Cardinality
and
The Nature of Infinity

Recap from Last Time

Functions

- A **function** f is a mapping such that every value in A is associated with a single value in B.
  - For every $a \in A$, there exists some $b \in B$ with $f(a) = b$.
  - If $f(a) = b_0$ and $f(a) = b_1$, then $b_0 = b_1$.
- If f is a function from A to B, we call A the **domain** of f andl B the **codomain** of f.
- We denote that f is a function from A to B by writing

$$\boldsymbol{f : A \to B}$$

Injective Functions

- A function $f : A \to B$ is called **injective** (or **one-to-one**) iff each element of the codomain has at most one element of the domain associated with it.
  - A function with this property is called an **injection**.
- Formally:

$$\textbf{If } f(x_0) = f(x_1)\textbf{, then } x_0 = x_1$$

- An intuition: injective functions label the objects from A using names from B.

Surjective Functions

- A function $f : A \to B$ is called **surjective** (or **onto**) iff each element of the codomain has at least one element of the domain associated with it.
  - A function with this property is called a **surjection**.
- Formally:

For any $b \in B$, there exists at least one $a \in A$ such that $f(a) = b$.

- An intuition: surjective functions cover every element of B with at least one element of A.

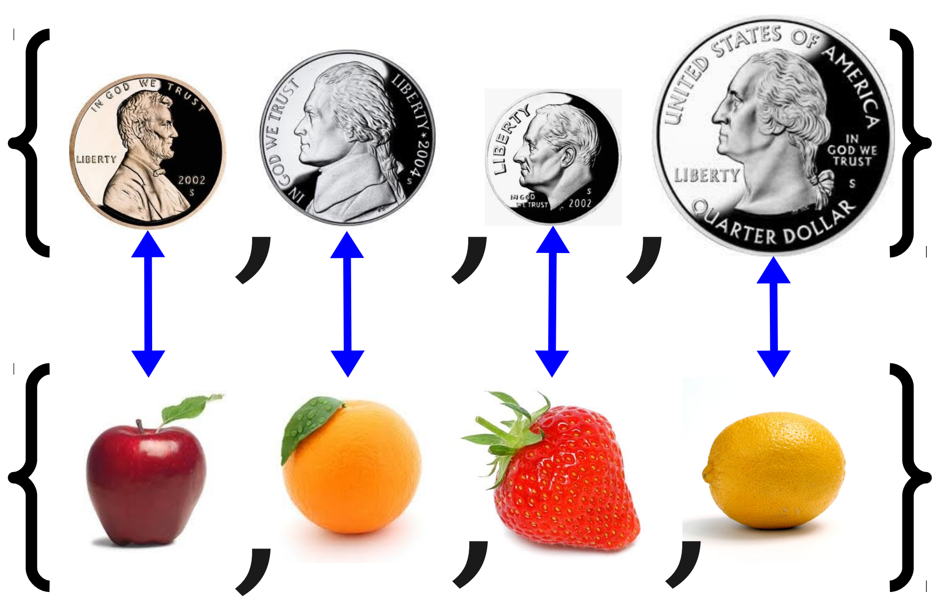
Bijections

- A function that associates each element of the codomain with a unique element of the domain is called **bijective**.
  - Such a function is a **bijection**.
- Formally, a bijection is a function that is both **injective** and **surjective**.
- A bijection is a one-to-one correspondence between two sets.

Comparing Cardinalities

- The relationships between set cardinalities are defined in terms of functions between those sets.
- $|S| = |T|$ is defined using bijections.

$|S| = |T|$ iff there is a bijection $f : S \to T$



The Nature of Infinity

Infinite Cardinalities

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$\mathbb{Z}$	...	-3	-2	-1	0	1	2	3	4	...

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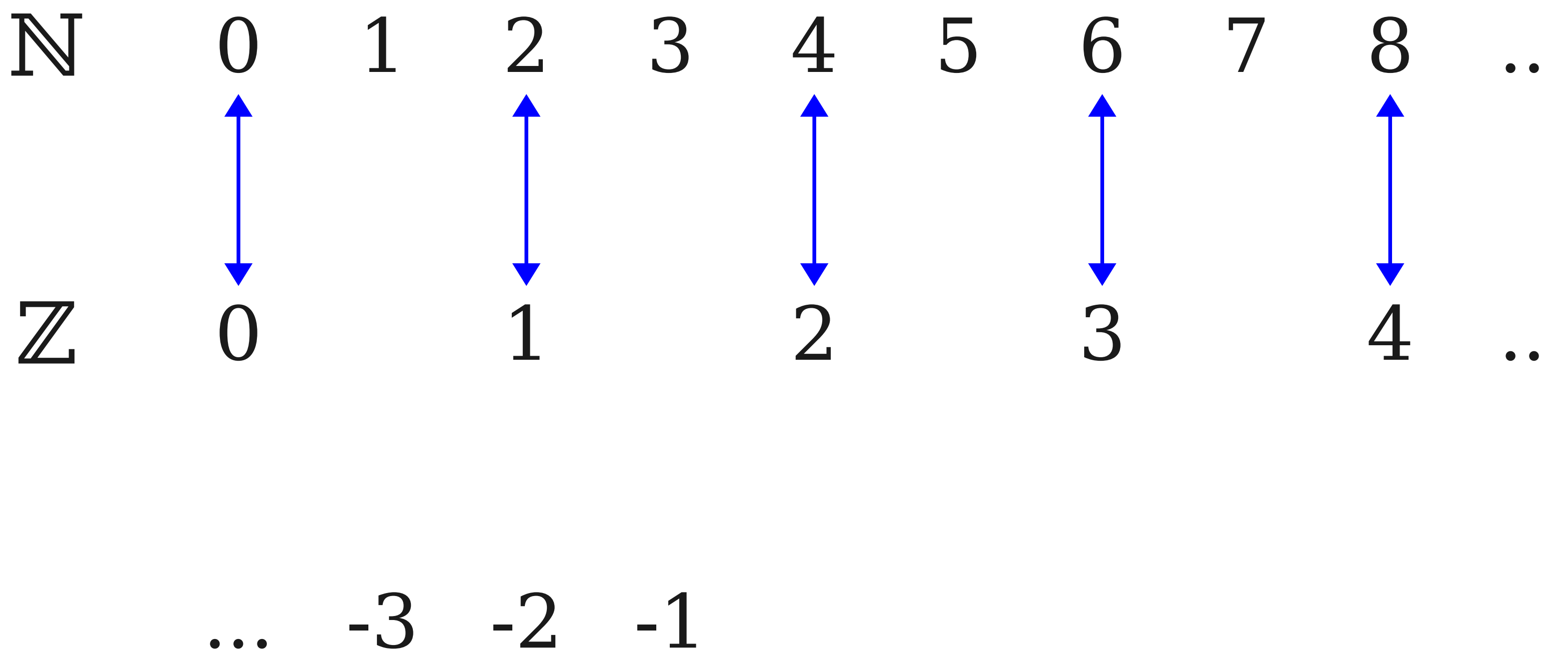

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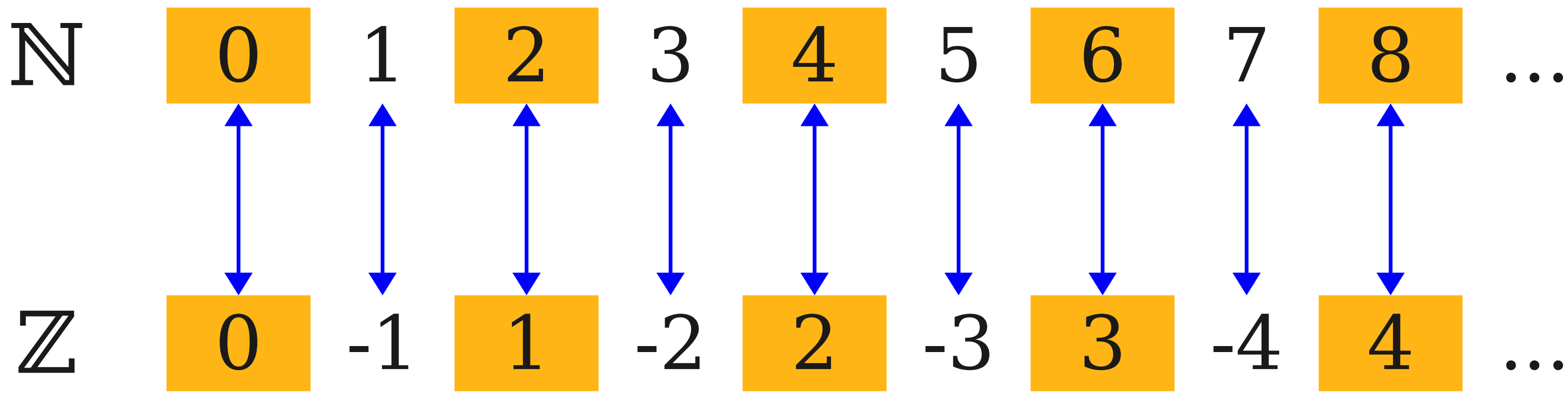

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Case 1: x and y are nonnegative.

