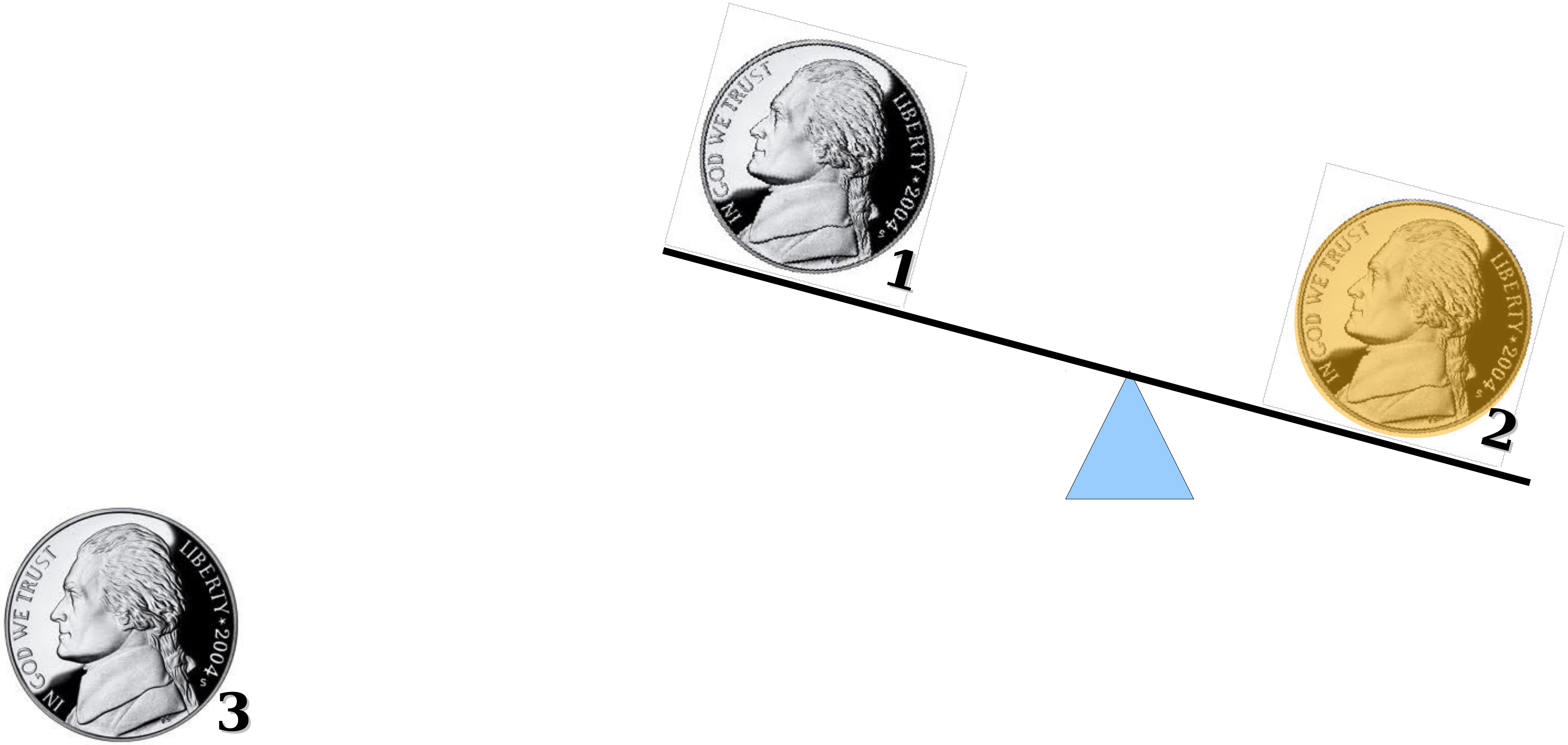

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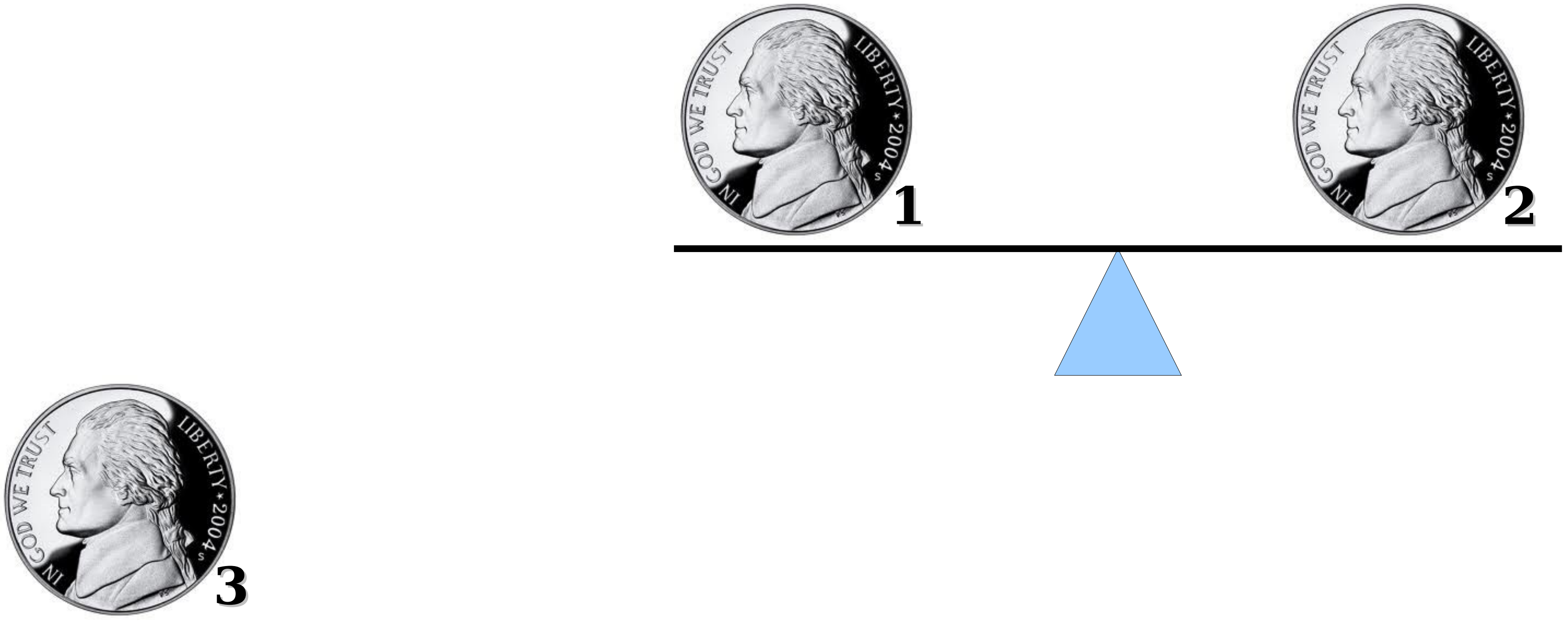

