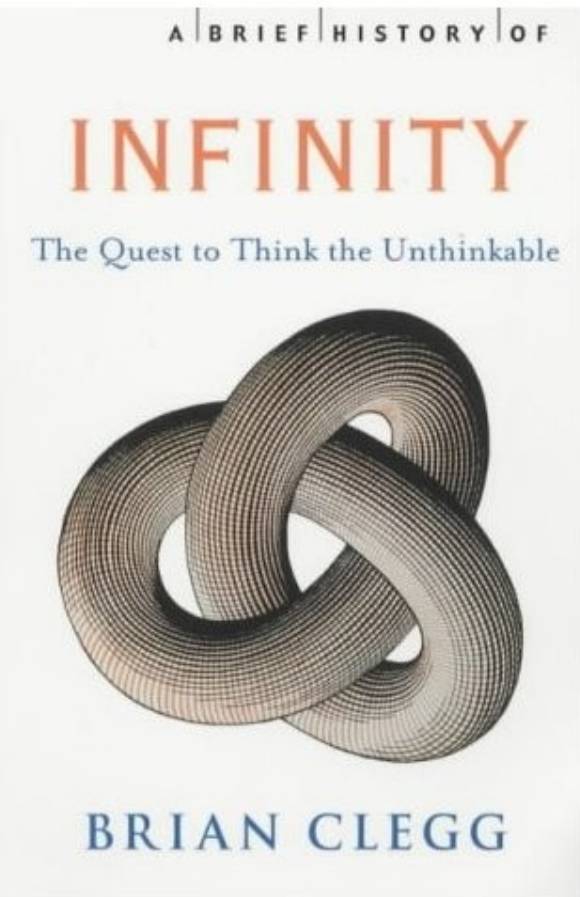
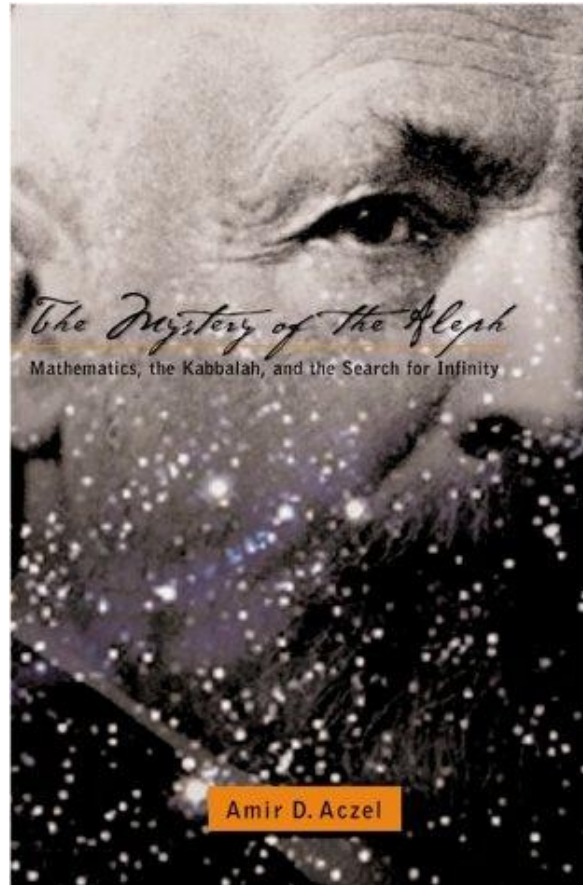