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First, we prove this is a legal function from $\mathbb{Z}$ to $\mathbb{N}$. Consider any $x \in \mathbb{Z}$. Note that if $x \geq 0$, then $f(x) = 2x$. Since in this case x is nonnegative, $2x$ is a natural number. Thus $f(x) \in \mathbb{N}$. Otherwise, $x < 0$, so $f(x) = -2x - 1 = 2(-x) - 1$. Since $x < 0$, we have $-x > 0$, so $-x \geq 1$. Then $f(x) = 2(-x) - 1 \geq 2 - 1 = 1$. Thus $f(x)$ is a positive integer, so $f(x) \in \mathbb{N}$. In either case $f(x) \in \mathbb{N}$, so $f : \mathbb{Z} \to \mathbb{N}$.

Next, we prove f is injective. Suppose that $f(x) = f(y)$. We will prove that $x = y$. Note that, by construction, $f(z)$ is even iff z is nonnegative. Since $f(x) = f(y)$, we know x and y must have the same sign. We consider two cases:

Case 1: x and y are nonnegative. Then $f(x) = 2x$ and $f(y) = 2y$. Since $f(x) = f(y)$, we have $2x = 2y$. Thus $x = y$.

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Finally, we prove f is surjective. Consider any $n \in \mathbb{N}$. We will prove that there is some $x \in \mathbb{Z}$ such that $f(x) = n$. We consider two cases:

Case 1: n is even. Then $n / 2$ is a nonnegative integer. Moreover, $f(n / 2) = 2(n / 2) = n$.

Case 2: n is odd. Then $-(n + 1) / 2$ is a negative integer. Moreover, $f(-(n + 1) / 2) = -2(-(n + 1) / 2) - 1 = n + 1 - 1 = n$.

Since f is injective and surjective, it is a bijection.

Theorem: $|\mathbb{Z}| = |\mathbb{N}|$.
Proof: We exhibit a bijection from $\mathbb{Z}$ to $\mathbb{N}$. Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as follows:

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Why This Matters

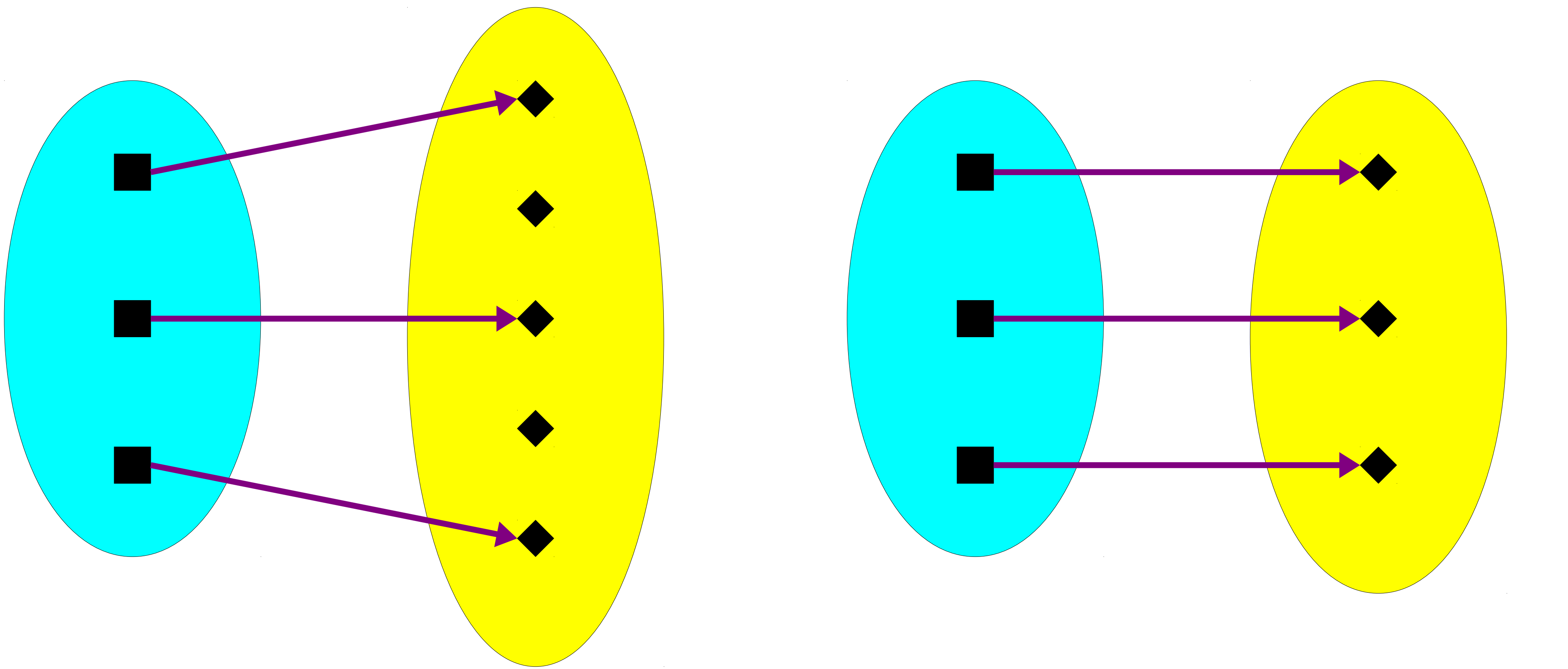
- Note the thought process from this proof:
  - Start by drawing a picture to get an intuition.
  - Convert the picture into a mathematical object (here, a function).
  - Prove the object has the desired properties.
- This technique is at the heart of mathematics.
- We will use it extensively throughout the rest of this lecture.

Cantor's Theorem Revisited

Comparing Cardinalities

- We define $|S| \leq |T|$ as follows:

  $|S| \leq |T|$ iff there is an injection $f : S \to T$

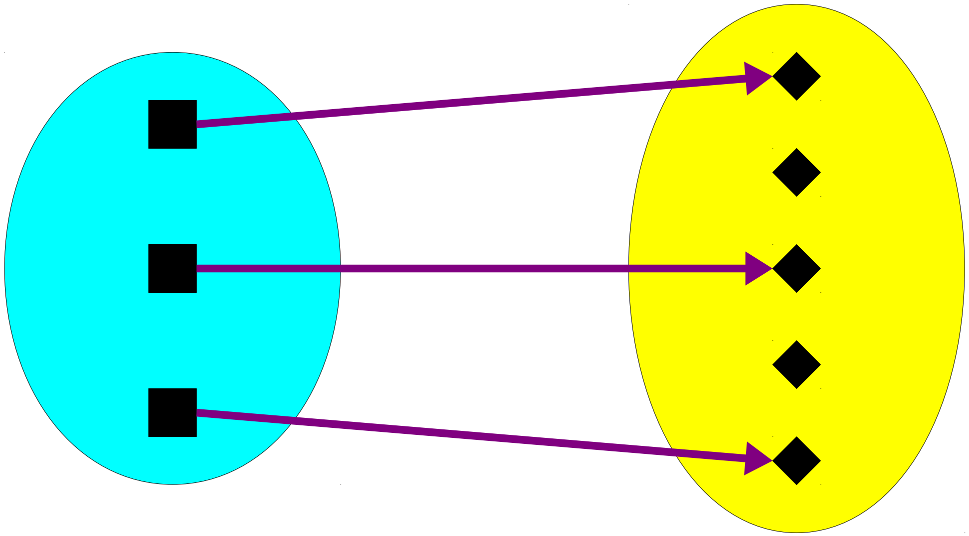


Comparing Cardinalities

- Formally, we define < on cardinalities as

$$|S| < |T| \text{ iff } |S| \leq |T| \text{ and } |S| \neq |T|$$

- In other words:
  - There is an injection from *S* to *T*.
  - There is no bijection between *S* and *T*.