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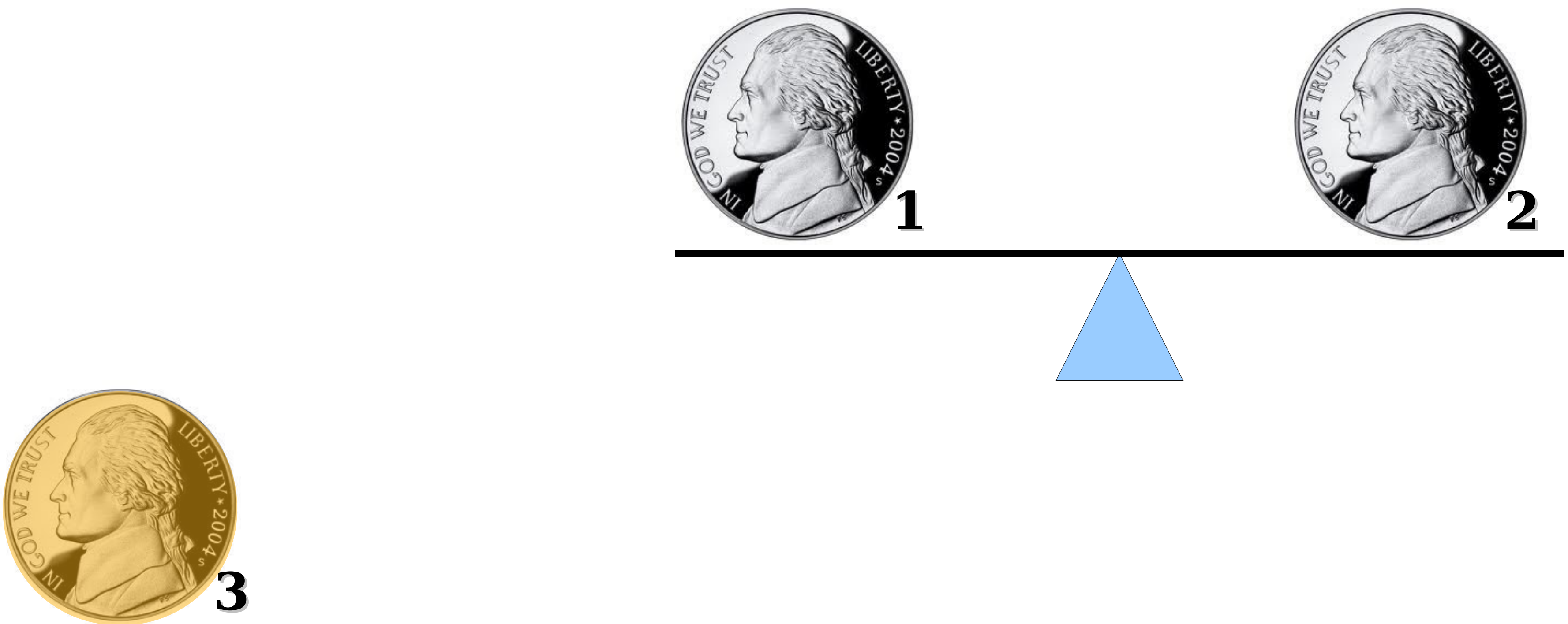
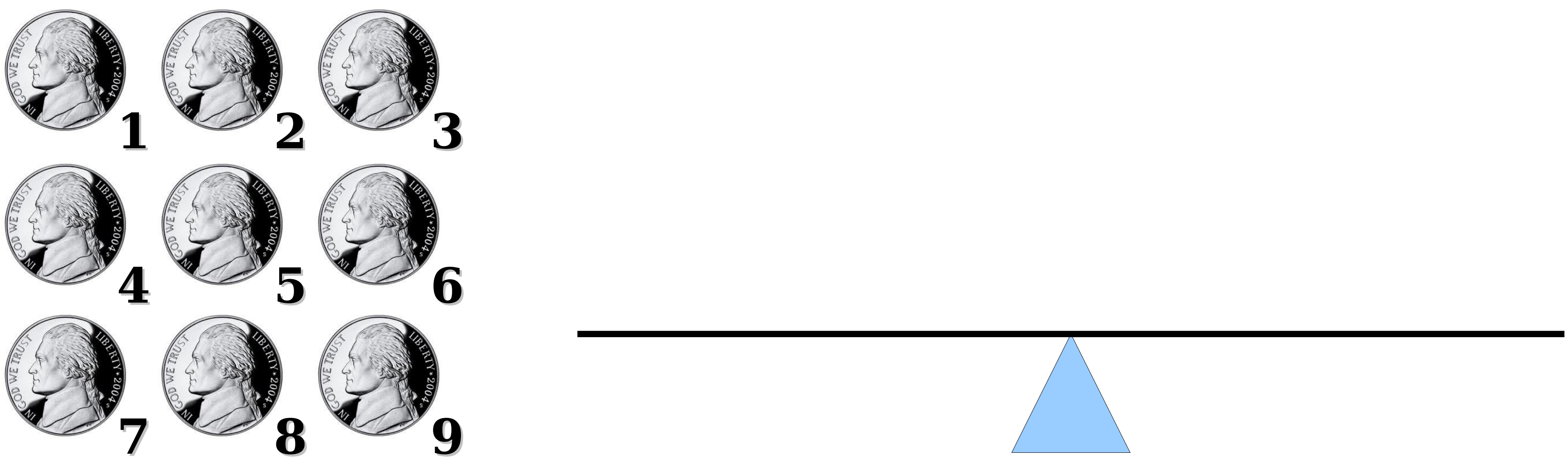


A Harder Problem

- You are given a set of **nine** seemingly identical coins, eight of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only **two** weighings on the balance, find the counterfeit coin.