Direct Proofs

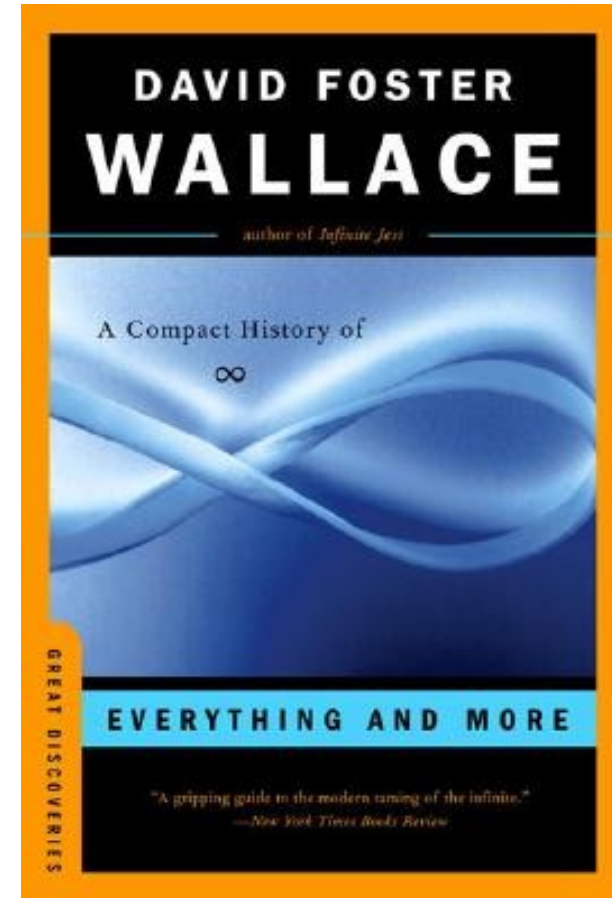
Recommended Reading



A Brief History of Infinity



The Mystery of the Aleph



Everything and More

Recommended Courses

Math 161: Set Theory

What is a Proof?

Induction and Deduction

- In the sciences, much reasoning is done **inductively**.
 - Conduct a series of experiments and find a rule that explains all the results.
 - Conclude that there is a general principle explaining the results.
 - Even if all data are correct, the conclusion might be incorrect.
- In mathematics, reasoning is done **deductively**.
 - Begin with a series of statements assumed to be true.
 - Apply logical reasoning to show that some conclusion necessarily follows.
 - If all the starting assumptions are correct, the conclusion necessarily must be correct.

Structure of a Mathematical Proof

- Begin with a set of initial assumptions.
 - Some will be explicitly stated, others assumed as background knowledge.
- Apply logical reasoning to derive the final result from those initial assumptions.
- Assuming all intermediary steps follow sound logical reasoning, the final result necessarily follows from the assumptions.
- It is a secondary question whether the initial assumptions are correct; that's the domain of the *philosophy of mathematics*.

Direct Proofs

Direct Proofs

- A **direct proof** is the simplest type of proof.
- Starting with an initial set of assumptions, apply simple logical steps to derive the result.
 - *Directly* prove that the result is true.
- Contrasts with **indirect proofs**, which we'll see on Friday.

Two Quick Definitions

- An integer n is **even** if there is some integer k such that $n = 2k$.
 - This means that 0 is even.
- An integer n is **odd** if there is some integer k such that $n = 2k + 1$.
- We'll assume the following for now:
 - Every integer is either even or odd.
 - No integer is both even and odd.

A Simple Direct Proof

Theorem: If n is an even integer, then n^2 is even.

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Proof: Let n be an even integer.

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Since n is even, there is some integer k such that $n = 2k$.

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This means that $n^2 = (2k)^2$

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This means that $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.

Since $2k^2$ is an integer, this means that there is some integer m (namely, $2k^2$) such that $n^2 = 2m$.

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Thus n^2 is even. ■

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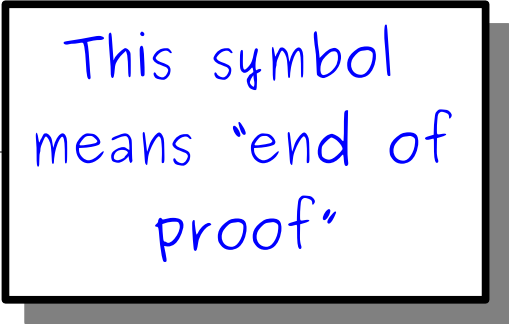
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Thus n^2 is even. 



This symbol
means "end of
proof"

A Simple Direct Proof

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This means

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Thus n^2 is even.

To prove a statement of the form

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Assume that P is true, then show that Q must be true as well.

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Since $2k^2$ is an even integer, there is some integer m such that $n^2 = 2m$.

Thus n^2 is even.

This is the definition of an even integer. When writing a mathematical proof, it's common to call back to the definitions.

A Simple Direct Proof

Theorem: If n is an even integer, then n^2 is even.

Proof: Let n be an even integer.

Since n is even, there is some integer k such that $n = 2k$.

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Since n is even, there is some integer k such that $n = 2k$.

This means that $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.

Since $2k^2$ is an integer, there is some integer m such that $n^2 = 2m$. Thus n^2 is even.

Notice how we use the value of k that we obtained above. Giving names to quantities, even if we aren't fully sure what they are, allows us to manipulate them. This is similar to variables in programs.

A Simple Direct Proof

Theorem: If n is an even integer, then n^2 is even.

Proof: Let n be an even integer.