Cantor's Theorem

- **Cantor's Theorem** states that

$$\text{For every set } S, |S| < |\wp(S)|$$

- This is how we concluded that there are more problems to solve than programs to solve them.
- We informally sketched a proof of this in the first lecture.
- Let's now formally prove Cantor's Theorem.

Lemma: For any set S, $|S| \leq |\wp(S)|$.

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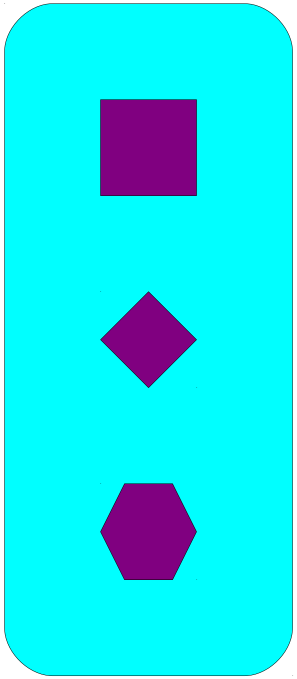

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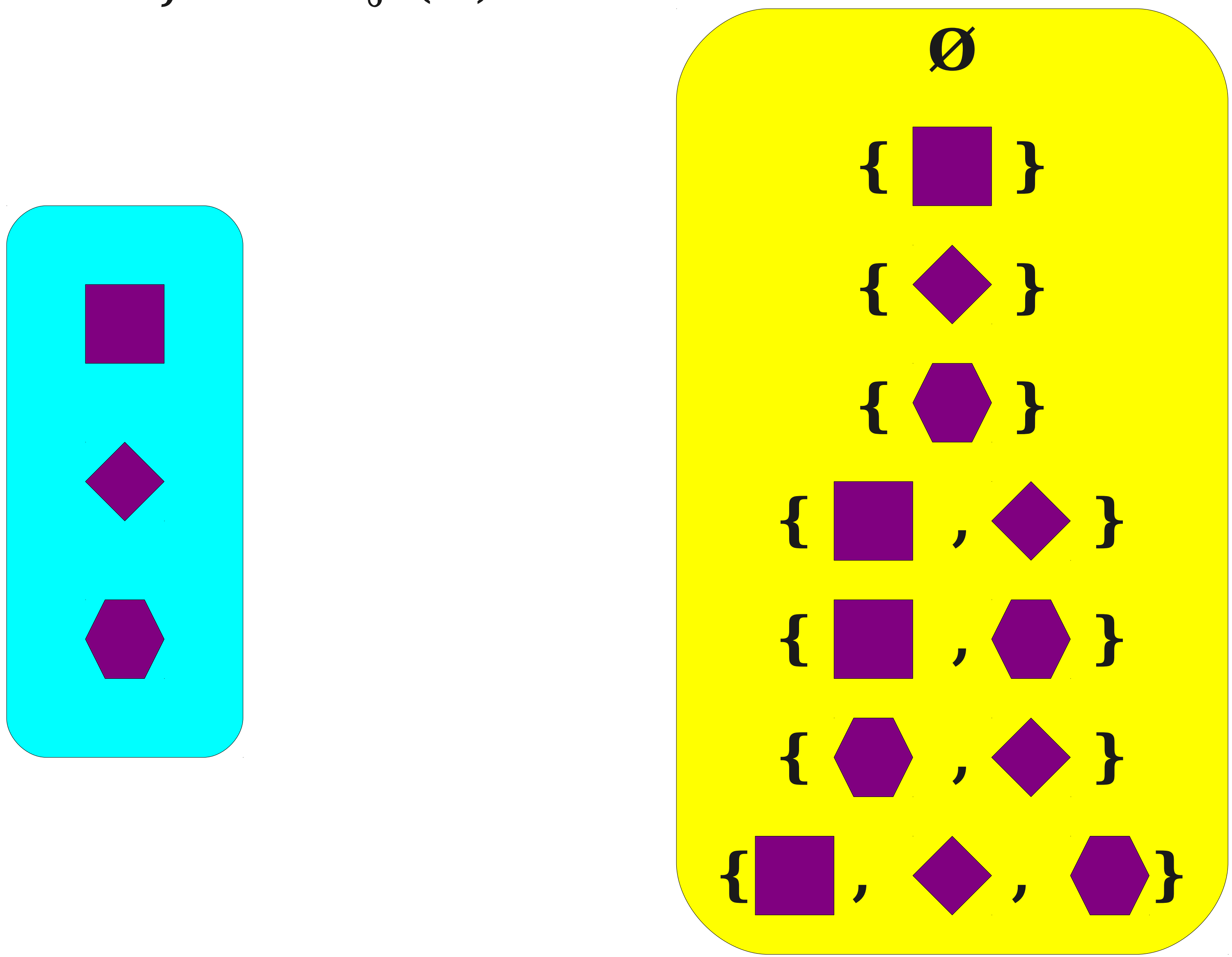

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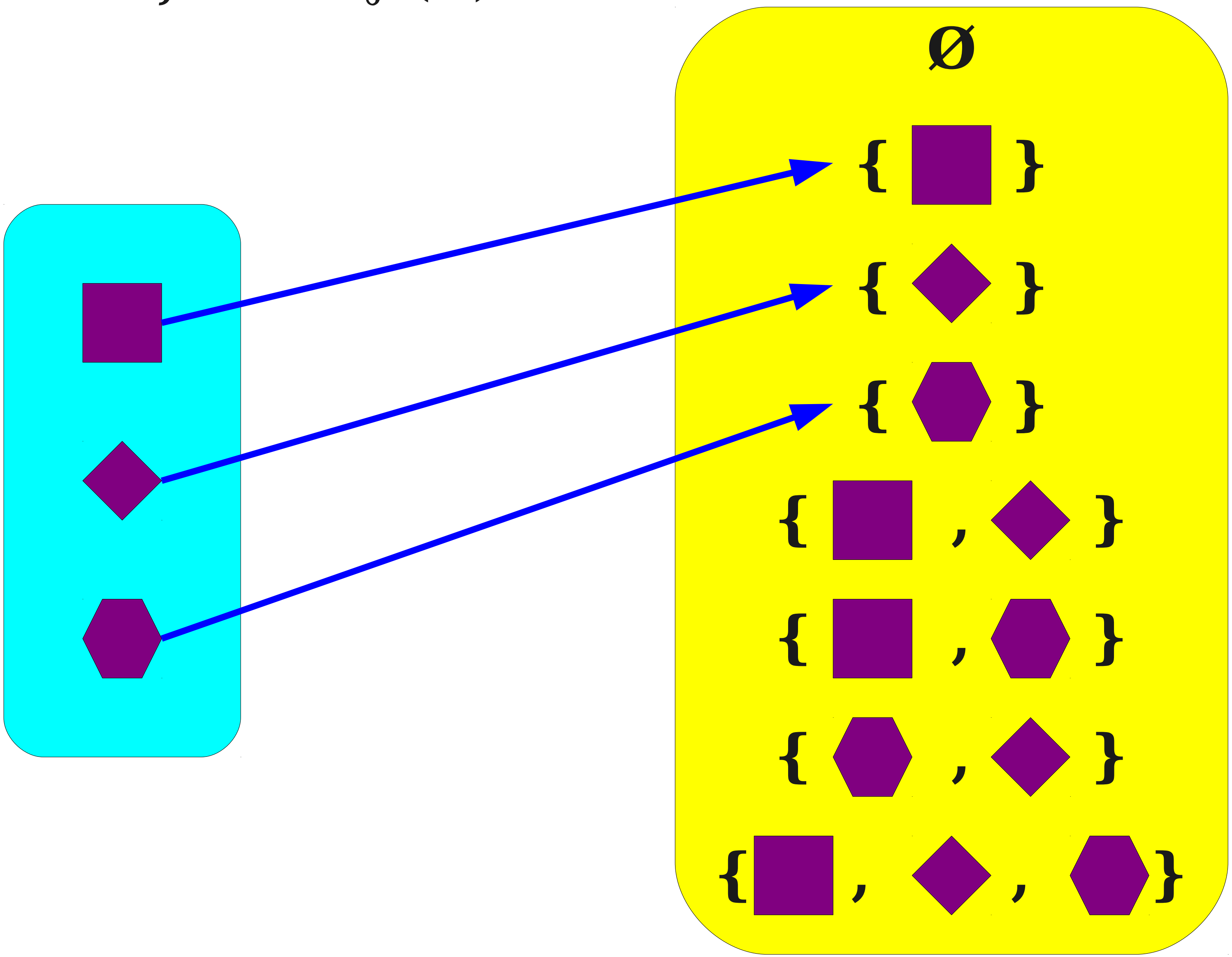

Lemma: For any set S, $|S| \leq |\wp(S)|$.