Finding the Counterfeit Coin



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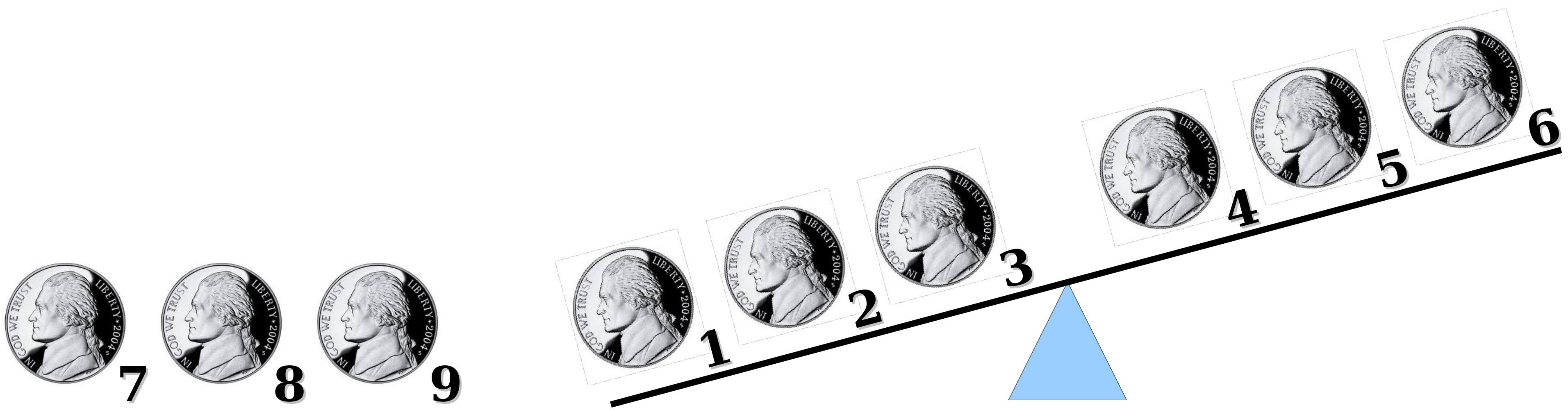

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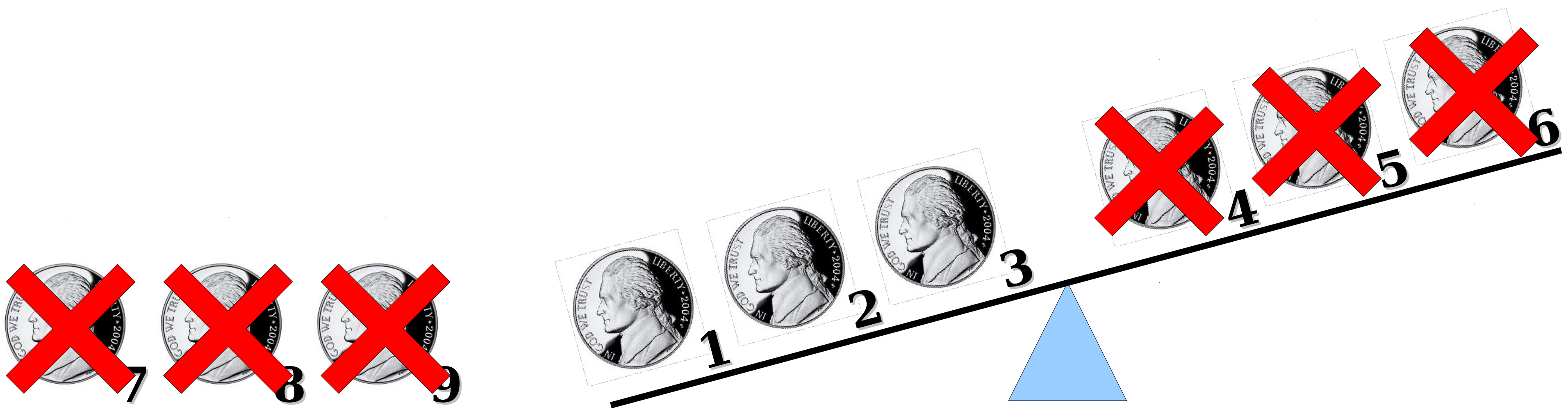

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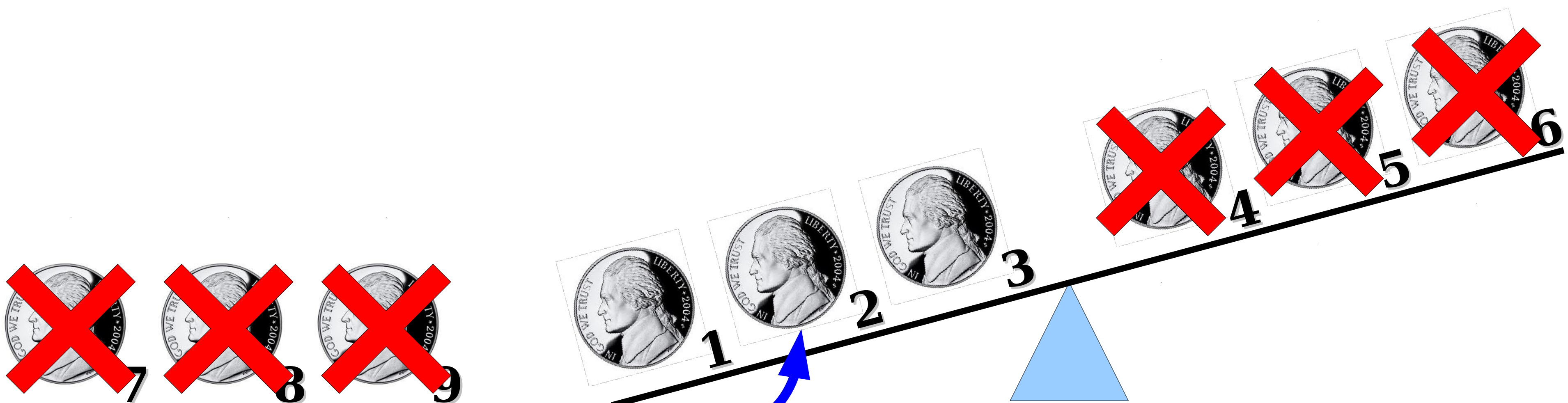
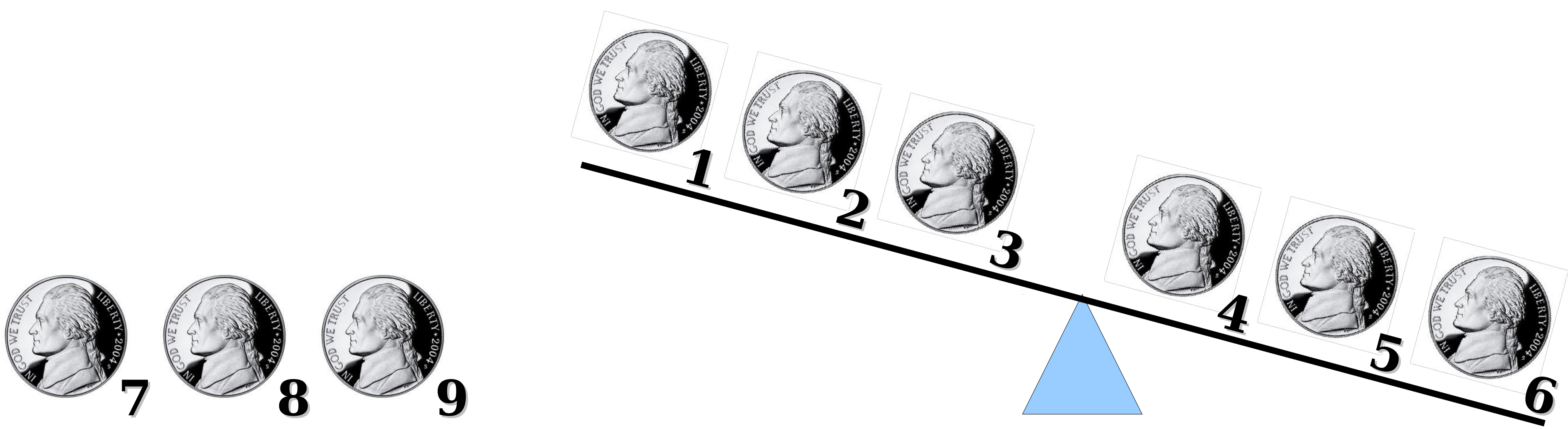