Since n is even, there is some integer k such that $n = 2k$.

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Theorem: If n is even, then n^2 is even. 1.

Proof: Let n be an even integer. n^2 is even. This means that we

Since n is even, we can write $n = 2k$ for some integer k . need to find some m such that

such that $n^2 = 2m$. Here, we're explicitly

This means we need to show that n^2 is even, showing how we can do that.

Since $2k^2$ is an integer, this means that there is some integer m (namely, $2k^2$) such that $n^2 = 2m$. 2).

Thus n^2 is even. ■

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This means

Since $2k$
there is
that $n^2 =$

Hey, that's what we were trying
to show! We're done now.

Thus n^2 is even. ■

An Important Result

- Set equality is defined as follows

$A = B$ precisely when every element of A belongs to B and vice-versa

- This definition makes it a bit tricky to prove that two sets are equal.
- It's often easier to use the following result to show that two sets are equal:

**For any sets A and B ,
if $A \subseteq B$ and $B \subseteq A$, then $A = B$.**

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then $A = B$.

How do we prove
that this is true for
any choice of sets?

Proving Something Always Holds

- Many statements have the form

For any X , $P(X)$ is true.

- Examples:

For all integers n , if n is even, n^2 is even.

For any sets A and B , if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

For all sets S , $|S| < |\wp(S)|$.

Everybody's looking forward to the weekend, weekend.

- How do we prove these statements when there are (potentially) infinitely many cases to check?

Arbitrary Choices

- To prove that $P(x)$ is true for all possible x , show that no matter what choice of x you make, $P(x)$ must be true.
- Start the proof by making an **arbitrary choice** of x :
 - “Let x be chosen arbitrarily.”
 - “Let x be an arbitrary even integer.”
 - “Let x be an arbitrary set containing 137.”
 - “Consider any x .”
- Demonstrate that $P(x)$ holds true for this choice of x .

Theorem: For any sets A and B , if $A \subseteq B$ and $B \subseteq A$,
then $A = B$.

Proof:

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Proof: Let A and B be arbitrary sets such that $A \subseteq B$ and
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We're showing here that regardless of what A and B you pick, the result will still be true.

Theorem: For any sets A and B , if $A \subseteq B$ and $B \subseteq A$,
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To prove a statement of the form

“If P , then Q ”

Assume that **P** is true, then show that **Q** must be true as well.

Theorem: For any sets A and B , if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Proof: Let A and B be arbitrary sets such that $A \subseteq B$ and $B \subseteq A$.

By definition, $A \subseteq B$ means that for all $x \in A$, we have $x \in B$.

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Thus whenever $x \in A$ we have $x \in B$ and whenever $x \in B$ we have $x \in A$.

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Consequently, $A = B$.

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Consequently, $A = B$. ■

An Incorrect Proof

Theorem: For any natural number n , the sum of all the positive divisors of n is always no greater than $2n$.

Proof: Consider an arbitrary natural number, say, 16. 16 has positive divisors 1, 2, 4, 8, and 16. Note that $1 + 2 + 4 + 8 + 16 = 31 \leq 2 \cdot 16$. Since our choice of n was arbitrary, we see that for an arbitrary natural number n , the sum of all the divisors of n is no greater than $2n$. ■