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The Key Step

- We now need to show that

For any set $\boldsymbol{S}$, $|\boldsymbol{S}| \neq |\wp(\boldsymbol{S})|$

- By definition, $|S| = |\wp(S)|$ iff there exists a bijection $f : S \to \wp(S)$.
- This means that

$|\boldsymbol{S}| \neq |\wp(\boldsymbol{S})|$ iff there is no bijection $\boldsymbol{f} : \boldsymbol{S} \to \wp(\boldsymbol{S})$

- Prove this by contradiction:
  - Assume that there is a bijection $f : S \to \wp(S)$.
  - Derive a contradiction by showing that f is not a bijection.

x_0

x_1

x_2

x_3

x_4

x_5

$\cdots$

$$x_0 \longleftrightarrow \{\, x_0, x_2, x_4, \ldots \,\}$$

$$x_1 \longleftrightarrow \{\, x_0, x_3, x_4, \ldots \,\}$$

$$x_2 \longleftrightarrow \{\, x_4, \ldots \,\}$$

$$x_3 \longleftrightarrow \{\, x_1, x_4, \ldots \,\}$$

$$x_4 \longleftrightarrow \{\, x_0, x_5, \ldots \,\}$$

$$x_5 \longleftrightarrow \{\, x_0, x_1, x_2, x_3, x_4, x_5, \ldots \,\}$$

$$\ldots$$

x_0	x_1	x_2	x_3	x_4	x_5	…

$$x_0 \longleftrightarrow \{\, x_0, x_2, x_4, \dots \,\}$$

$$x_1 \longleftrightarrow \{\, x_0, x_3, x_4, \dots \,\}$$

$$x_2 \longleftrightarrow \{\, x_4, \dots \,\}$$

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$$x_5 \longleftrightarrow \{\, x_0, x_1, x_2, x_3, x_4, x_5, \dots \,\}$$

…

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longleftrightarrow \{\, x_0, \quad x_2, \quad x_4, \quad \ldots \,\}$

$x_1 \longleftrightarrow \{\, x_0, x_3, x_4, \ldots \,\}$

$x_2 \longleftrightarrow \{\, x_4, \ldots \,\}$

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$\ldots$

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \longleftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**

$x_1 \longleftrightarrow \{\ x_0,\ x_3,\ x_4,\ \dots\ \}$

$x_2 \longleftrightarrow \{\ x_4,\ \dots\ \}$

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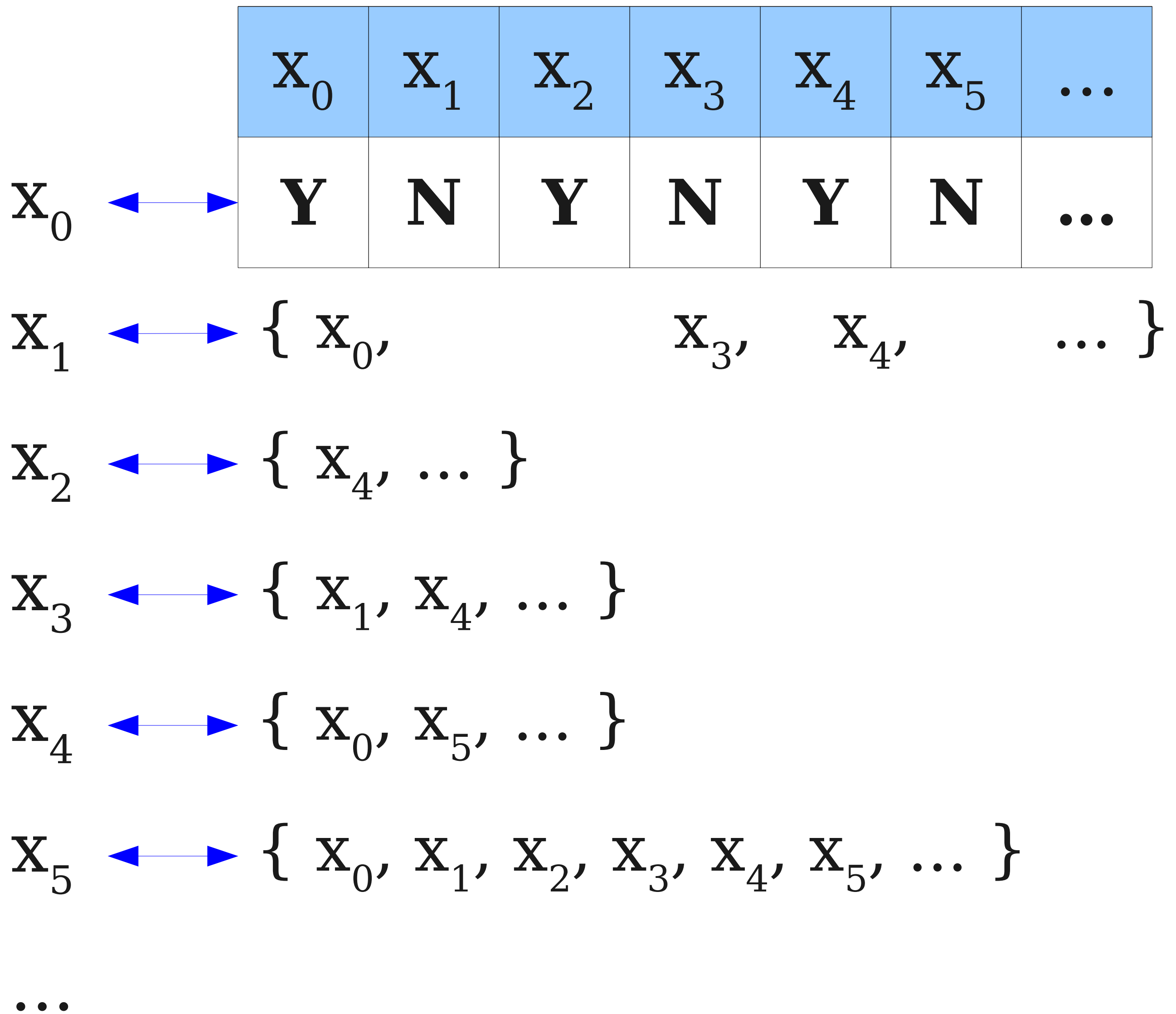
…



x_0 x_1 x_2 x_3 x_4 x_5 ...
x_0 Y N Y N Y N ...
x_1 { x_0, x_3, x_4, ... }
x_2 { x_4, ... }
x_3 { x_1, x_4, ... }
x_4 { x_0, x_5, ... }
x_5 { x_0, x_1, x_2, x_3, x_4, x_5, ... }
...

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \longleftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \longleftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**

$x_2 \longleftrightarrow \{\, x_4, \ldots \,\}$

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…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \leftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \leftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**

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…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \leftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \leftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
$x_2 \leftrightarrow$	**N**	**N**	**N**	**N**	**Y**	**N**	**…**

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…

	x_0	x_1	x_2	x_3	x_4	x_5	...
$x_0 \longleftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	...
$x_1 \longleftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	...
$x_2 \longleftrightarrow$	**N**	**N**	**N**	**N**	**Y**	**N**	...