Now we have one weighing to find the counterfeit out of these three coins.

Finding the Counterfeit Coin



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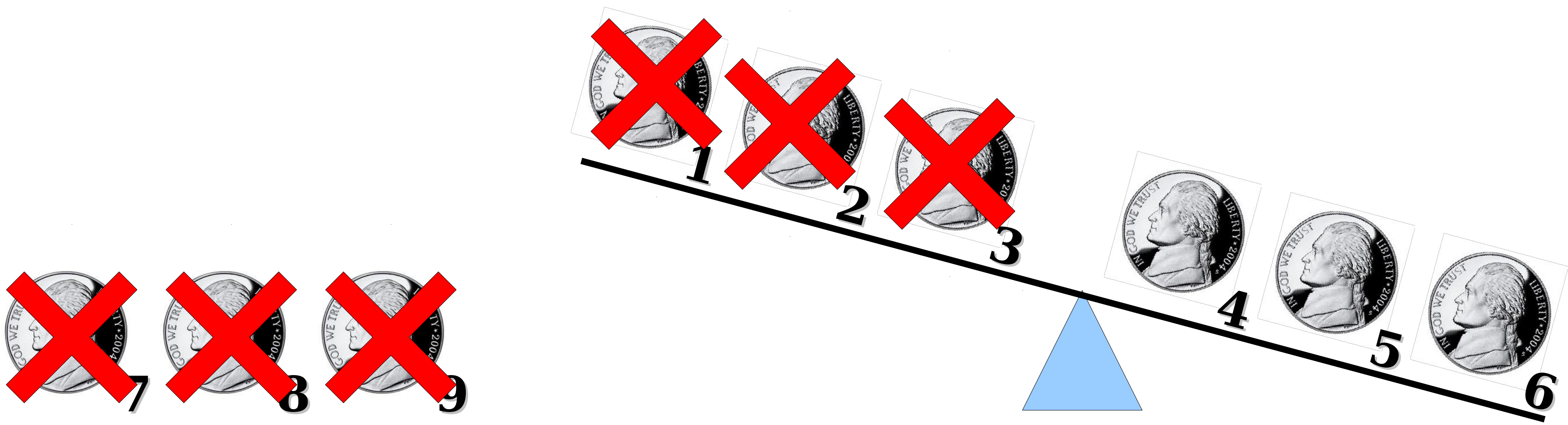

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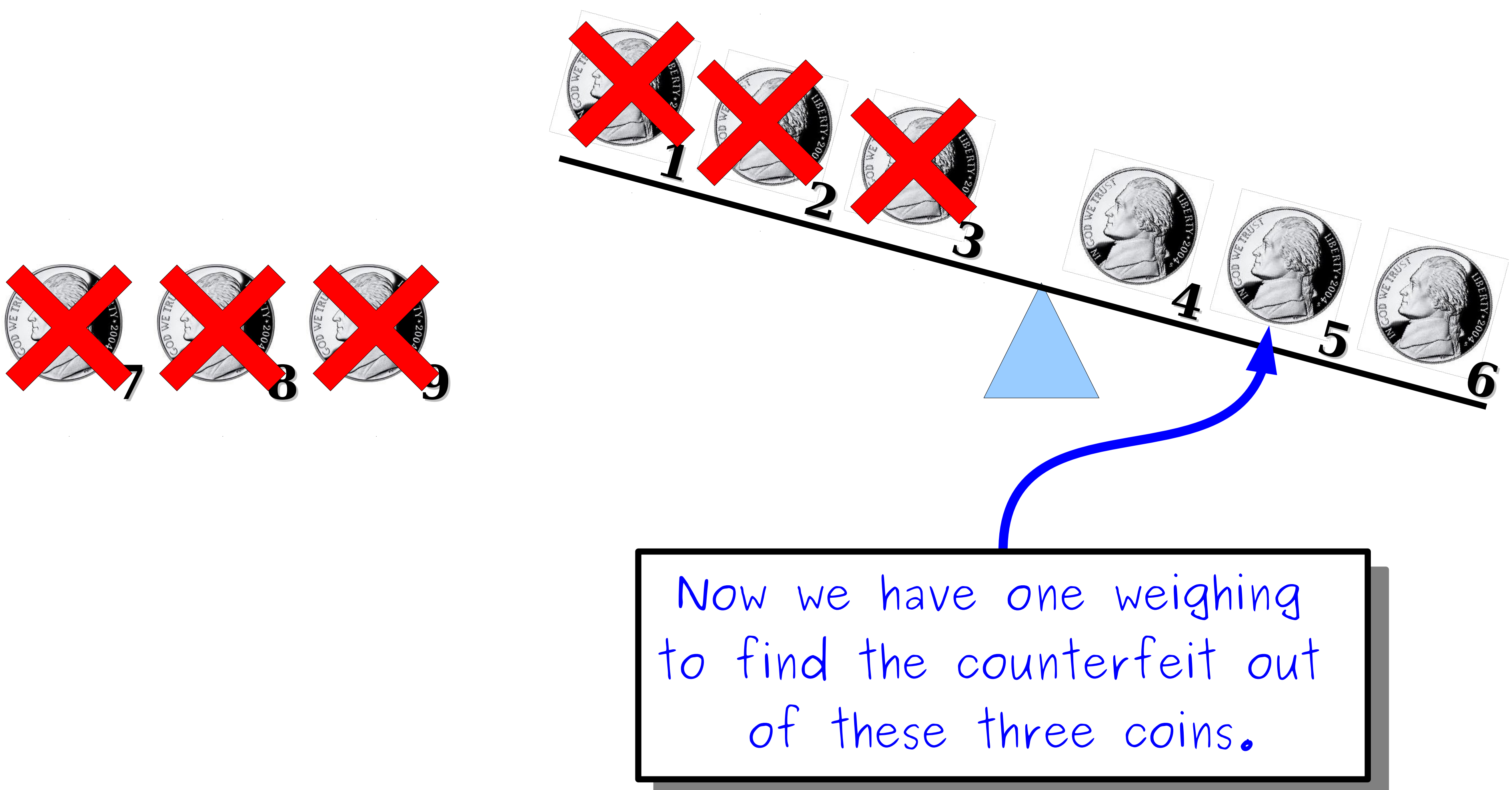