An Incorrect Proof

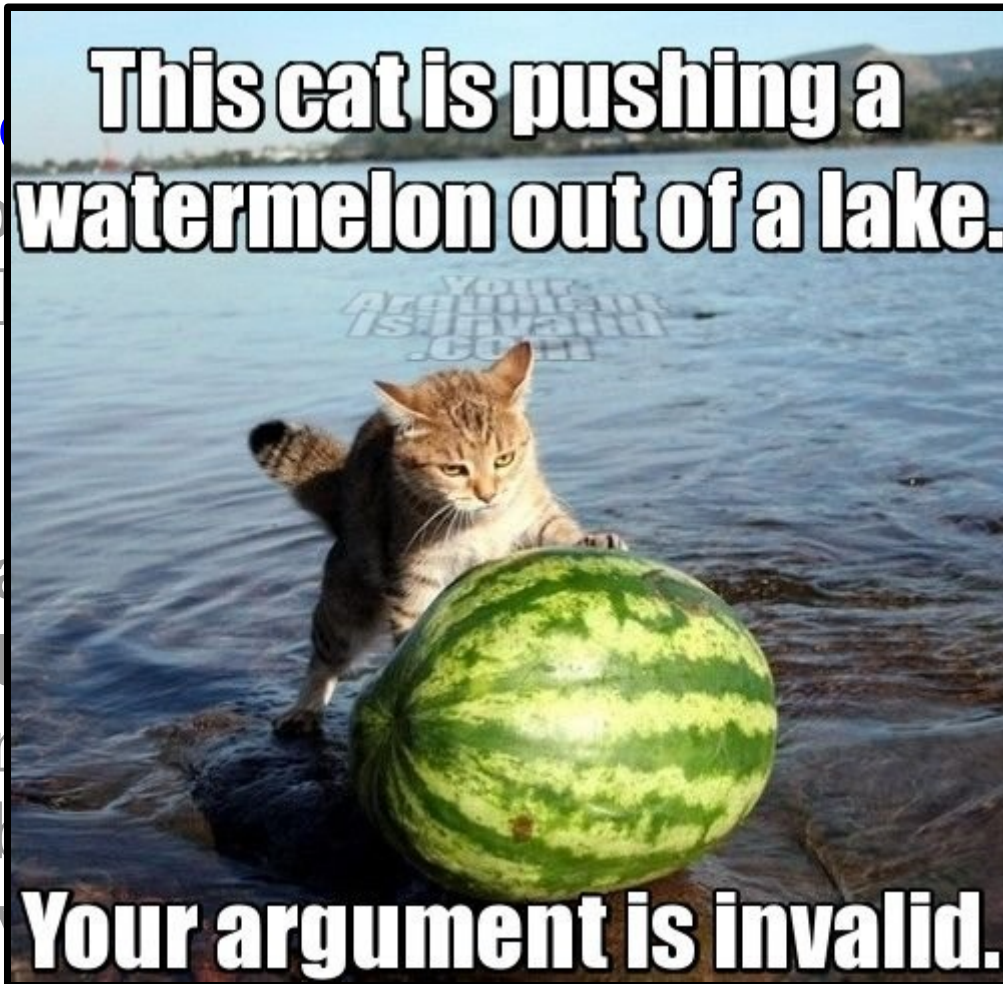
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An Incorrect Proof

Theorem: For all the positive integers n , 2^n is greater than the sum of all the positive divisors of n .

Proof: Consider $n = 16$. 16 has divisors 1, 2, 4, 8, and 16. Note that $1 + 2 + 4 + 8 + 16 = 31$. Since our theorem claims that 2^n is greater than the sum of the divisors of n , for an argument to be valid, we must show that 2^n is greater than the sum of all the divisors of n .



For example, the sum of the divisors of 16 is no

greater than 16. For example, say, $n = 8$, and 16. $2^8 = 256$, and $1 + 2 + 4 + 8 + 16 = 31$. Note that $256 > 31$. We see that 2^n is greater than the sum of the divisors of n . \blacksquare

ar·bi·trar·y

adjective /'ärbi,trerē/

1. Based on random choice or personal whim, rather than any reason or system - *“his mealtimes were entirely arbitrary”*
2. (of power or a ruling body) Unrestrained and autocratic in the use of authority - *“arbitrary rule by King and bishops has been made impossible”*
3. (of a constant or other quantity) Of unspecified value

ar·bi·trar·y

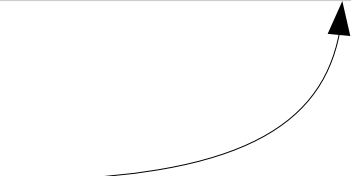
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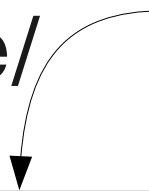
Use this
definition



ar·bi·trar·y

adjective /'ärbi,trerē/

Not this
one!

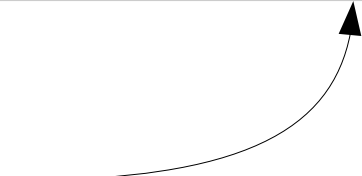


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Use this
definition



To prove something is true for all x ,
don't choose an x and base the proof
off of your choice.

Instead, leave x unspecified
and show that no matter what x is,
the specified property must hold.

Another Incorrect Proof

Theorem: For any sets A and B , $A \subseteq A \cap B$.