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...

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \leftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \leftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
$x_2 \leftrightarrow$	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
$x_3 \leftrightarrow$	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**

$x_4 \leftrightarrow \{\, x_0, x_5, \dots \,\}$

$x_5 \leftrightarrow \{\, x_0, x_1, x_2, x_3, x_4, x_5, \dots \,\}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \leftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \leftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
$x_2 \leftrightarrow$	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
$x_3 \leftrightarrow$	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
$x_4 \leftrightarrow$	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**

$x_5 \leftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	...
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	...
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	...
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	...
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	...
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	...
...							

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	...
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	...
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	...
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	...
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	...
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	...
...	...	...	...	...	...	...	...

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

Y	**N**	**N**	**N**	**N**	**Y**	…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

Y	**N**	**N**	**N**	**N**	**Y**	…

Flip all Y's to N's and vice-versa to get a new set

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Flip all Y's to N's and vice-versa to get a new set

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

$$\{\ \mathbf{x_1},\ \mathbf{x_2},\ \mathbf{x_3},\ \mathbf{x_4},\ \dots\ \}$$

Flip all Y's to N's and vice-versa to get a new set

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

Flip all Y's to N's and vice-versa to get a new set

N	**Y**	**Y**	**Y**	**Y**	**N**	…

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	...
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	...
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	...
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	...
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	...
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	...
...	...	...	...	...	...	...	...

N	**Y**	**Y**	**Y**	**Y**	**N**	...

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	Y	N	Y	N	Y	N	…
x_1	Y	N	N	Y	Y	N	…
x_2	N	N	N	N	Y	N	…
x_3	N	Y	N	N	Y	N	…
x_4	Y	N	N	N	N	Y	…
x_5	Y	Y	Y	Y	Y	Y	…
…	…	…	…	…	…	…	…

N	Y	Y	Y	Y	N	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

Which row in the table is paired with this set?

N	**Y**	**Y**	**Y**	**Y**	**N**	…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

Which row in the table is paired with this set?

N	**Y**	**Y**	**Y**	**Y**	**N**	…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

Formalizing the Diagonal Argument

- Proof by contradiction; assume there is a bijection $f : S \to \wp(S)$.
- The diagonal argument shows that f cannot be a bijection:
  - Construct the table given the bijection f.
  - Construct the complemented diagonal.
  - Show that the complemented diagonal cannot appear anywhere in the table.
  - Conclude, therefore, that f is not a bijection.

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  Conclude, therefore, that f is not a bijection.

- For finite sets this is fine, but what if the set is infinitely large?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**Y**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**N**	**Y**	**N**	**N**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**N**	**Y**	**N**	…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**Y**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**N**	**Y**	**N**	**N**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**N**	**Y**	**N**	…

$$f(x_0) = \{\ x_0, x_2, x_4, \ldots\ \}$$

$$f(x_1) = \{\ x_0, x_3, x_4, \ldots\ \}$$

$$f(x_2) = \{\ x_4, \ldots\ \}$$

$$f(x_3) = \{\ x_1, x_3, x_4, \ldots\ \}$$

$$f(x_4) = \{\ x_1, x_5, \ldots\ \}$$

$$f(x_5) = \{\ x_1, x_4, x_5, \ldots\ \}$$

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**Y**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**N**	**Y**	**N**	**N**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

	N	**Y**	**Y**	**N**	**Y**	**N**	…

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	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**Y**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**N**	**Y**	**N**	**N**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

	N	**Y**	**Y**	**N**	**Y**	**N**	…

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	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	...
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	...
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	...
x_3	**N**	**Y**	**N**	**Y**	**Y**	**N**	...
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	...
x_5	**N**	**Y**	**N**	**N**	**Y**	**Y**	...
...	...	...	...	...	...	...	...

N	**Y**	**Y**	**N**	**Y**	**N**	...

$$f(x_0) = \{\, x_0, x_2, x_4, \ldots \,\}$$

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	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	...
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	...
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	...
x_3	**N**	**Y**	**N**	**Y**	**Y**	**N**	...
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	...
x_5	**N**	**Y**	**N**	**N**	**Y**	**Y**	...
...	...	...	...	...	...	...	...

N	**Y**	**Y**	**N**	**Y**	**N**	...

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The **diagonal set** D is the set

$$D = \{\ x \in S \mid x \notin f(x)\ \}$$

There is no longer a dependence on the existence of the two-dimensional table.

Lemma: For any set S, $|S| \neq |\wp(S)|$.

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Case 1: $y \in f(y)$.

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***Theorem* (Cantor's Theorem)**: For any set S, we have $|S| < |\wp(\mathrm{S})|$.

Proof: Consider any set S. By our first lemma, we have that $|S| \leq |\wp(S)|$. By our second lemma, we have that $|S| \neq |\wp(S)|$. Thus $|S| < |\wp(S)|$. ■

Why All This Matters

- The intuition behind a result is often more important than the result itself.
- Given the intuition, you can usually reconstruct the proof.
- Given just the proof, it is almost impossible to reconstruct the intuition.
- Think about compilation – you can more easily go from a high-level language to machine code than the other way around.

Cantor's *Other* Diagonal Argument

What is $|\mathbb{R}|$?

Theorem: $|\mathbb{N}| < |\mathbb{R}|$.

Sketch of the Proof

- To prove that $|\mathbb{N}| < |\mathbb{R}|$, we will use a modification of the proof of Cantor's theorem.
- First, we will directly prove that $|\mathbb{N}| \leq |\mathbb{R}|$.
- Second, we will use a proof by diagonalization to show that $|\mathbb{N}| \neq |\mathbb{R}|$.

Theorem: $|\mathbb{N}| \leq |\mathbb{R}|$.

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Consider the function $f(n) = n$. Since all natural numbers are real numbers, this is a valid function from $\mathbb{N}$ to $\mathbb{R}$. Moreover, it is injective. To see this, consider any $n_0, n_1 \in \mathbb{N}$ such that $f(n_0) = f(n_1)$. We will prove that $n_0 = n_1$. To see this, note that $n_0 = f(n_0) = f(n_1) = n_1$. Thus $n_0 = n_1$, as required, so f is injective. ■

$|\mathbb{N}| \neq |\mathbb{R}|$

- Now, we need to show that $|\mathbb{N}| \neq |\mathbb{R}|$.
- To do this, we will use a proof by diagonalization similar to the one for Cantor's Theorem.
  - Assume there is a bijection $f : \mathbb{N} \to \mathbb{R}$.
  - Construct a two-dimensional table from f.
  - Construct a “diagonal number” from the table.
  - Show the diagonal number is not in the table.
  - Conclude f is not a bijection.

0	$\longleftrightarrow$	8.	6	7	5	3	0	...
1	$\longleftrightarrow$	3.	1	4	1	5	9	...
2	$\longleftrightarrow$	0.	1	2	3	5	8	...
3	$\longleftrightarrow$	-1.	0	0	0	0	0	...
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5	$\longleftrightarrow$	1.	6	1	8	0	3	...
...	$\longleftrightarrow$	...	...	...	...	...	...	...

		d_0	d_1	d_2	d_3	d_4	d_5	...
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Set all nonzero values to 0 and all 0s to 1.

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In either case, $f(n) \neq d$. This contradicts the fact that $f(n) = d$.

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Since $d \in \mathbb{R}$, there must be some $n \in \mathbb{N}$ such that $f(n) = d$. So consider $f(n)_n$ and d_n. We consider two cases:

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The Power of Diagonalization

- A large number of fundamental results in computability and complexity theory are based on diagonal arguments.
- We will see at least three of them in the remainder of the quarter.