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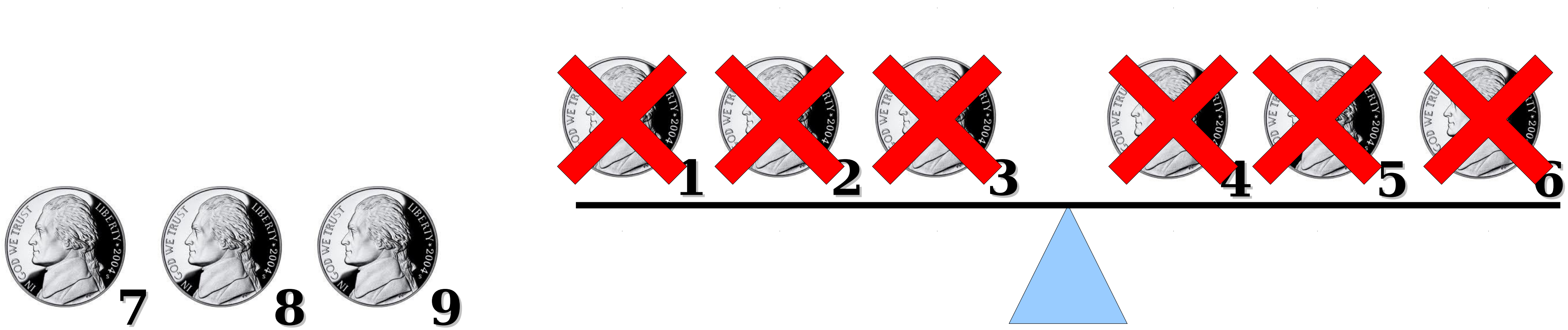
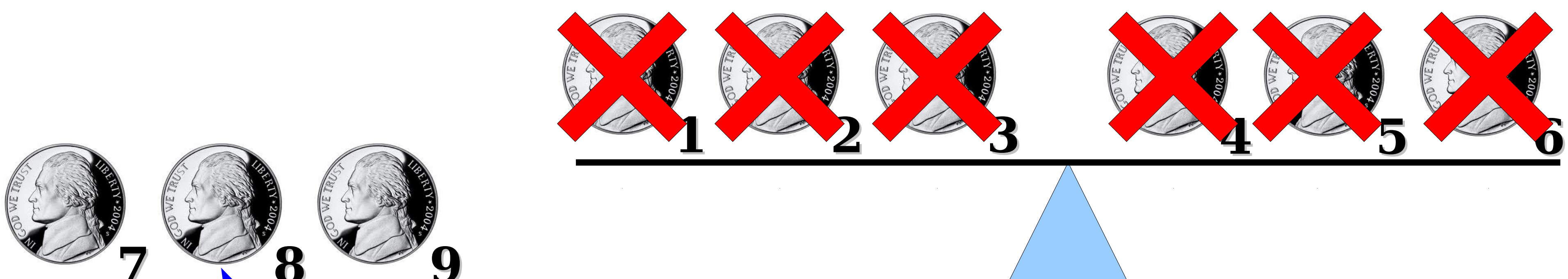


Finding the Counterfeit Coin



Now we have one weighing
to find the counterfeit out
of these three coins.

If we have n weighings on the scale, what is the largest number of coins out of which we can find the counterfeit?

A Pattern

- Assume out of the coins that are given, exactly one is counterfeit and weighs more than the other coins.
- If we have no weighings, how many coins can we have while still being able to find the counterfeit?
  - **One coin**, since that coin has to be the counterfeit!
- If we have one weighing, we can find the counterfeit out of **three** coins.
- If we have two weighings, we can find the counterfeit out of **nine** coins.

So far, we have

$$\mathbf{1, 3, 9 = 3^0, 3^1, 3^2}$$

Does this pattern continue?

