AA278A Homework 2: Stability of Hybrid Systems
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Assigned May 3; Due May 19

If you make use of results from lecture notes or elsewhere, please state the result and the reference.

Problem 1: When Globally Quadratic Lyapunov Theory Fails.

Consider the linear hybrid system example from Lecture 6:

- $Q = \{q_1, q_2\}$, $X = \mathbb{R}^2$
- Init $= Q \times \{x \in X : ||x|| > 0\}$
- $f(q_1, x) = A_1 x$ and $f(q_2, x) = A_2 x$, with:

$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}$$

- Dom $= \{q_1, \{x \in \mathbb{R}^2 : x_1 x_2 \geq 0\}\} \cup \{q_2, \{x \in \mathbb{R}^2 : x_1 x_2 \leq 0\}\}$
- $R(q_1, \{x \in \mathbb{R}^2 : x_1 x_2 \leq 0\}) = (q_2, x)$ and $R(q_2, \{x \in \mathbb{R}^2 : x_1 x_2 \geq 0\}) = (q_1, x)$

Simulation indicates the equilibrium $x_e = 0$ to be stable; yet there is no solution P to the LMI conditions:

$$P = P^T > 0 \quad (1)$$

$$A_i^T P + P A_i < 0 \quad (2)$$

for $i = 1, 2$. As we saw in class, this makes sense, since when the system matrices A_1 and A_2 are interchanged, $x_e = 0$ is unstable.

Using the piecewise quadratic Lyapunov function theorem (Theorem 9) of Lecture 6, prove that $x_e = 0$ of the hybrid system described above is asymptotically stable. HINT 1: The same Lyapunov function actually works across both discrete states. HINT 2: This problem may be done very quickly using MATLAB's LMI toolbox (instructions at the end of Lecture 6).

Problem 2: Stabilizing unstable systems through switching. Give an example of a hybrid automaton that has unstable dynamics in each discrete mode, but for which the equilibrium $x_e = 0$ is stable. Illustrate your example through Matlab simulation. Prove, using one of the stability theorems from class, that your example is stable.

Problem 3.

Consider the following switching system

$$\dot{x} = A_q x$$

where $q \in \{1, 2\}$ and

$$\begin{aligned} A_1 &= \begin{bmatrix} -a_1 & b_1 \\ 0 & -c_1 \end{bmatrix} \\ A_2 &= \begin{bmatrix} -a_2 & b_2 \\ 0 & -c_2 \end{bmatrix} \end{aligned}$$

Assume that a_i, b_i, and c_i, for $i = 1, 2$ are real numbers and that $a_i, c_i > 0$. Show that the switched system is asymptotically stable.

Problem 4: Common Lyapunov Function for Commuting A-matrices. Assume that $K > 1$ matrices A_q, $q \in \{1, \ldots K\}$ are given. Consider the switched linear system $\dot{x} = A_q x$ where $\sigma : \mathbb{R}^n \to Q$ such that for all $(q, x) \in Q \times \mathbb{R}^n$, $f(q, x) = A_q x$. Assume that the matrices A_q are stable (ie. with eigenvalues in the open left half of the complex plane). Also, assume that, for all $i, j \in \{1, \ldots K\}$, $A_i A_j = A_j A_i$. Now, let $P_1, \ldots, P_m$ be the unique symmetric positive definite matrices that satisfy the Lyapunov equations:

$$A_1^T P_1 + P_1 A_1 = -I \qquad (3)$$

$$A_i^T P_i + P_i A_i = -P_{i-1}, \qquad i = 2, \ldots, m \qquad (4)$$

Derive an explicit integral formula for P_m which only depends on the A_i, $i = 1, \ldots m$. Then show that the function $V(q, x) = x^T P_m x$ is a common Lyapunov function for the systems $\dot{x} = A_i x$, $i = 1, \ldots m$.