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Consider any arbitrary $x \in A \cap B$. This means that $x \in A$ and $x \in B$, so $x \in A$ as required. ■

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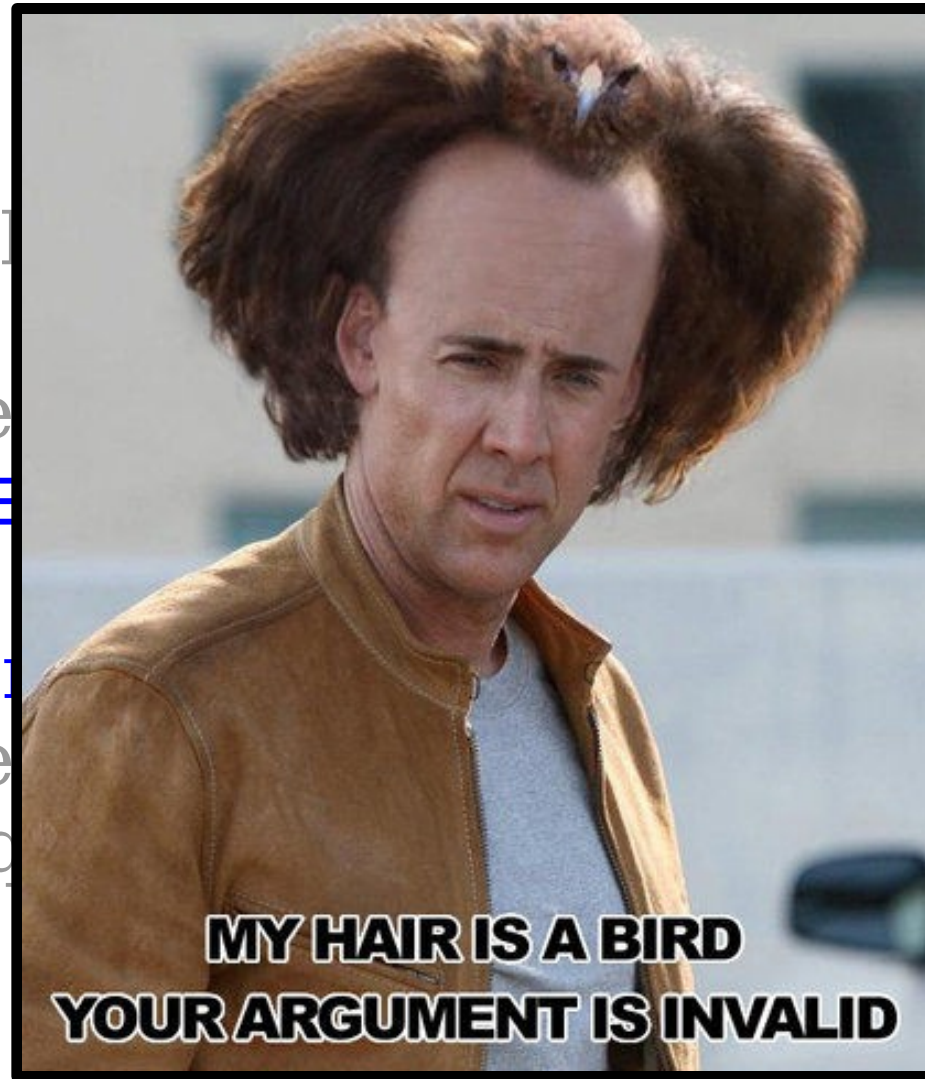
Consider any arbitrary $x \in A \cap B$. This means that $x \in A$ and $x \in B$, so $x \in A$ as required. ■

Another Incorrect Proof

Theorem: If $A \cap B$ is non-empty, then $A \cap B$ is a subset of A .

Proof: We assume $A \cap B$ is non-empty. Then, there exists $x \in A \cap B$.

Consider $x \in A \cap B$. This implies $x \in A$ and $x \in B$. This is a contradiction, so $x \in A$ as required.



If you want to prove that P implies Q ,
assume P and prove Q .

Don't assume Q and then prove P !

An Entirely Different Proof

Theorem: There exists a natural number $n > 0$ such that the sum of all natural numbers less than n is equal to n .

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An Entirely Different Proof

Theorem: **There exists** a natural number $n > 0$
such that the sum of all natural
numbers less than n is equal to n .

This is a fundamentally different type of proof that what we've done before. Instead of showing that every object has some property, we want to show that some object has a given property.

Universal vs. Existential Statements

- A **universal statement** is a statement of the form
For all x , $P(x)$ is true.
- We've seen how to prove these statements.

Universal vs. Existential Statements

- A **universal statement** is a statement of the form
For all x , $P(x)$ is true.
- We've seen how to prove these statements.
- An **existential statement** is a statement of the form
There exists an x for which $P(x)$ is true.
- How do you prove an existential statement?