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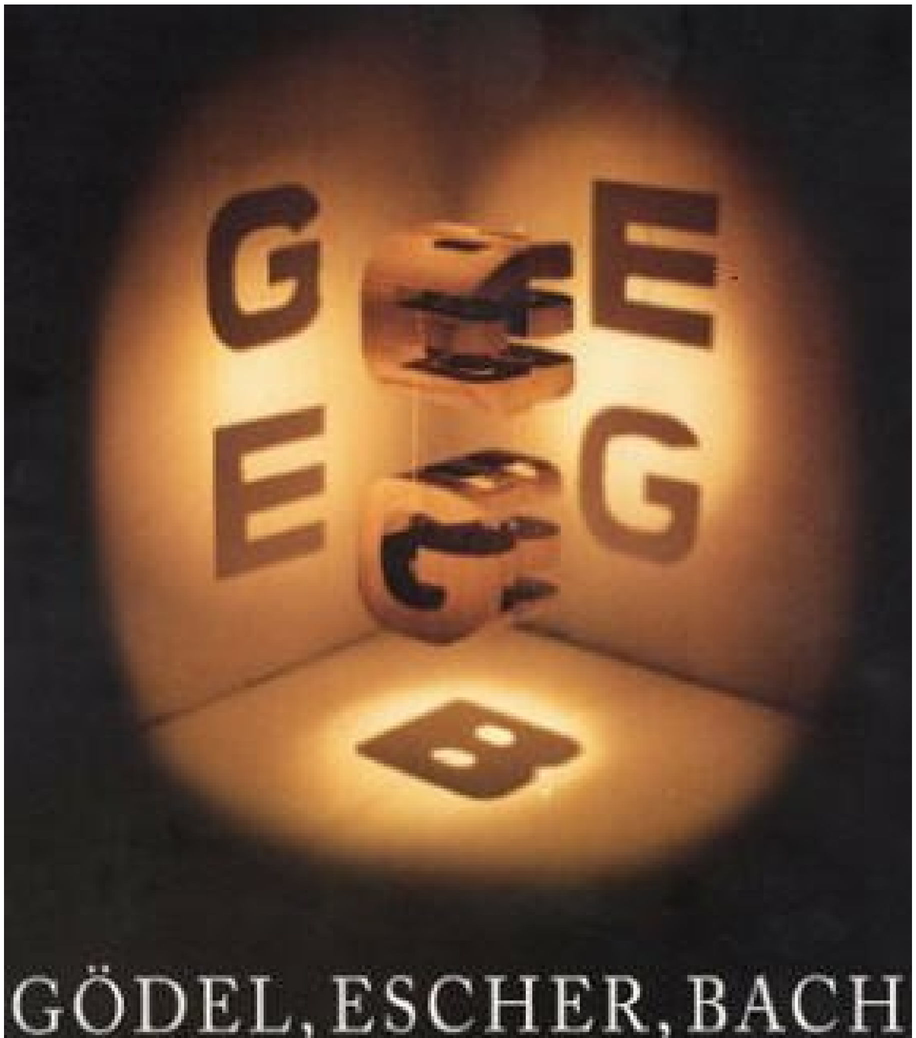
The **MU** Puzzle

Gödel, Escher Bach: An Eternal Golden Braid



- Pulitzer-Prize winning book exploring recursion, computability, and consciousness.
- Written by Douglas Hofstadter, computer scientist at Indiana University.
- A great (but dense!) read.

The **`MU`** Puzzle

- Begin with the string **`MI`**.
- Repeatedly apply one of the following operations:
  - Double the contents of the string after the **`M`**: for example, **`MIIU`** becomes **`MIIUIIU`** or **`MI`** becomes **`MII`**.
  - Replace **`III`** with **`U`**: **`MIIII`** becomes **`MUI`** or **`MIU`**
  - Append **`U`** to the string if it ends in **`I`**: **`MI`** becomes **`MIU`**
  - Remove any **`UU`**: **`MUUU`** becomes **`MU`**
- **Question**: How do you transform **`MI`** to **`MU`**?

A) Double the contents of the string after **`M`**.

B) Replace **`III`** with **`U`**.

C) Remove **`UU`**

D) Append **`U`** if the string ends in **`I`**.

MI

A) Double the contents of the string after **M**.

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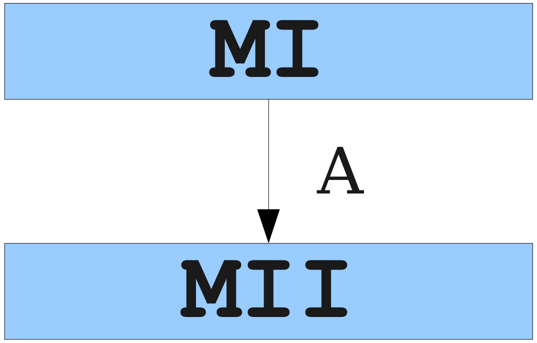

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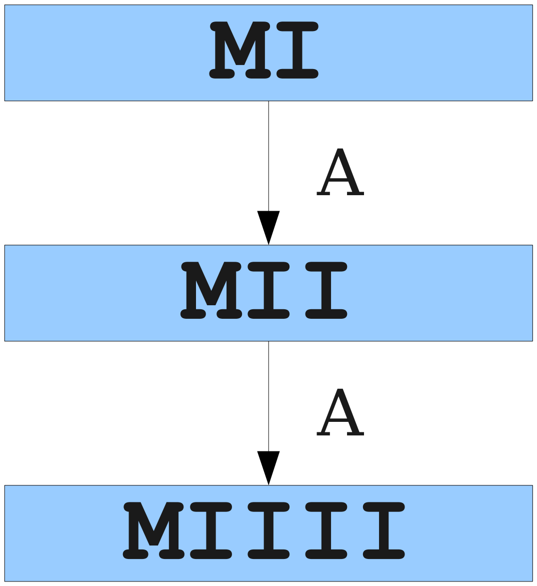

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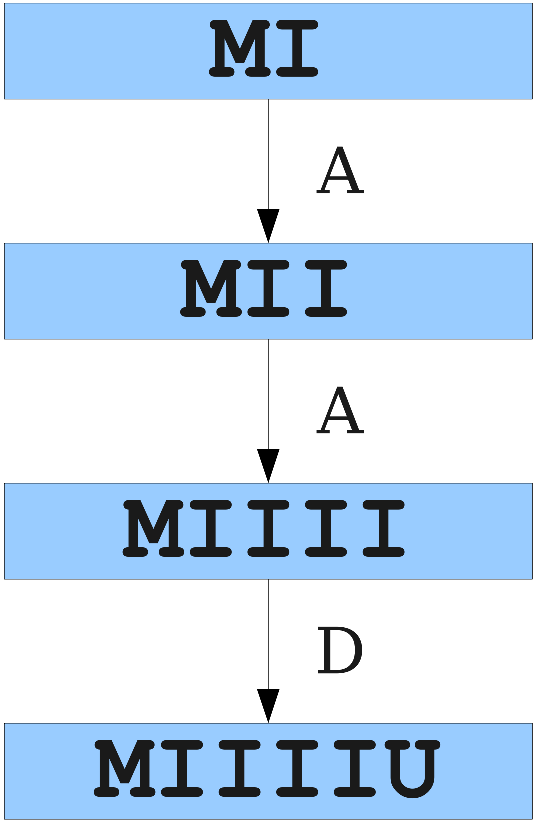