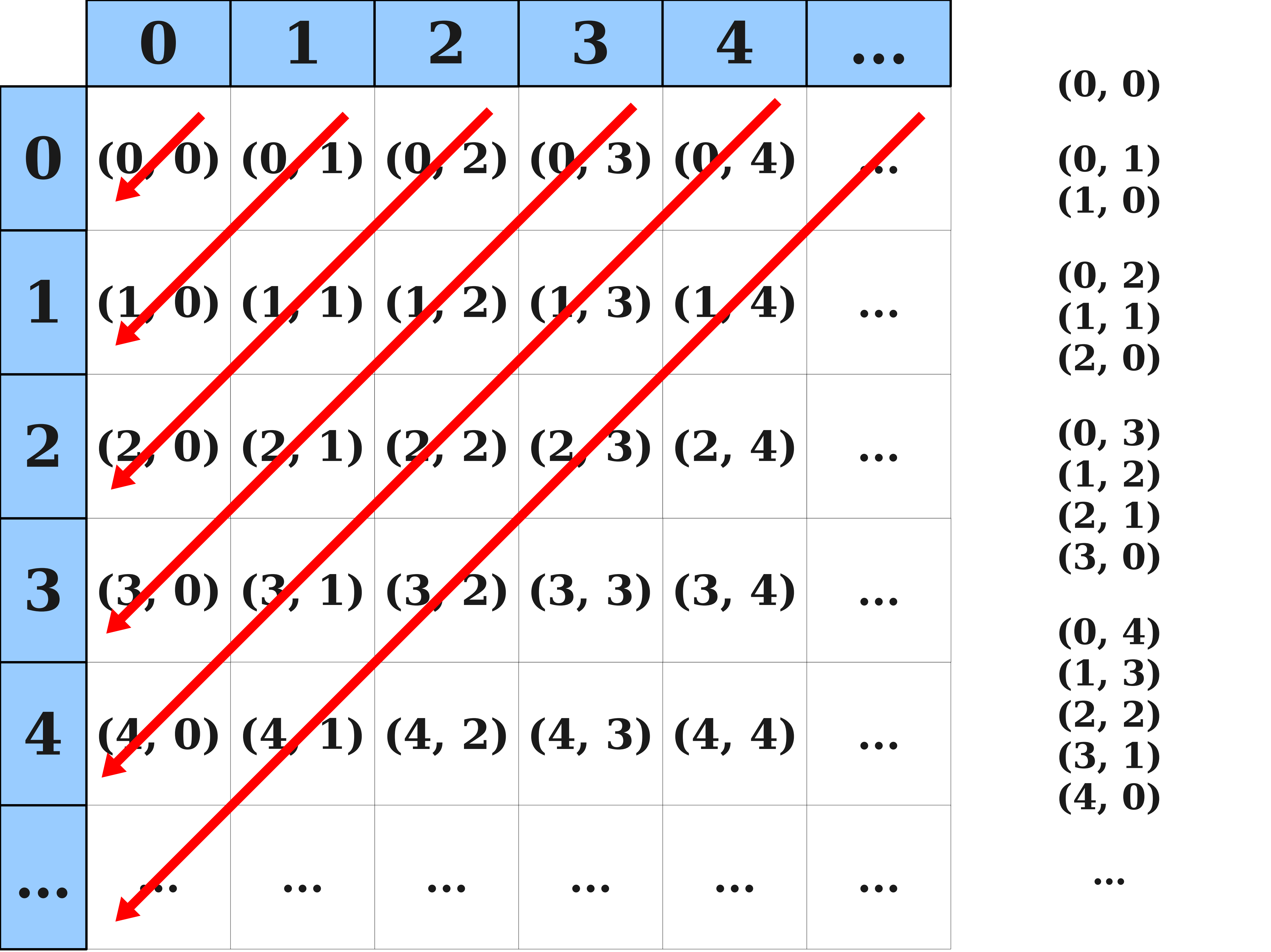
An Interesting Historical Aside

- The diagonalization proof that $|\mathbb{N}| \neq |\mathbb{R}|$ was Cantor's original diagonal argument; he proved Cantor's theorem later on.
- However, this was ***not*** the first proof that $|\mathbb{N}| \neq |\mathbb{R}|$. Cantor had a different proof of this result based on infinite sequences.
- Come talk to me after class if you want to see the original proof; it's absolutely brilliant!

Cantor's *Other Other* Diagonal Argument

(This one is different!)

What is $|\mathbb{N}^2|$?



0 1 2 3 4 ...
0 (0, 0) (0, 1) (0, 2) (0, 3) (0, 4) ...
1 (1, 0) (1, 1) (1, 2) (1, 3) (1, 4) ...
2 (2, 0) (2, 1) (2, 2) (2, 3) (2, 4) ...
3 (3, 0) (3, 1) (3, 2) (3, 3) (3, 4) ...
4 (4, 0) (4, 1) (4, 2) (4, 3) (4, 4) ...
... ...
(0, 0)
(0, 1)
(1, 0)
(0, 2)
(1, 1)
(2, 0)
(0, 3)
(1, 2)
(2, 1)
(3, 0)
(0, 4)
(1, 3)
(2, 2)
(3, 1)
(4, 0)
...

Diagonal 0

$f(0, 0) = 0$

Diagonal 1

$f(0, 1) = 1$
$f(1, 0) = 2$

Diagonal 2

$f(0, 2) = 3$
$f(1, 1) = 4$
$f(2, 0) = 5$

Diagonal 3

$f(0, 3) = 6$
$f(1, 2) = 7$
$f(2, 1) = 8$
$f(3, 0) = 9$

Diagonal 4

$f(0, 4) = 10$
$f(1, 3) = 11$
$f(2, 2) = 12$
$f(3, 1) = 13$
$f(4, 0) = 14$

$$f(a, b) = \begin{array}{c}\text{The number of elements on} \\ \text{all previous diagonals} \\ + \\ \text{The index of the current} \\ \text{pair on its diagonal}\end{array}$$

Diagonal 0

$f(0, 0) = 0$

Diagonal 1

$f(0, 1) = 1$
$f(1, 0) = 2$

Diagonal 2

$f(0, 2) = 3$
$f(1, 1) = 4$
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$$f(a, b) = \sum_{i=1}^{a+b} i + \text{The index of the current pair on its diagonal}$$

Diagonal 0

$f(0, 0) = 0$

Diagonal 1

$f(0, 1) = 1$
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Diagonal 2

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$$f(a, b) = (a + b)(a + b + 1) / 2$$

$+$

The index of the current pair on its diagonal

Diagonal 0

$f(0, 0) = 0$

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$$f(a, b) = \begin{array}{c} (a + b)(a + b + 1) / 2 \\ + \\ a \end{array}$$