Proving an Existential Statement

- We will see several different ways to prove “there is some x for which $P(x)$ is true.”
- Simple approach: Just go and find some x for which $P(x)$ is true!
 - In our case, we need to find a positive natural number n such that that sum of all smaller natural numbers is equal to n .
 - Can we find one?

An Entirely Different Proof

Theorem: There exists a natural number $n > 0$ such that the sum of all natural numbers less than n is equal to n .

An Entirely Different Proof

Theorem: There exists a natural number $n > 0$ such that the sum of all natural numbers less than n is equal to n .

Proof: Take $n = 3$.

There are three natural numbers smaller than 3: 0, 1, and 2.

We have $0 + 1 + 2 = 3$.

Thus 3 is a natural number greater than zero equal to the sum of all smaller natural numbers. ■

Extended Example: **XOR**

Logical Operators

- A **bit** is a value that is either 0 or 1.
- The set $\mathbb{B} = \{0, 1\}$ is the set of all bits.
- A **logical operator** is an operator that takes in some number of bits and produces a new bit as output.
- Example: Logical NOT, denoted $\neg x$:

$$\neg 0 = 1$$

$$\neg 1 = 0$$

Logical XOR

- The **exclusive OR** operator (**XOR**) operates on two bits and produces 0 if the bits are the same and 1 if they are different.
 - Since XOR operates on two values, it is called a **binary operator**.
- We denote the XOR of a and b by $a \oplus b$.
- Formally, XOR is defined as follows:

$$0 \oplus 0 = 0$$

$$0 \oplus 1 = 1$$

$$1 \oplus 0 = 1$$

$$1 \oplus 1 = 0$$

Fun with XOR

- The XOR operator has numerous uses throughout computer science.
 - Applications in cryptography, data structures, error-correcting codes, networking, machine learning, etc.
- XOR is useful because of four key properties:
 - XOR has an **identity element**.
 - XOR is **self-inverting**.
 - XOR is **associative**.
 - XOR is **commutative**.

Identity Elements

- An **identity element** for a binary operator \star is some value z such that for any a :

$$a \star z = z \star a = a$$

Identity Elements

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Identity Elements

An **identity element** for a binary operator \star is some value z such that **for any a :**

$$a \star z = z \star a = a$$

In math-speak, the term
“**for any a** ” is synonymous
with “for every a ” or
“**for every possibly choice of a .**”
It does not mean
“**for some specific choice of a .**”

Identity Elements

- An **identity element** for a binary operator \star is some value z such that for any a :

$$a \star z = z \star a = a$$

- Example: 0 is an identity element for +:

$$a + 0 = 0 + a = a$$

- Example: 1 is an identity element for \times :

$$a \times 1 = 1 \times a = a$$

Theorem: 0 is an identity element for \oplus .

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Proof: We will prove that for any $b \in \mathbb{B}$ that $b \oplus 0 = b$ and that $0 \oplus b = b$.

Theorem: 0 is an identity element for \oplus .

Proof: We will prove that for any $b \in \mathbb{B}$ that $b \oplus 0 = b$ and that $0 \oplus b = b$. To do this, consider an arbitrary $b \in \mathbb{B}$.

Theorem: 0 is an identity element for \oplus .

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Theorem: 0 is an identity element for \oplus .

Proof: We will prove that for any $b \in \mathbb{B}$ that $b \oplus 0 = b$ and that $0 \oplus b = b$. To do this, consider an arbitrary $b \in \mathbb{B}$. We consider two cases:

Case 1: $b = 0$.

Case 2: $b = 1$.

Theorem: 0 is an identity element for \oplus .

Proof: We will prove that for any $b \in \mathbb{B}$ that $b \oplus 0 = b$ and that $0 \oplus b = b$. To do this, consider an arbitrary $b \in \mathbb{B}$. We consider two cases:

Case 1: $b = 0$.

Case 2: $b = 1$.

This is called a **proof by cases** (alternatively, a **proof by exhaustion**) and works by showing that the theorem is true regardless of what specific outcome arises.

Theorem: 0 is an identity element for \oplus .

Proof: We will prove that for any $b \in \mathbb{B}$ that $b \oplus 0 = b$ and that $0 \oplus b = b$. To do this, consider an arbitrary $b \in \mathbb{B}$. We consider two cases:

Case 1: $b = 0$. Then we have

$$b \oplus 0 = 0 \oplus 0$$

Case 2: $b = 1$.

Theorem: 0 is an identity element for \oplus .