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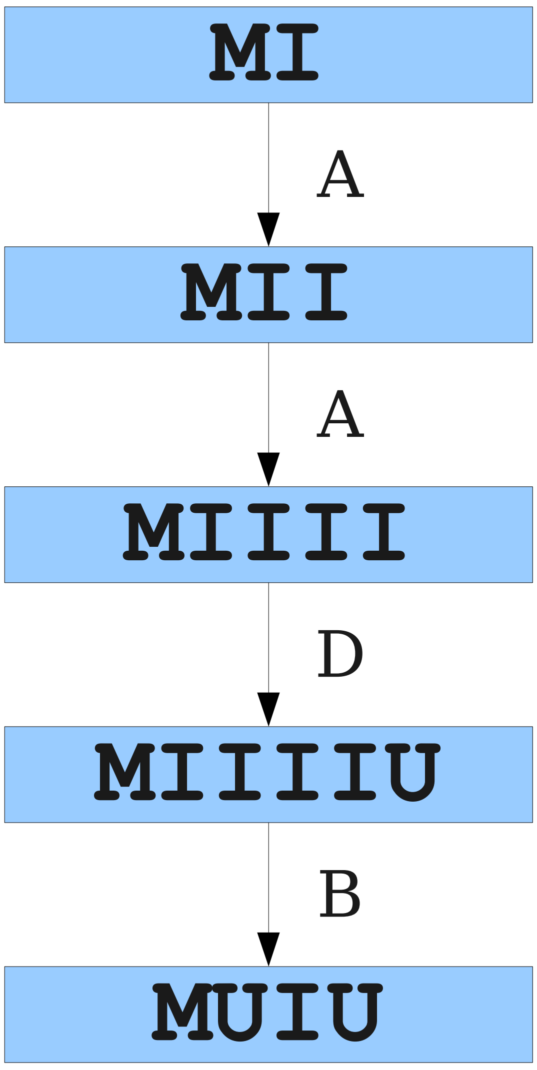

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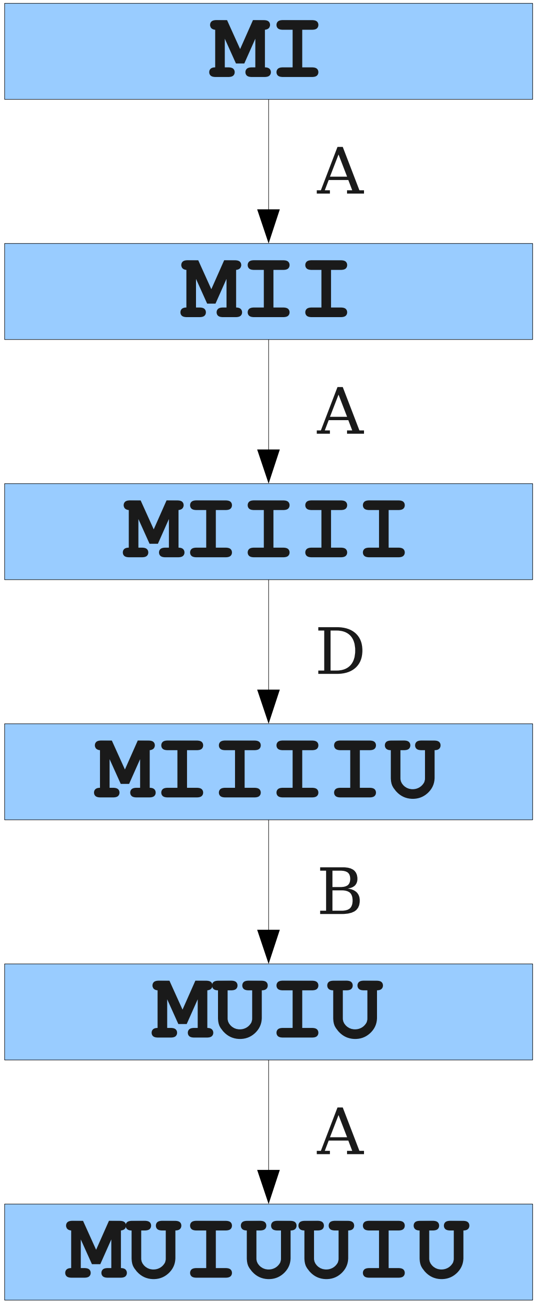

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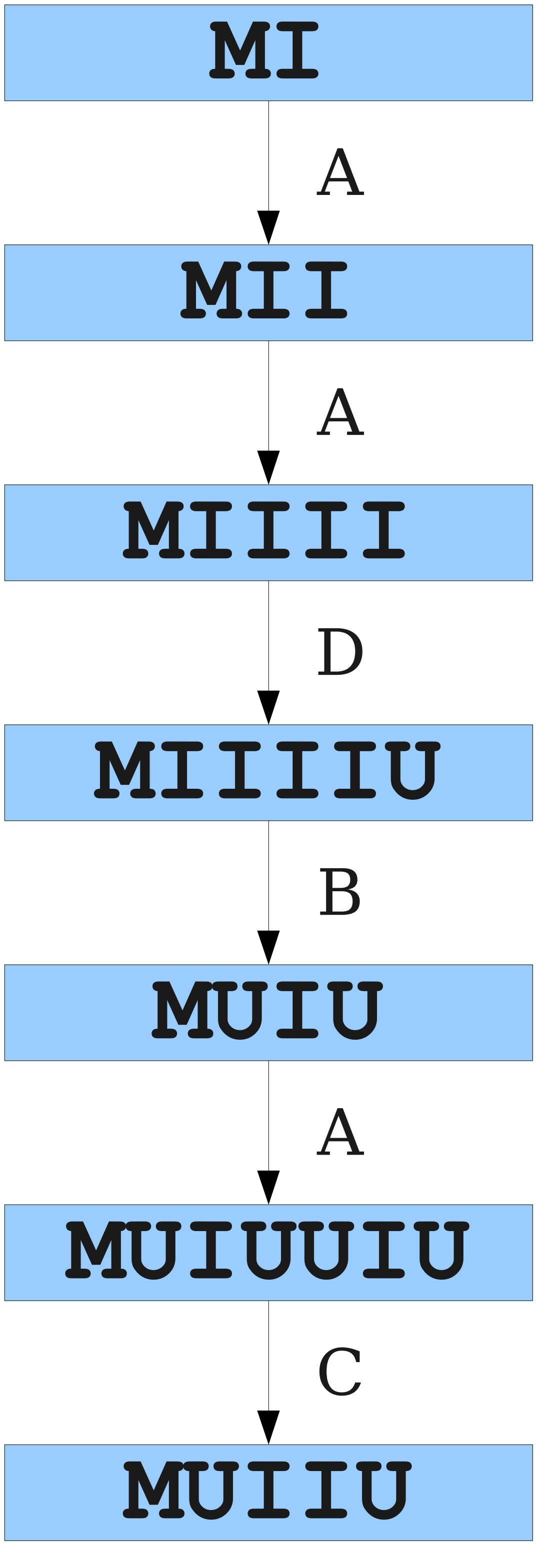