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$f(0, 0) = 0$

Diagonal 1

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Diagonal 2

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Diagonal 4

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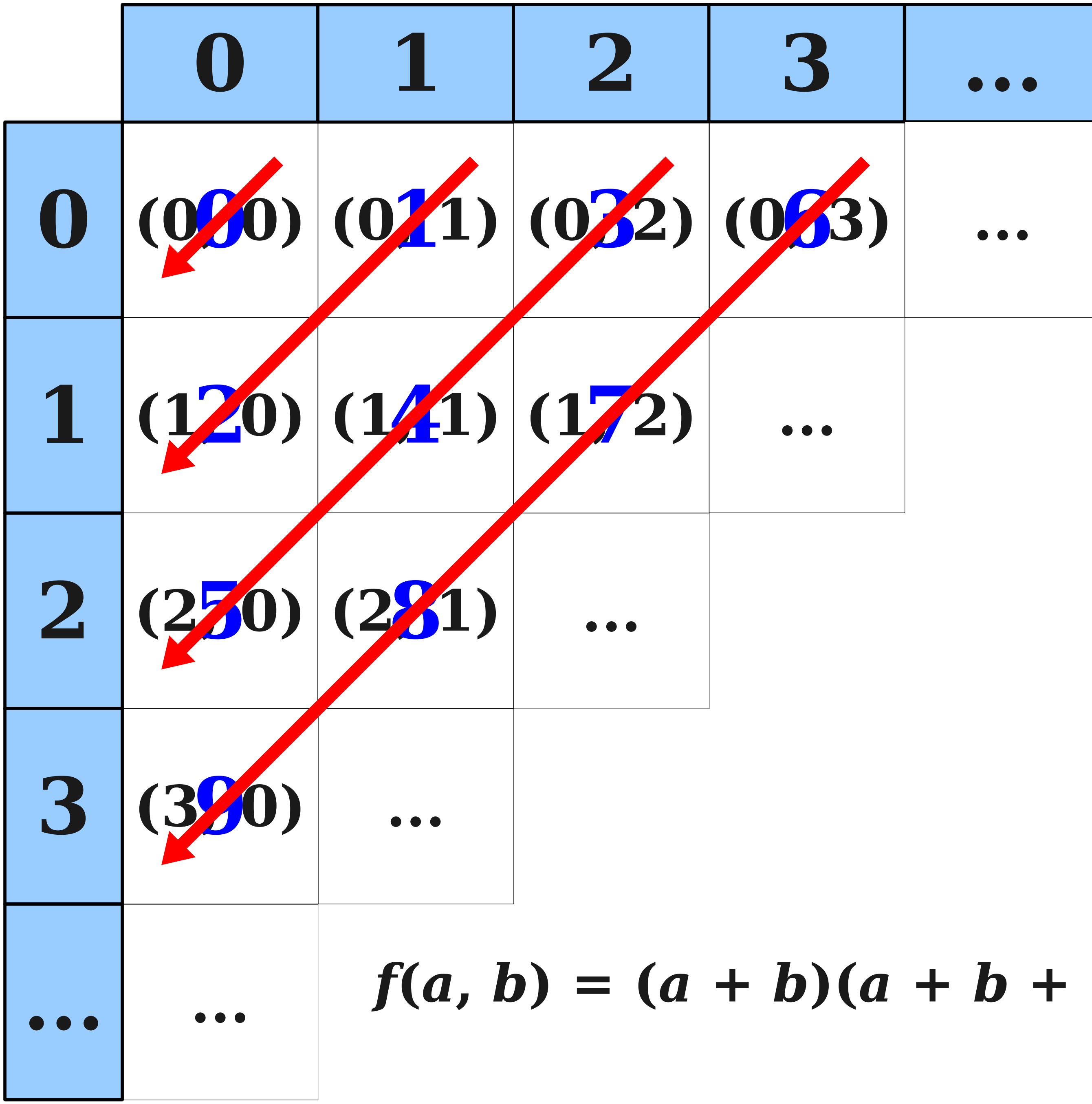
$$f(a, b) = (a + b)(a + b + 1) / 2 + a$$

This function is called
Cantor's Pairing Function.



$$f(a, b) = (a + b)(a + b + 1) / 2 + a$$

Theorem: $|\mathbb{N}^2| = |\mathbb{N}|$.

Formalizing the Proof

- We need to show that this function f is injective and surjective.
- These proofs are nontrivial, but have beautiful intuitions.
- I've included the proofs at the end of these slides if you're curious.

Next Time

- **The Pigeonhole Principle**
  - Pleasing and poignant pigeon-powered proofs!

Appendix: Proof that $|\mathbb{N}^2| = |\mathbb{N}|$

Proving Surjectivity

- Given just the definition of our function:

  $$\boldsymbol{f(a, b) = (a + b)(a + b + 1) / 2 + a}$$

  It is not at all clear that every natural number can be generated.
- However, given our intuition of how the function works (crawling along diagonals), we can start to formulate a proof of surjectivity.

Proving Surjectivity

$$\boldsymbol{f(a, b) = (a + b)(a + b + 1) / 2 + a}$$

- What pair of numbers maps to 137?
- We can figure this out by first trying to figure out what diagonal this would be in.

Proving Surjectivity

$$f(a, b) = (a + b)(a + b + 1) / 2 + a$$

- What pair of numbers maps to 137?
- We can figure this out by first trying to figure out what diagonal this would be in.

	0	**1**	**2**
0	**(0, 0)**	**(0, 1)**	**(0, 2)**
1	**(1, 0)**	**(1, 1)**	**(1, 2)**
2	**(2, 0)**	**(2, 1)**	**(2, 2)**

Proving Surjectivity

$$f(a, b) = (a + b)(a + b + 1) / 2 + a$$

- What pair of numbers maps to 137?
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Proving Surjectivity

$$\boldsymbol{f(a, b) = (a + b)(a + b + 1) \,/\, 2 + a}$$

- What pair of numbers maps to 137?
- We can figure this out by first trying to figure out what diagonal this would be in.

	0	**1**	**2**
0	**(0, 0)**	**(0, 1)**	**(0, 2)**
1	**(1, 0)**	**(1, 1)**	**(1, 2)**
2	**(2, 0)**	**(2, 1)**	**(2, 2)**

Total number of elements before

Row 0: 0
Row 1: 1
Row 2: 3
Row 3: 6
Row 4: 10
...
Row m: $m(m + 1) / 2$

Proving Surjectivity

$$\boldsymbol{f(a, b) = (a + b)(a + b + 1) / 2 + a}$$

- What pair of numbers maps to 137?
- We can figure this out by first trying to figure out what diagonal this would be in.
  - Answer: Diagonal 16, since there are 136 pairs that come before it.
- Now that we know the diagonal, we can figure out the index into that diagonal.
  - 137 – 136 = 1.
- So we'd expect the first entry of diagonal 16 to map to 137.

$$\boldsymbol{f(1, 15) = 16 \times 17 / 2 + 1 = 136 + 1 = 137}$$

Generalizing Into a Proof

- We can generalize this logic as follows.
- To find a pair that maps to n:
  - Find which diagonal the number is in by finding the largest d such that

    $$d(d + 1) / 2 \leq n$$

  - Find which index the in that diagonal it is in by subtracting the starting position of that diagonal:

    $$k = n - d(d + 1) / 2$$

  - The kth entry of diagonal d is the answer:

    $$f(k, d - k) = n$$

Lemma: Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$ be a function from $\mathbb{N}^2$ to $\mathbb{N}$. Then f is surjective.

Lemma: Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$ be a function from $\mathbb{N}^2$ to $\mathbb{N}$. Then f is surjective.

Proof: Consider any $n \in \mathbb{N}$.

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Proof: Consider any $n \in \mathbb{N}$. We will show that there exists a pair $(a, b) \in \mathbb{N}^2$ such that $f(a, b) = n$.

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Consider the largest $d \in \mathbb{N}$ such that $d(d + 1) / 2 \leq n$.

Intuitively, d is the diagonal containing n.

Lemma: Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$ be a function from $\mathbb{N}^2$ to $\mathbb{N}$. Then f is surjective.

Proof: Consider any $n \in \mathbb{N}$. We will show that there exists a pair $(a, b) \in \mathbb{N}^2$ such that $f(a, b) = n$.

Consider the largest $d \in \mathbb{N}$ such that $d(d + 1) / 2 \leq n$. Then, let $k = n - d(d + 1) / 2$.

Intuition: k is the position within this diagonal.

Now, we need to rigorously establish that we came up with a legal pair, and that the pair actually maps to n.

Lemma: Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$ be a function from $\mathbb{N}^2$ to $\mathbb{N}$. Then f is surjective.

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Consider the largest $d \in \mathbb{N}$ such that $d(d + 1) / 2 \leq n$. Then, let $k = n - d(d + 1) / 2$. Since $d(d + 1) / 2 \leq n$, we have that $k \in \mathbb{N}$.

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We need to formalize our intuition by showing that d gives an index on this diagonal.

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> If m and n are natural numbers or integers, then $m < n$ iff $m + 1 \leq n$. This fact is remarkably useful in proofs on $\mathbb{N}$ or $\mathbb{Z}$.

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$$d + 1 \leq k$$
$$d + 1 \leq n - d(d + 1) / 2$$
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$$(2(d + 1) + d(d + 1)) / 2 \leq n$$

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$$
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&d + 1 + d(d + 1) / 2 \leq n \\
&(2(d + 1) + d(d + 1)) / 2 \leq n \\
&(d + 1)(d + 2) / 2 \leq n
\end{aligned}
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Lemma: Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$ be a function from $\mathbb{N}^2$ to $\mathbb{N}$. Then f is surjective.

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But this means that d is not the largest natural number satisfying the inequality $d(d + 1) / 2 \leq n$, a contradiction.

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\end{aligned}$$

But this means that d is not the largest natural number satisfying the inequality $d(d + 1) / 2 \leq n$, a contradiction. Thus our assumption must have been wrong, so $k \leq d$.

Since $k \leq d$, we have that $0 \leq k - d$, so $k - d \in \mathbb{N}$.