Proof: We will prove that for any $b \in \mathbb{B}$ that $b \oplus 0 = b$ and that $0 \oplus b = b$. To do this, consider an arbitrary $b \in \mathbb{B}$. We consider two cases:

Case 1: $b = 0$. Then we have

$$\begin{aligned} b \oplus 0 &= 0 \oplus 0 \\ &= 0 \end{aligned}$$

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$$b \oplus 0 = 0 \oplus 0 = 0 \quad 0 \oplus b = 0 \oplus 0 = 0$$

In a proof by cases, after demonstrating each case, you should summarize the cases afterwards to make your point clearer.

Case 2:

$$b \oplus$$

$$= b$$

$$= b$$

In both cases, we find $b \oplus 0 = 0 \oplus b = b$.

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In both cases, we find $b \oplus 0 = 0 \oplus b = b$. Thus 0 is an identity element for \oplus . ■

Self-Inverting Operators

- A binary operator \star with identity element z is called **self-inverting** when for any a , we have

$$a \star a = z$$

- Is $+$ self-inverting?
- Is $-$ self-inverting?

XOR is Self-Inverting

Theorem: \oplus is self-inverting.

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Case 1: $b = 0$. Then $b \oplus b = 0 \oplus 0$

Case 2: $b = 1$.

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Associative Operators

- A binary operator \star is called **associative** when for any a , b and c , we have

$$a \star (b \star c) = (a \star b) \star c$$

- Is $+$ associative?
- Is $-$ associative?
- Is \times associative?

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Proof: Consider any $a, b, c \in \mathbb{B}$. We will prove that $a \oplus (b \oplus c) = (a \oplus b) \oplus c$. To do this, we consider two cases:

Case 1: $c = 0$.

Case 2: $c = 1$.

Theorem: \oplus is associative.

Proof: Consider any $a, b, c \in \mathbb{B}$. We will prove that $a \oplus (b \oplus c) = (a \oplus b) \oplus c$. To do this, we consider two cases:

Case 1: $c = 0$. Then we have that

$$a \oplus (b \oplus c) = a \oplus (b \oplus 0)$$

Case 2: $c = 1$.

Theorem: \oplus is associative.

Proof: Consider any $a, b, c \in \mathbb{B}$. We will prove that $a \oplus (b \oplus c) = (a \oplus b) \oplus c$. To do this, we consider two cases:

Case 1: $c = 0$. Then we have that

$$\begin{aligned} a \oplus (b \oplus c) &= a \oplus (b \oplus 0) \\ &= a \oplus b \end{aligned} \quad (\text{since } 0 \text{ is an identity})$$

Case 2: $c = 1$.

Theorem: \oplus is associative.

Proof: Consider any $a, b, c \in \mathbb{B}$. We will prove that $a \oplus (b \oplus c) = (a \oplus b) \oplus c$. To do this, we consider two cases:

Case 1: $c = 0$. Then we have that

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Case 2: $c = 1$. Then we have that

$$\begin{aligned} a \oplus (b \oplus c) &= a \oplus (b \oplus 1) \\ &= ? \end{aligned}$$

When You Get Stuck

- When writing proofs, you are bound to get stuck at some point.
- When this happens, it can mean multiple things:
 - What you're proving is incorrect.
 - You are on the wrong track.
 - You're on the right track, but you need to prove an additional result to get to your goal.
- Unfortunately, there is no general way to determine which case you are in.
- You'll build this intuition through experience.

Where We're Stuck

- Right now, we have the expression

$$a \oplus (b \oplus 1)$$

and we don't know how to simplify it.

- Let's focus on the $(b \oplus 1)$ part and see what we find:
 - $0 \oplus 1 = 1$
 - $1 \oplus 1 = 0$
- It seems like $b \oplus 1 = \neg b$. Could we prove it?

Relations Between Proofs

- Proofs often build off of one another: large results are almost often accomplished by building off of previous work.
 - Like writing a large program – split the work into smaller methods, across different classes, etc. instead of putting the whole thing into **main**.
- A result that is proven specifically as a stepping stone toward a larger result is called a **lemma**.
- Our result that $b \oplus 1 = \neg b$ serves as a lemma in our larger proof that \oplus is associative.

Lemma: For any $b \in \mathbb{B}$, we have $b \oplus 1 = \neg b$.

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Case 1: $b = 0$.

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Lemma: For any $b \in \mathbb{B}$, we have $b \oplus 1 = \neg b$.