Try It!

Starting with **`MI`**, apply these operations to make **`MU`**:

A) Double the contents of the string after **`M`**.

B) Replace **`III`** with **`U`**.

C) Remove **`UU`**

D) Append **`U`** if the string ends in **`I`**.

Not a single person in this room
was able to solve this puzzle.

Are we even sure that there is a solution?

Counting `I`'s

Counting **I**'s

MI	1
MII	2
MIIII	4
MIIIIU	4
MIIIIUIIIIU	8
MIIIIUUIU	5
MIIIIUUIUIIIIUUIU	10
MUIUUIUIIIIUUIU	7

The Key Insight

- Initially, the number of `I`'s is **not** a multiple of three.
- To make `MU`, the number of `I`'s must end up as a multiple of three.
- Can we *ever* make the number of `I`'s a multiple of three?

Lemma: Beginning with **`MI`** and applying any legal sequence of moves, the number of **`I`**s never becomes a multiple of three.

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Lemma: Beginning with **`MI`** and applying any legal sequence of moves, the number of **`I`**s never becomes a multiple of three.

Proof: By induction. Let $P(n)$ be "Starting with **`MI`** and making n moves, the number of **`I`**s is not a multiple of 3." We prove $P(n)$ holds for all $n \in \mathbb{N}$. As a base case, we prove $P(0)$, that after making no moves the number of **`I`**s is not a multiple of 3. **`MI`** has one **`I`** in it, which is not a multiple of 3.

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Case 1: "Double the string after the **`M`**."

Case 2: "Delete **`UU`**" or "append **`U`**."

Case 3: "Delete **`III`**."

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Case 1: "Double the string after the **`M`**." Then we end up with either $2(3k+1) = 6k+2 = 3(2k)+2$ or $2(3k+2) = 6k+4 = 3(2k+1) + 1$ **`I`**s, neither of which is a multiple of 3.

Case 2: "Delete **`UU`**" or "append **`U`**."

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Lemma: Beginning with **`MI`** and applying any legal sequence of moves, the number of **`I`**s never becomes a multiple of three.

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Lemma: Beginning with **`MI`** and applying any legal sequence of moves, the number of **`I`**s never becomes a multiple of three.

Proof: By induction. Let $P(n)$ be "Starting with **`MI`** and making n moves, the number of **`I`**s is not a multiple of 3." We prove $P(n)$ holds for all $n \in \mathbb{N}$. As a base case, we prove $P(0)$, that after making no moves the number of **`I`**s is not a multiple of 3. **`MI`** has one **`I`** in it, which is not a multiple of 3.

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Case 1: "Double the string after the **`M`**." Then we end up with either $2(3k+1) = 6k+2 = 3(2k)+2$ or $2(3k+2) = 6k+4 = 3(2k+1) + 1$ **`I`**s, neither of which is a multiple of 3.

Case 2: "Delete **`UU`**" or "append **`U`**." The number of **`I`**s is unchanged.

Case 3: "Delete **`III`**." The number of **`I`**s either changes from $3k + 1$ to $3k+1 - 3 = 3(k-1)+1$ or from $3k+2$ to $3k+2 - 3 = 3(k-1) + 2$, neither of which is a multiple of 3.

Lemma: Beginning with **MI** and applying any legal sequence of moves, the number of **I**s never becomes a multiple of three.

Proof: By induction. Let $P(n)$ be "Starting with **MI** and making n moves, the number of **I**s is not a multiple of 3." We prove $P(n)$ holds for all $n \in \mathbb{N}$. As a base case, we prove $P(0)$, that after making no moves the number of **I**s is not a multiple of 3. **MI** has one **I** in it, which is not a multiple of 3.

For the inductive step, assume for some $n \in \mathbb{N}$ that $P(n)$ holds: after any sequence of n moves, the number of **I**s is not a multiple of 3. We prove $P(n+1)$: after $n+1$ moves, the number of **I**s is not a multiple of 3. Any sequence of $n+1$ moves is a sequence of n moves followed by an $(n+1)$st move. By the inductive hypothesis, after the first n moves, the number of **I**s is not a multiple of 3, so before the $(n+1)$st move, the number of **I**s equals either $3k+1$ or $3k+2$ for some $k \in \mathbb{N}$. Consider the $(n+1)$st move:

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Case 2: "Delete **UU**" or "append **U**." The number of **I**s is unchanged.

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Thus after the $(n+1)$st move, the number of **I**s is not a multiple of three, so $P(n+1)$ holds, completing the induction. ■

Theorem: The **`MU`** puzzle has no solution.

Proof: By contradiction; assume it has a solution. By our lemma, the number of **`I`**'s in the final string must not be a multiple of three. However, for the solution to be valid, the number of **`I`**'s must be 0, which is a multiple of three. We have reached a contradiction, so our assumption was wrong and the **`MU`** puzzle has no solution. ■

Algorithms and Loop Invariants

- The proof we just made had the form
  - "If P is true before we perform an action, it is true after we perform an action."
- We could therefore conclude that after any series of actions of any length, if P was true beforehand, it is true now.
- In algorithmic analysis, this is called a **loop invariant**.
- Proofs on algorithms often use loop invariants to reason about the behavior of algorithms.
  - Take CS161 for more details!

Next Time

- **Variations on Induction**
  - Starting induction later.
  - Taking larger steps.
  - Complete induction.