AA278A Homework 1: Hybrid System Modeling.

Assigned April 11; Due April 26

Problem 1: Consider the discontinuous differential equation

$$\dot{x}_1 = -\text{sgn}(x_1) + 2\text{sgn}(x_2) \quad (1)$$
$$\dot{x}_2 = -2\text{sgn}(x_1) - \text{sgn}(x_2) \quad (2)$$

where $x(0) \neq (0,0)$, and

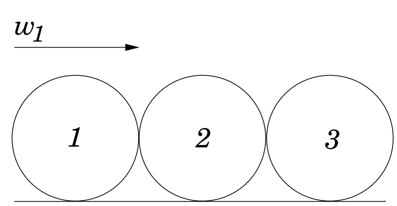
$$\text{sgn}(z) = \begin{cases} 1 & \text{if } z > 0 \\ -1 & \text{if } z < 0 \\ \text{undefined} & \text{otherwise} \end{cases} \quad (3)$$

This system defines a hybrid automaton with four discrete modes having invariants corresponding to the four quadrants.

(a) Specify a non-blocking and deterministic hybrid automaton modeling the system.

(b) Does H accept Zeno executions for every initial state?

Problem 2: Consider three balls with unit masses and suppose that they are touching at time $t = 0$. The initial velocity of ball 1 is $w_1(0) = 1$ and balls 2 and 3 are at rest. Assume that the impact is a sequence of simple inelastic impacts. The first inelastic collision occurs between balls 1 and 2, resulting in $w_1(0+) = w_2(0+) = 0.5$ and $w_3(0+) = 0$. Since $w_2(0+) > w_3(0+)$, ball 2 hits ball 3 instantaneously giving $w_1(0++) = 0.5$ and $w_2(0++) = w_3(0++) = 0.25$. Now $w_1(0++) > w_2(0++)$ so ball 1 hits ball 2 again resulting in a new inelastic collision. This leads to an infinite sequence of collisions.



(a) Model the inelastic collisions of the three ball system as a hybrid automaton with a single discrete mode and three continuous variables (x_1, x_2, x_3) representing the velocities of the balls.

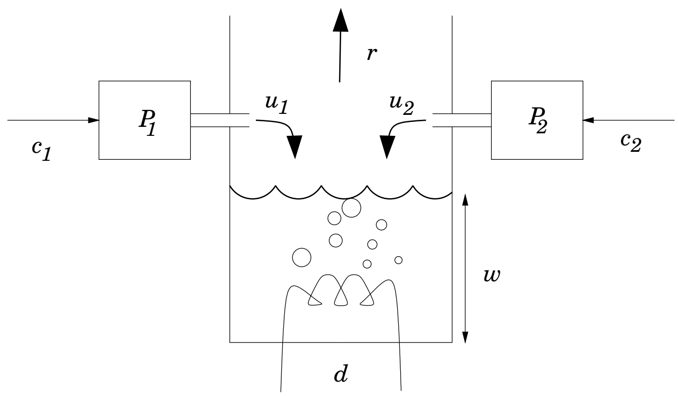
(b) Show that H accepts a Zeno execution corresponding to the sequence of collisions described above.

Problem 3: You may choose to do EITHER (a) or (b).

(a) A "benchmark" problem in hybrid systems analysis and control is that of the steam boiler [1].

The steam boiler consists of a tank containing water and a heating element that causes the water to boil and escape as steam. The water level in the boiler is denoted by w, and, for the sake of simplicity, consider only $w > 0$. The water is replenished by two pumps which at time t pump water into the boiler at rates $u_1(t)$ and $u_2(t)$ respectively. The water boils off at a rate r with d a variable that controls the rate of evaporation: $\dot{r} = d$. It is assumed the value of d at any given time is unknown, yet it is known to lie within given bounds. At every time t, pump i can be either be on ($u_i(t) = P_i$) or off ($u_i(t) = 0$). There is a delay T_i between the time pump i is ordered to switch on and the time u_i switches to P_i. There is no delay when the pumps are switched off.

The requirement is that the pumps are switched on and off so that the water level remains between two values M_1 and M_2.



(i) Derive a deterministic, non-blocking, hybrid automaton for the steam boiler.

(ii) Simulate the system for parameters: $P_i = 2.5$ and $T_i = 5$ (for $i = 1, 2$), $r \in [0, 4]$, $d \in [0, 0.5]$, $M_1 = 1, M_2 = 20$. Using your intuition, can you devise a pumping strategy that keeps the system within the allowable bounds, for possible worst case disturbance?

(b) Consider the inverted pendulum system in the paper handed out this week:
Kuipers and Ramamoorthy, "*Qualitative Modeling and Heterogeneous Control of Global System Behavior*", HSCC2002, LNCS 2289 pp 294-307.

Using the model and control scheme outlined in the paper, implement a controlled inverted pendulum simulation in Matlab. Is there a benefit to this hybrid control scheme against a continuous nonlinear control scheme?

Problem 4: Consider the water tank hybrid automaton given in Figure 4 of Lecture 3. Assume that $\max\{v_1, v_2\} < w < v_1 + v_2$, so that the water tank hybrid automaton is Zeno.

Temporal regularization of this system refers to the situation in which there is a delay $\epsilon > 0$ between the time the inflow is commanded to switch from one tank to the other, and the time the switch actually takes place.

Derive a new hybrid automaton for the water tank which incorporates this regularization, and show that this regularized automaton accepts a unique non-Zeno execution for each initial state.

References

[1] J. R. Abrial, E. Borger, and H. Langmaack. The steam boiler case study project: An introduction. In J. R. Abrial, E. Borger, and H. Langmaack, editors, *Formal Methods for Industrial Applications: Specifying and Programming the Steam Boiler Control*, LNCS 1165. Springer Verlag, 1996.