Proof: Consider any $b \in \mathbb{B}$. We consider two cases:

Case 1: $b = 0$. Then

$$b \oplus 1 = 0 \oplus 1$$

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Proof: Consider any $b \in \mathbb{B}$. We consider two cases:

Case 1: $b = 0$. Then

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In both cases, we find that $b \oplus 1 = \neg b$, which is what we needed to show.

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Theorem: \oplus is associative.

Proof: Consider any $a, b, c \in \mathbb{B}$. We will prove that $a \oplus (b \oplus c) = (a \oplus b) \oplus c$. To do this, we consider two cases:

Case 1: $c = 0$. Then we have that

$$\begin{aligned} a \oplus (b \oplus c) &= a \oplus (b \oplus 0) \\ &= a \oplus b && \text{(since 0 is an identity)} \\ &= (a \oplus b) \oplus 0 && \text{(since 0 is an identity)} \\ &= (a \oplus b) \oplus c \end{aligned}$$

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Commutative Operators

- A binary operator \star is called **commutative** when the following is always true:

$$a \star b = b \star a$$

- Is $+$ commutative?
- Is $-$ commutative?

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$$x \oplus b = a \oplus (b \oplus b) \quad (\text{since } \oplus \text{ is associative})$$

$$x \oplus b = a \oplus 0 \quad (\text{since } \oplus \text{ is self-inverting})$$

$$x \oplus b = a \quad (\text{since } 0 \text{ is an identity of } \oplus)$$

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The only properties of \oplus that we used here are that it is associative, has an identity, and is self-inverting. This same proof works for any operator with these three properties!

Binary operators that have this property give rise to **boolean groups** (but you don't need to know that for this class).

This means that $a \oplus b = x = b \oplus a$. Therefore, \oplus is commutative. ■

Application: **Encryption**

Bitstrings

- A **bitstring** is a finite sequence of 0s and 1s.
- Internally, computers represent all data as bitstrings.
 - For details on how, take CS107 or CS143.

Bitstrings and \oplus

- We can generalize the \oplus operator from working on individual bits to working on bitstrings.
- If A and B are bitstrings of length n , then we'll define $A \oplus B$ to be the bitstring of length n formed by applying \oplus to the corresponding bits of A and B .
- For example:

$$\begin{array}{r} 110110 \\ \oplus 011010 \\ \hline 101100 \end{array}$$

Encryption

- Suppose that you want to send me a secret bitstring M of length n .
- You should be able to read the message, but anyone who intercepts the secret message should not be able to read it.
- How might we accomplish this?

\oplus and Encryption

- In advance, you and I share a randomly-chosen bitstring K of length n (called the **key**) and keep it secret.
- To send me message M secretly, you send me the string $C = M \oplus K$.
 - C is called the **ciphertext**.
- To decrypt the ciphertext C , I compute the string $C \oplus K$. This is

$$\begin{aligned} C \oplus K &= (M \oplus K) \oplus K \\ &= M \oplus (K \oplus K) \\ &= M \end{aligned}$$

\oplus and Encryption

- Suppose that you don't have the key and get the message $M \oplus K$.
- If K is chosen to be truly random, then every bit in $M \oplus K$ appears to be truly random.
- Intuition: Let b be a original bit from the message and k be the corresponding bit in the key.
 - If $k = 0$, then $b \oplus k = b \oplus 0 = b$.
 - If $k = 1$, then $b \oplus k = b \oplus 1 = \neg b$.
- Since the key bit is truly random, the bits in the original string are flipped totally randomly.
- Can formalize the math; take CS109 for details!

An Example

PUPPIES

M	01010000010101010101000001010000010010010100010101010011
K	11011100101110111100010011010101111001101111011111000010
C	10001100111011101001010010000101101011111011001010010001

€1”...©² ‘

An Example

€î”...©² ‘

C	10001100111011101001010010000101101011111011001010010001
K	11011100101110111100010011010101111001101111011111000010
M	01010000010101010101000001010000010010010100010101010011

PUPPIES

An Example

€î”...©² ‘

C	10001100111011101001010010000101101011111011001010010001
K?	01011100010101010101000001010000010010010100010101010011
M?	01001100010011110100110001000110010000010100100101001100

LOLFAIL

Some Caveats

- This scheme is **very insecure** if you encrypt multiple messages using the same key.
 - Good exercise: Figure out why this is!
- This scheme guarantees security if the key is random, but it isn't tamperproof.
 - You'll see why this is on the problem set.
- General good advice: ***never implement your own cryptography!***
- Take CS255 for more details!

Next Time

- **Indirect Proofs**
 - Proof by contradiction.
 - Proof by contrapositive.