We have a valid pair! All that's left to do now is to show that index k on diagonal d maps to n.

Lemma: Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$ be a function from $\mathbb{N}^2$ to $\mathbb{N}$. Then f is surjective.

Proof: Consider any $n \in \mathbb{N}$. We will show that there exists a pair $(a, b) \in \mathbb{N}^2$ such that $f(a, b) = n$.

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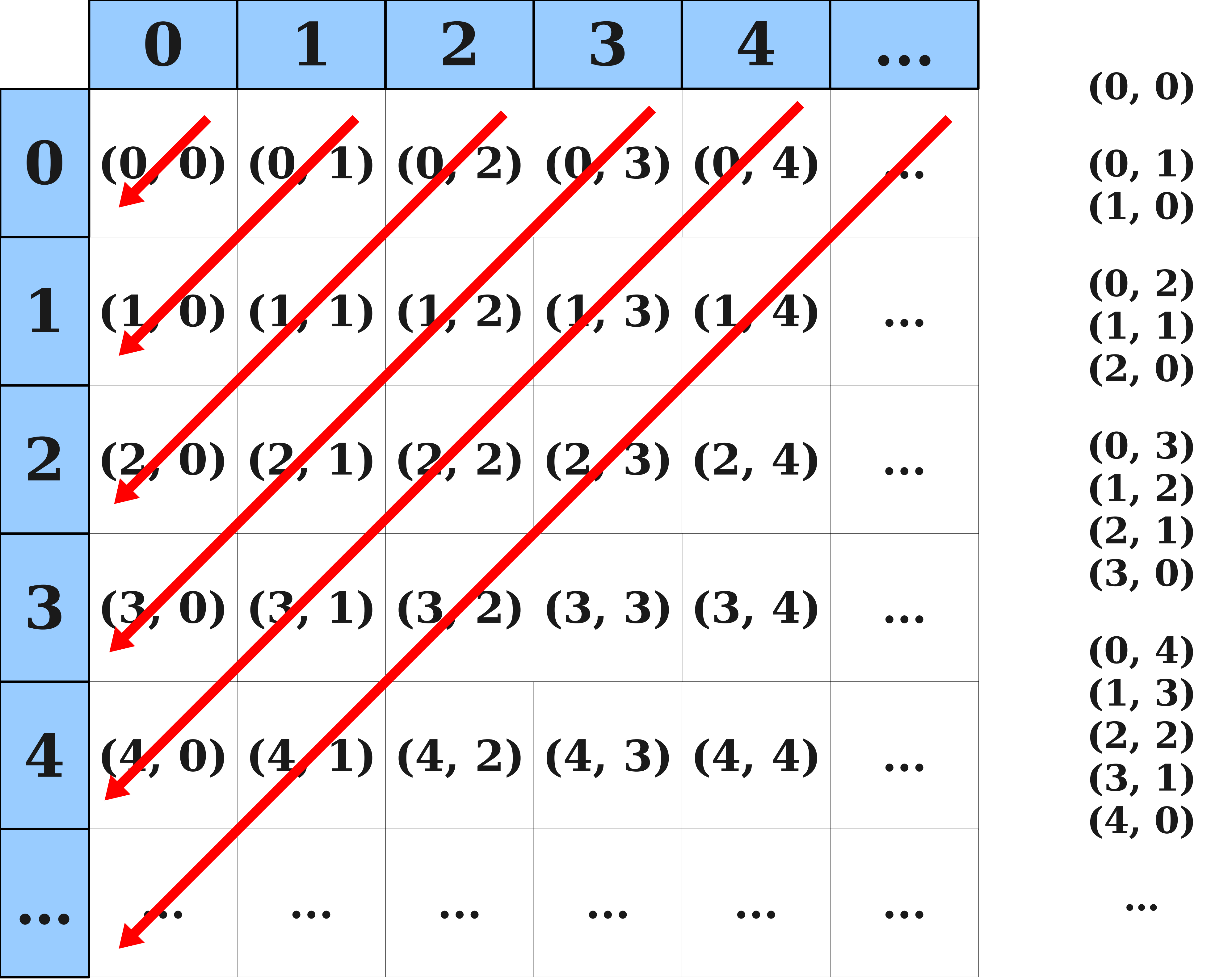
Proving Injectivity

- Given the function

$$\boldsymbol{f(a, b) = (a + b)(a + b + 1) / 2 + a}$$

- It is not at all obvious that f is injective.
- We'll have to use our intuition to figure out why this would be.



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Proving Injectivity

$$f(a, b) = (a + b)(a + b + 1) / 2 + a$$

- Suppose that $f(a, b) = f(c, d)$. We need to prove $(a, b) = (c, d)$.
- Our proof will proceed in two steps:
  - First, we'll prove that (a, b) and (c, d) have to be in the same diagonal.
  - Next, using the fact that they're in the same diagonal, we'll show that they're at the same position within that diagonal.
  - From this, we can conclude $(a, b) = (c, d)$.

Lemma: Suppose $f(a, b) = (a + b)(a + b + 1) / 2 + a$. Then the largest $m \in \mathbb{N}$ for which $m(m + 1) / 2 \leq f(a, b)$ is given by $m = a + b$.

The point of this lemma is to let us "read off" what diagonal we are in just by looking at **a** and **b**. We will need this in a second.

Lemma: Suppose $f(a, b) = (a + b)(a + b + 1) / 2 + a$. Then the largest $m \in \mathbb{N}$ for which $m(m + 1) / 2 \leq f(a, b)$ is given by $m = a + b$.

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$$\begin{aligned} m'(m' + 1) / 2 &\geq (a + b + 1)(a + b + 2) / 2 \\ &= ((a + b)(a + b + 2) + 2(a + b + 1)) / 2 \\ &= (a + b)(a + b + 1) / 2 + a + b + 1 \\ &> (a + b)(a + b + 1) / 2 + a \\ &= f(a, b) \end{aligned}$$

Thus m' does not satisfy the inequality. Consequently, $m = a + b$ is the largest natural number satisfying the inequality.

Lemma: Suppose $f(a, b) = (a + b)(a + b + 1) / 2 + a$. Then the largest $m \in \mathbb{N}$ for which $m(m + 1) / 2 \leq f(a, b)$ is given by $m = a + b$.

Proof: First, we show that $m = a + b$ satisfies the above inequality. Note that if $m = a + b$, we have

$$\begin{aligned} f(a, b) &= (a + b)(a + b + 1) / 2 + a \\ &\geq (a + b)(a + b + 1) / 2 \\ &= m(m + 1) / 2 \end{aligned}$$

So m satisfies the inequality.

Next, we will show that any $m' \in \mathbb{N}$ with $m' > a + b$ will not satisfy the inequality. Take any $m' \in \mathbb{N}$ where $m' > a + b$. This means that $m' \geq a + b + 1$. Consequently, we have

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Thus m' does not satisfy the inequality. Consequently, $m = a + b$ is the largest natural number satisfying the inequality. ■

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First, we will show that $a + b = c + d$.

Intuitively, this proves that (a, b) and (c, d) belong to the same diagonal.

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$$\begin{aligned} f(a, b) &= (a + b)(a + b + 1) / 2 + a \\ &< (c + d)(c + d + 1) / 2 \end{aligned}$$

This step works because we know that any number $\boldsymbol{n}$ bigger than $\boldsymbol{a + b}$ doesn't satisfy

$$\boldsymbol{n(n + 1) / 2 \leq f(a, b)}$$

This means that

$$\boldsymbol{f(a, b) < n(n + 1) / 2.}$$

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But this means that $f(a, b) < f(c, d)$, contradicting that $f(a, b) = f(c, d)$.

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Now that we've got these points in the same diagonal, we just need to show that they have the same index.

Theorem: Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$. Then f is injective.

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$$\begin{aligned}
f(a, b) &= (a + b)(a + b + 1) / 2 + a \\
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But this means that $f(a, b) < f(c, d)$, contradicting that $f(a, b) = f(c, d)$. We have reached a contradiction, so our assumption must have been wrong. Thus $a + b = c + d$. Given this, we have that

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Since $a = c$ and $a + b = c + d$, we have that $b = d$. Thus $(a, b) = (c, d)$, as required.

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